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FROM THE TRIALITY VIEWPOINT

LUCAS M. CHAVES, A. RIGAS

Abstract. *In this note we give some elementary applications of the concept of triality to the geometry of low dimensional manifolds. We exhibit explicit identifications between the compact, simply connected Lie groups, relations between principal bundles over S^7, a new view of Hopf maps and expressions of the Killing - Cartan orthogonal projections from the Lie algebra of $Spin(8)$ onto the Lie algebra of $Spin(7)$ and G_2. Some basic material, from the books "Spinors and Calibrations "[H] and "Compact Projective Planes" [S-B-G-H-L-S] is repeated here for the sake of completeness.*

1 Triality

Let H be the quaternion algebra [L-M] and K the Cayley algebra defined in $R^8 = H \oplus H$, by $\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac-\bar{d}b \\ da+b\bar{c} \end{pmatrix}$, where all products are quaternionic. It is well known that K is a non associative division algebra with unit $1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. One can fix a basis for $K, 1 = e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} i \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} j \\ 0 \end{pmatrix}, \ldots, e_7 = \begin{pmatrix} 0 \\ k \end{pmatrix}$, where 1,i,j,k are the usual orthonormal basis of the quaternions, the e_l's anticommute for $l = 1, \ldots, 7$, and their square is -1.

Conjugation in K is given by $\overline{\begin{pmatrix} a \\ b \end{pmatrix}} = \begin{pmatrix} \bar{a} \\ -b \end{pmatrix}$. Right and left multiplications define the linear morphisms $L_\alpha(\eta) = \alpha\eta$ and $R_\alpha(\eta) = \eta\alpha$, for α, η in K, that are isometries relative to the euclidean scalar product on $R^8 = K$, if $||\alpha|| = 1$.

Triality, first observed by Study and other algebraic geometers and formalized by Elie Cartan [C] in the early 20's, can be summarized in the following **Triality Principle**:

(T) For all $A \in SO(8)$ there is a unique pair, modulo common change of sign, $(B,C) \in SO(8) \times SO(8)$, such that, for all $\xi, \eta \in K$, $A(\xi\eta) = B(\xi)C(\eta)$, where both products are Cayley multiplications.

Proof: The reflection Re_ξ in $R^8 = K$, with respect to the hyperplane perpendicular to a unitary $\xi \in S^7$, is given by $Re_\xi(x) = -\xi\bar{x}\xi$. Each $A \in SO(8)$ can be written as a product of an even number of reflections and one of the Moufang identities [Mo] is $\xi(ab)\xi = (\xi a)(b\xi)$ for all $\xi, a, b \in K$. All triples (A,B,C) that satisfy **(T)** form a group and we can suppose that A is a product of just two reflections $A(xy) = Re_\xi Re_\eta(xy) = \xi(\eta(xy)\eta)\xi = [\xi(\eta x)][(y\eta)\xi] = [L_\xi L_\eta(x)][R_\xi R_\eta(y)] = B(x)C(y)$, where $B = L_\xi L_\eta, C = R_\xi R_\eta$. To show uniqueness of the pair $\pm(B,C)$, let $A(xy) = B_1(x)C_1(y) = B_2(x)C_2(y)$ for all $x, y \in K$. For $w = B_1(x), u = C_1(y)$, $wu = (B_2(B_1^{-1}w))(C_2(C_1^{-1}(u)) = B_3(w)C_3(u), \forall u, w \in K$. If $u = 1$, $B_3(w) = wb$, where $b =$

$\overline{C_3(1)}$ and similarly $C_3(u) = au$ with $a = \overline{B_3(1)}$. So $wu = (wb)(au), \forall w, u$ and $w = u = 1$ implies $b = a^{-1}$. Replacing w by va we get $(va)u = v(au)$ for all $u, v \in K$, which implies that a is real and therefore $a = \pm 1$, $b = a$ and $(B_3, C_3) = \pm(I, I)$. □

