Note di Matematica Vol. 19 - n. 2, 173-181 (1999)

FORCED OSCILLATIONS OF SOLUTIONS OF IMPULSIVE NONLINEAR HYPERBOLIC DIFFERENTIAL-DIFFERENCE EQUATIONS

DRUMI BAINOV, EMIL MINCHEV

Abstract. *Sufficient conditions for forced oscillations of the solutions of impulsive nonlinear hyperbolic differential-difference equations are obtained.*
KEYWORDS AND PHRASES: forced oscillations, impulsive hyperbolic differential-difference equations.
AMS SUBJECT CLASSIFICATION: 35B05.

1 Introduction

The oscillation theory is a traditional direction of the qualitative theory of partial differential equations (PDE) and it is an object of permanent interest [8], [10], [11]. On the other hand, the theory of impulsive PDE underwent rapid development in the last four years [1]–[7], [9]. In view of the numerous applications of the impulsive PDE to science and technology it is important to investigate the oscillation theory for this new class of PDE.

The present paper deals with the oscillation properties of impulsive nonlinear hyperbolic differential-difference equations. Sufficient conditions are obtained such that every solution satisfying certain boundary condition