Recall that $SO(n)$ is connected, but not simply connected, its fundamental group is Z_2, for $n \geq 3$, and that its universal (double) covering group is denoted by $Spin(n)$. The Triality Principle **(T)** furnishes an explicit form of representing $Spin(8)$ in $SO(8) \times SO(8)$, since A and B determine C uniquely. $Spin(8) = \{(A, B, C,) \in SO(8) \times SO(8) \times SO(8) | A(\xi\eta) = B(\xi)C(\eta)$ for all ξ, η in $K\}$. Connectedness of $Spin(8)$ follows by observing that the path $(A_t, B_t, C_t) \in Spin(8)$ with $A_t = Re_{e_1} \circ Re_{e_1 \exp(t\pi e_1)}$, $B_t = L_{e_1} \circ L_{e_1 \exp(t\pi e_1)}$, $C_t = R_{e_1} \circ R_{e_1 \exp(t\pi e_1)}$, $0 \leq t \leq 1$, joins the point $(I, -I, -I)$ to (I, I, I), where Re_x denotes the reflection in K relative to the hyperplane perpendicular to x and I is the identity element of $SO(8)$.

A covering map $Spin(8) \longrightarrow SO(8)$ is just $(A, B, C) \longmapsto A$.

Let $\widetilde{A}$ be defined by $\widetilde{A}(x) = \overline{A(\overline{x})}$.

Proposition 1 *If* $(A, B, C) \in Spin(8)$ *then* $(C, \widetilde{B}, A), (\widetilde{A}, \widetilde{C}, \widetilde{B}), (\widetilde{C}, \widetilde{A}, B), (\widetilde{B}, C, \widetilde{A})$ *and* $(B, A, \widetilde{C})$ *are also in* $Spin(8)$.

Proof: For any $\eta \in K$, (T) implies $A(\overline{\eta}(\eta\xi)) = B(\overline{\eta})C(\eta\xi)$,
$\| \eta \|^2 A(\xi) = B(\overline{\eta})C(\eta\xi)$, $\| \eta \|^2 \overline{B(\overline{\eta})}A(\xi) = \overline{B(\overline{\eta})}B(\overline{\eta})C(\eta\xi)$,
$\| \eta \|^2 \overline{B(\overline{\eta})}A(\xi) = \| B(\overline{\eta}) \|^2 C(\eta\xi)$,
$\widetilde{B}(\eta)A(\xi) = C(\eta\xi)$, since $\| B(\overline{\eta}) \| = \| \overline{\eta} \| = \| \eta \|$. So, $(C, \widetilde{B}, A) \in Spin(8)$.
The rest of the proof is completely analogous. □

Observe now that the center of $Spin(8)$ has four elements and is isomorphic to $Z_2 \times Z_2$. The automorphism group of $Spin(8)$ modulo the subgroup of internal automorphisms is parametrized by the group of automorphisms of $Z_2 \times Z_2$, that is S_3, the permutation group of three elements [L-M, pg. 55]. An explicit description of these six external automorphisms is the following:

the identity *id*,
$\delta(A, B, C) = (C, \widetilde{B}, A)$,
$\tau(A, B, C) = (\widetilde{A}, \widetilde{C}, \widetilde{B})$,
$\gamma(A, B, C) = \tau \circ \delta(A, B, C) = (\widetilde{C}, \widetilde{A}, B)$,
$\gamma^2(A, B, C) = (\widetilde{B}, C, \widetilde{A})$,
$\delta \circ \gamma(A, B, C) = (B, A, \widetilde{C})$.

The automorphism γ expresses all basic properties of triality and has order 3, while δ and τ have order 2. Rigorously speaking it is this group S_3 that represents the Triality Principle.

Let G_2 be the automorphism group of K, i.e., A is in G_2 iff $A \in SO(8)$ and $A(\xi\eta) = A(\xi)A(\eta)$, for all $\xi, \eta \in K$. So G_2 can be seen as the subgroup of $Spin(8)$ of the form (A, A, A). As $A(x) = A(1)A(x)$ for all $x \in K$ it follows that $A(1) = 1$ and it is easy to see that A is orthogonal, so $G_2 \subseteq O(8)$. In fact $G_2$is connected and simply connected [P, pg. 310].

Lemma 2 $A \in SO(7) \Leftrightarrow A = \widetilde{A}$.

Proof: For $\eta = \eta_0 + \eta_1$ in $R \cdot e_0 \oplus Im(K)$, $\widetilde{A}(\eta) = \overline{A(\eta_0 - \eta_1)}$
$= \overline{A(\eta_0)} - \overline{A(\eta_1)} = A(\eta_0) + A(\eta_1) = A(\eta)$. □

Proposition 3 *G_2 is the fixed point subgroup of γ.*

Proof: If $\gamma(A,B,C)=(A,B,C)$ then $(A,B,C)=(A,\widetilde{A},\widetilde{A})$, with $A\in SO(7)$.
By Lemma 2 then $(A,B,C)=(A,A,A)$. Obviously, γ fixes G_2. □

An immediate Corollary of this is that the Killing-Cartan orthogonal projection from the Lie algebra $\widehat{Spin(8)}$ onto the Lie algebra $\widehat{G_2}$ is given precisely by averaging over the infinitesimal version of the subgroup of S_3 generated by γ. We postpone giving a precise statement of this and its proof until §2, where we deal with infinitesimal triality. If $Spin(7)$ is the subgroup of $Spin(8)$ defined by $A\in SO(7)$, observing that $(I,-I,-I)$ is in $Spin(7)$, its connectedness follows as that of $Spin(8)$, replacing $e_1\exp(t\pi e_1)$ by $e_1\exp(t\pi e_3)$.

One can easily show now:

Proposition 4 *(i) The fixed point subgroup of the automorphism τ is $Spin(7)$ defined by $Spin(7)=\{(A,B,C)\in Spin(8)\mid A(1)=1\}$.*

(ii) δ fixes $\gamma(Spin(7))$.

We can define analogously the following subgroups of $Spin(7)$,
$Spin(6)=\{A(e_1)=e_1 \text{ or } A\in SO(6)\}$
$Spin(5)\subseteq Spin(6)$ by $\{A(e_2)=e_2$, i.e., $A\in SO(5)\}$
$Spin(4)\subseteq Spin(5)$ by $\{A(e_3)=e_3$, i.e., $A\in SO(4)\}$
$Spin(3)\subseteq Spin(4)$ by $\{A(e_4)=e_4$, i.e., $A\in SO(3)\}$.

Proposition 5 $Spin(7)=\{(A,B,\widetilde{B})\in Spin(8)\}$.

Proof: $(A,B,C)\in Spin(7)$ if and only if $1=A(1)$. For all $x\neq 0$ one has $A(1)=\frac{1}{\|x\|^2}A(x\bar{x})=\|x\|^{-2}\cdot B(x)C(\bar{x})$, i.e., $\overline{B(x)}=C(\bar{x})$, since B is orthogonal. So $\widetilde{B}=C$. □

An easy exercise shows $A(1)=1\Leftrightarrow A=\widetilde{A}$. Similarly one can also show that $Spin(7)\frown\gamma(Spin(7))=Spin(7)\frown\gamma^2(Spin(7)=\gamma(Spin(7))\frown\gamma^2(Spin(7))=G_2$.

Some classical fibrations related to *Spin* groups become quite simple when using the above triality representations.

Proposition 6 *The well-known principal bundle projections can be described as indicated below:*

a) $Spin(7)\dashrightarrow Spin(8)\to S^7$, $(A,B,C)\mapsto A(1)$.
b) $Spin(6)\dashrightarrow Spin(7)\to S^6$, $(A,B,\widetilde{B})\mapsto A(e_1)$.
c) $Spin(5)\dashrightarrow Spin(6)\to S^5$, $(A,B,\widetilde{B})\mapsto A(e_2)$.
d) $Spin(4)\dashrightarrow Spin(5)\to S^4$, $(A,B,\widetilde{B})\mapsto A(e_3)$.
e) $Spin(3)\dashrightarrow Spin(4)\to S^3$, $(A,B,\widetilde{B})\mapsto A(e_4)$.
f) $G_2\dashrightarrow Spin(8)\to S^7\times S^7$, $(A,B,C)\mapsto(A(1),B(1))$.
g) $G_2\dashrightarrow\gamma(Spin(7))\to S^7$, $(B,A,B)\mapsto B(1)$.
h) $G_2\dashrightarrow Spin(7)\to S^7$, $(A,B,\widetilde{B})\mapsto B(1)$.

Proof: We will just show g). The rest are proven the same way.
If $B(1)=1$ then $1=B(1)=A(1)B(1)=A(1)$, so $B(x)=B(x\cdot 1)$
$=A(x)B(1)=A(x)$, for all x so $A=B$. □

2 The exceptional isomorphisms

Here we show how triality provides an explicit and unified way to all low dimensional, compact, simply connected Lie group identifications. In particular,
$Spin(6) \equiv SU(4)$, $Spin(5) \equiv Sp(2)$, $Spin(4) \equiv S^3 \times S^3$ and $Spin(3) \equiv S^3$.

Recall that $Sp(2) = \{A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, a, ..., d \in H \mid A^\star A = AA^\star = I\}$. This group is represented in $SO(8)$ by all matrices that commute with two given anticommuting complex structures [Ch]. The same way $SU(4) = \{A \in C(4) \mid AA^\star = I\}$ is represented by all matrices in $SO(8)$ that anticommute with a given complex structure.

Proposition 7 *The map $(A,B,\widetilde{B}) \mapsto B$ defines group isomorphisms between*
a) $Spin(6)$ and $SU(4) \subseteq SO(8)$,
b) $Spin(5)$ and $Sp(2) \subseteq SO(8)$.

Proof: We will just show b), the proof of a) being easier along the same line.

Let $(A,B,\widetilde{B}) \in Spin(6)$ and apply γ to obtain $(B,A,B) \in Spin(8)$. So, $B(e_1\eta) = A(e_1)B(\eta) = e_1B(\eta)$, i.e., B commutes with the complex structure L_{e_1}, left Cayley multiplication by e_1, and therefore B belongs to the subgroup $SU(4)'$ of $SO(8)$ defined by L_{e_1}. Similarly, if $(A,B,\widetilde{B})$ is in $Spin(5)$, B commutes with L_{e_1} and L_{e_2} so it belongs to the subgroup $Sp(2)'$ of $SO(8)$ defined by commuting with the pair of anticommuting complex structures L_{e_1} and L_{e_2}. It is clear that this projection is a group morphism and that if $B = I$ then $(A,B,\widetilde{B}) = (I,I,I)$. Dimension counting shows it to be an isomorphism in each case. □

The complex structures usually considered to define $SU(4)$ and $Sp(2)$ in $SO(8)$ are not related to Cayley multiplication. For example, $Sp(2)$ is defined as all matrices in $SO(8)$ that commute with right quaternionic multiplication, say by $q \in H$, on $H \oplus H$, that sends $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} aq \\ bq \end{pmatrix}$. In this case the two complex structures in $SO(8)$ can be $C_i\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ai \\ bi \end{pmatrix}$ and $C_j\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} aj \\ bj \end{pmatrix}$. In our case, $L_{e_1}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ia \\ bi \end{pmatrix}$ and $L_{e_2}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ja \\ bj \end{pmatrix}$. The group $O(8)$ acts transitively on pairs of anticommuting complex structures in R^8 by conjugation with isotropy subgroup $Sp(2)$ [P, pg.269]. For $(A,B,\widetilde{B}) \in Spin(4)$ we have $A(e_3) = e_3$ as well as the conditions defining $Spin(5)$, so $A \in \begin{pmatrix} I_4 & 0 \\ 0 & SO(4) \end{pmatrix}$. It is well known [Cu, pg.144] that for each such A there exists a unique, modulo common sign, ordered pair of unit quaternions (p,q), with $A = \begin{pmatrix} I_4 & 0 \\ 0 & l_p \circ r_{\bar{q}} \end{pmatrix}$, i.e., $A\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ p\eta\bar{q} \end{pmatrix}$. Triality implies then, that $B = \begin{pmatrix} r_{\bar{q}} & 0 \\ 0 & l_p \end{pmatrix}$ and $\widetilde{B} = \begin{pmatrix} l_q & 0 \\ 0 & l_p \end{pmatrix}$. Similarly, $Spin(3)$ will consist of all $(A,B,\widetilde{B})$ in $Spin(4)$ with $A(e_4) = e_4$. This implies $p1\bar{q} = 1$ or $p = q$ and $Spin(3)$ consists of all $(\begin{pmatrix} I_4 & 0 \\ 0 & l_p \circ r_{\bar{p}} \end{pmatrix}, \begin{pmatrix} r_{\bar{p}} & 0 \\ 0 & l_p \end{pmatrix}, \begin{pmatrix} l_p & 0 \\ 0 & l_p \end{pmatrix})$, where l and r stand for left and right quaternionic multiplication. Observe that this $Spin(3)$ is not a subgroup of G_2.

Similarly, $SO(4)/Sp(1)$ can be identified with pairs of anticommuting complex structures on R^4. There exist, therefore, elements $D \in O(8)$ with $D \circ L_{e_i} \circ D^{-1} = C_i$, $i = 1,2$. It is convenient to choose $D \in \begin{pmatrix} SO(4) & 0 \\ 0 & I_4 \end{pmatrix} \subseteq SO(8)$, so that the effect on the second $H-$ summand that is identical for L_{e_s} and C_s , $s = i, j$ remains unchanged. Let β denote quaternionic conjugation in H and l_s, respectively r_s, $s = i, j, k$ denote left, respectively right, quaternionic multiplication by s. Then we have,

Lemma 8 $(l_k \circ \beta)^{-1} \circ l_s \circ (l_k \circ \beta) = r_s$, $s = i, j$.

Proof: For $s = i$ and any $a \in H$, $(l_k \circ \beta)^{-1} \circ l_i \circ (l_k \circ \beta)(a) = \overline{l_k^{-1}(ik\overline{a})} = -\overline{(kik\overline{a})} = \overline{ik^2\overline{a}} = \overline{-i\overline{a}} = -a\overline{i} = ai = r_i(a)$. The same holds for $s = j$. □

Corollary 9 *If* $d = l_k \circ \beta \in O(4)$ *and* $D = \begin{pmatrix} d^{-1} & 0 \\ 0 & I_4 \end{pmatrix}$, *then* $D \circ L_{e_s} \circ D^{-1} = C_s$, $s = 1,2$ *and* $D \circ Sp(2)' \circ D^{-1} = Sp(2)$. *A similar consideration identifies* B *of* $(A, B, \widetilde{B}) \in Sp(2)$ *with an element of* $SU(4)$.

3 Relations with Hopf maps

The group $Sp(2)$ is the total space of an S^3- principal bundle over S^7 by projecting, say, on the first column, $S^3 \cdots Sp(2) \to S^7$ and in this case S^3 acts from the right as the subgroup $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$, $\| q \| = 1$. We will translate this bundle to a $Spin(5)$ setting and then look at some of the consequences.

The map $\Phi : S^3 \times S^3 \to G_2$ defined by $\Phi(p,q)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} pa\overline{p} \\ qb\overline{p} \end{pmatrix}$, determines an inclusion of $SO(4) = (S^3 \times S^3)/Z_2$ in G_2 as a subgroup, $(SO(4), G_2)$ is a symmetric pair [C-R1]. If $S^3_q = \Phi(1 \times S^3) \cong S^3$, then $S^3_q \subseteq Spin(5)$.

Proposition 10 *We have the principal* S^3*- bundle* $S^3_q \cdots Spin(5) \to S^7$, *with* $\pi(A, B, \widetilde{B}) = B(1)$.

Proof: We must show that the action of S^3_q is compatible with the projection π . If $B(1) = 1$, then $\widetilde{B}(1) = 1$ too and $A(x) = B(x)\widetilde{B}(1) = B(1)\widetilde{B}(x)$, so $(A, B, \widetilde{B}) = (A, A, A) \in G_2$. Since $A(e_1) = e_1$ and $A(e_2) = e_2$, it follows that $A(e_3) = e_3$ and so A lives in $\begin{pmatrix} I_4 & 0 \\ 0 & SO(4) \end{pmatrix} \subseteq SO(8)$. So, $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ A'(b) \end{pmatrix}$ for some $A' \in SO(4)$. But $\begin{pmatrix} -\overline{y}x \\ 0 \end{pmatrix} = A\begin{pmatrix} -\overline{y}x \\ 0 \end{pmatrix} = A\left(\begin{pmatrix} 0 \\ x \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix}\right) = A\begin{pmatrix} 0 \\ x \end{pmatrix} A\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ A'(x) \end{pmatrix}\begin{pmatrix} 0 \\ A'(y) \end{pmatrix} = \begin{pmatrix} -\overline{A'(y)}A'(x) \\ 0 \end{pmatrix}$ and $\overline{y}x = \overline{A'(y)}A'(x)$. Let $y = 1$, $A'(x) = A'(1)x$, $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ A'(1)b \end{pmatrix} \in S^3_q$. □

The projection on the second column of $Sp(2)$, corresponding to the action of $S^3_p = \Phi(S^3 \times 1) = \{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mid \| p \| = 1\} \cong S^3$, again from the right, is also a principal S^3-

bundle over S^7 and one may combine these two, together with the classical Hopf projection $h : S^7 \to S^4$ to obtain the following commutative diagram

$$\begin{array}{ccccccc} & & S^3 & & S^3 & & \\ & & \vdots & \pi' & \vdots & & \\ S^3 & \cdots & Sp(2) & \longrightarrow & S^7 & & \\ & \pi & \downarrow & & \downarrow & -\iota_4 \circ h & \\ S^3 & \cdots & S^7 & \longrightarrow & S^4 & & \\ & & & h & & & \end{array}$$

Diagram 1.

Here π' is the second column projection, all solid arrows are principal S^3 - bundle projections, $h\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \| a \|^2 - \| b \|^2 \\ 2a\bar{b} \end{pmatrix}$ is the classical Hopf map and $-\iota_4$ is the antipodal map of S^4. By abuse of notation we write $-h$ for $-\iota_4 \circ h$. The commutativity of this diagram is precisely the characterization of $Sp(2)$ as all 2×2 quaternionic matrices with $A^\star A = I = AA^\star$. The compositions $h \circ \pi$ and $-h \circ \pi'$ result in the bundle projection $Spin(4) \cdots Sp(2) \longrightarrow S^4$, which is another way of observing that $Sp(2)$ is isomorphic to $Spin(5)$. Observe that $S_p^3 \times S_q^3$ is precisely $Spin(4) \subseteq Spin(7)$, as

$$Spin(4) = \{\left(\begin{pmatrix} I_4 & 0 \\ 0 & l_p \circ r_{\bar{q}} \end{pmatrix}, \begin{pmatrix} r_{\bar{q}} & 0 \\ 0 & l_p \end{pmatrix}, \begin{pmatrix} l_q & 0 \\ 0 & l_p \end{pmatrix} \right) |$$

$p, q \in H$, unitary$\}$. This is the same as all $(A, B, \widetilde{B}) \in Spin(7)$ with $A(e_s) = e_s$, $s = 1,2,3$.

The bundle $Spin(4) \cdots Spin(5) \longrightarrow S^4$ is defined by $(A, B, \widetilde{B}) \longmapsto A(e_3)$. We want to express Diagram 1 in terms of Spins and triality, therefore Cayley numbers, instead of in terms of quaternions.

Proposition 11 *For* $(A, B, \widetilde{B}) \in Spin(5)$ *we have*
$A(e_3) = (e_1 B(1))(\overline{B(1)} e_2) \in S^4 \subseteq R^5 \equiv span\{e_3, ..., e_7\} \subseteq R^8 \equiv K.$

Proof: Observe that $A(e_3) = A(e_1 e_2) = B(e_1)\widetilde{B}(e_2)$. Applying the triality automorphism we have $\gamma(A, B, \widetilde{B}) = (B, \widetilde{A}, B)$ and $\gamma^2(A, B, \widetilde{B}) = (\widetilde{B}, \widetilde{B}, \widetilde{A})$, therefore, $B(e_1) = B(e_1 \cdot 1) = \widetilde{A}(e_1)B(1) = A(e_1)B(1) = e_1 B(1)$ and

$\widetilde{B}(e_2) = \widetilde{B}(1 \cdot e_2) = \widetilde{B}(1)\widetilde{A}(e_2) = \overline{B(1)}A(e_2) = \overline{B(1)}e_2$.

Consequently, $A(e_3) = (e_1 B(1))(\overline{B(1)} e_2)$.

As $A(e_3)$ is perpendicular to $1 = A(1)$, $e_1 = A(e_1)$ and $e_2 = A(e_2)$ the result follows.

□

Remark 12 *In [C-R3] we show that the map* $\alpha \mapsto (e_1\alpha)(\overline{\alpha} e_2)$ *from* S^7 *to* $S^4 \subseteq R^5 = span$ $\{e_3, ..., e_7\}$ *is essentially the Hopf map h expressed in terms of Cayley numbers. Its non triviality is due to the non associativity of this product, as the formula clearly shows. From the above we have*

Corollary 13 *The following diagram is commutative and it is the Spin-version of Diagram 1.*

$$
\begin{array}{cccccccc}
Spin(4) & & S^3_p & & & & S^3 & \\
 & \ddots & \vdots & & & & \vdots & \\
S^3_q & \cdots & Spin(5) & & \longrightarrow & & S^7 & \\
 & & & (A,B,\widetilde{B}) & \longmapsto & \widetilde{B}(e_4) & & \\
 & & \downarrow & \downarrow & & \downarrow & \downarrow & -h \\
 & & & B(1) & \longmapsto & A(e_3) & & \\
S^3 & \cdots & S^7 & & \longrightarrow & & S^4 & \\
 & & & \alpha & \longmapsto & (e_1\alpha)(\overline{\alpha}e_2) & &
\end{array}
$$

Diagram 2.

Proof: We just have to show that $-h(\widetilde{B}(e_4)) = A(e_3)$. But $A(e_3) = A(-e_5e_6) = -B(e_5)\widetilde{B}(e_6) = B(e_5)\overline{B(e_6)}$. From $\gamma(A,B,\widetilde{B}) = (B,\widetilde{A},B)$ we have

$B(e_5) = \widetilde{A}(e_1)B(e_4) = e_1B(e_4)$ and $B(e_6) = B(e_2e_4) = \widetilde{A}(e_2)B(e_4) = e_2B(e_4)$ which implies $\overline{B(e_6)} = -\overline{B(e_4)}e_2$. Therefore, $A(e_3) = -(e_1B(e_4))(\overline{B(e_4)}e_2) = -h(\widetilde{B}(e_4))$. □

4 Infinitesimal Triality

By taking the first derivative of (T) we get an expression for the triality relation in the Lie algebra $\widehat{Spin(8)}$: Let $\gamma(t) = (A(t),B(t),C(t))$ be a curve in $Spin(8)$, with $\gamma(0) = (I,I,I)$ and $\gamma'(0) = (A_0,B_0,C_0)$.Then we have for any $\xi,\eta \in K$,

$$A_0(\xi\eta) = B_0(\xi)\cdot\eta + \xi\cdot C_0(\eta)$$

Call this relation ($\mathbf{T}'$).

It is convenient to write the relation above as

$\widehat{Spin(8)} = \{(X,X^\lambda,X^\rho) \in \widehat{SO(8)} \times \widehat{SO(8)} \times \widehat{SO(8)}\}$,

where $X(\xi\eta) = X^\lambda(\xi)\eta + \xi X^\rho(\eta)$ for any $\xi,\eta \in K$. As conjugation commutes with derivatives, we have $\widehat{Spin(7)} = \{(X,X^\lambda,\widetilde{X^\lambda}) \in \widehat{Spin(8)}\}$.

The automorphisms γ,δ, and τ of Spin(8) define, by linearity, Lie algebra automorphisms of $\widehat{Spin(8)}$.

Proposition 14 *The maps* $\frac{1}{2}(id+\tau) = M$ *and* $\frac{1}{3}(id+\gamma+\gamma^2) = \Lambda$ *from* $\widehat{Spin(8)}$ *are the Killing - Cartan projections onto a)* $\widehat{Spin(7)}$ *and b)* $\widehat{G_2}$.

Proof: We will only show b). Part a) is completely analogous.

First note that $\Lambda^2 = \Lambda$, since γ has order 3. The image of Λ is equal to $\widehat{G_2}$, since $\Lambda(X,X^\lambda,X^\rho) = \frac{1}{3}(X+\widetilde{X}^\lambda+\widetilde{X}^\rho, X^\lambda+\widetilde{X}+X^\rho, X^\rho+X^\lambda+\widetilde{X})$ is always of the form $(W,\widetilde{W},\widetilde{W})$. As $W(1) = W(1\cdot 1) = 1\cdot\widetilde{W}(1) + \widetilde{W}(1)\cdot 1 = 2\cdot\widetilde{W}(1)$ and

$\| W(1) \| = \| \widetilde{W}(1) \|$ we have that $W(1) = 0$ and therefore $\widetilde{W} = W$. As Λ fixes every element of $\widehat{G_2}$ we have the equality, as claimed. The kernel of Λ is orthogonal to $\widehat{G_2}$ with respect to the Killing - Cartan metric, by

$\langle (X, X^\lambda, X^\rho), (Z, Z, Z) \rangle = -Trace(XZ + X^\lambda Z + X^\rho Z)$
$= -Trace(X + X^\lambda + X^\rho)Z = 0$, as (X, X^λ, X^ρ) is in $\ker(\Lambda)$. □

5 Further Remarks

For $\alpha \in S^7$ we have $L_\alpha, R_\alpha \in SO(8)$ and the Cayley conjugation
$C_\alpha = L_\alpha \circ R_{\overline{\alpha}} \in SO(7)$. In [T-S-Y] was proved that the map $\alpha \longmapsto C_\alpha$ generates $\pi_7(SO(7)) \cong Z$ and $\pi_7(SO) \cong Z$ (where SO is the infinite special orthogonal group) and that $\alpha \longmapsto L_\alpha$ (similarly, $\alpha \longmapsto R_\alpha$) generate the second component of $\pi_7(SO(8)) \cong Z \oplus Z$. The Moufang identities [Mo] show that $\Psi(\alpha) = (L_\alpha \circ R_{\overline{\alpha}}, L_\alpha \circ R_{\alpha^2}, L_{\overline{\alpha^2}} \circ R_{\overline{\alpha}}) \in Spin(7)$ and $\Theta(\alpha) = (L_\alpha, L_\alpha \circ R_\alpha, L_{\overline{\alpha}}) \in Spin(8)$, for all $\alpha \in S^7$. The maps Ψ and Θ provide explicit constructions for homotopy generators in several other cases [C-R1], [C-R2], [C-R3], [W].

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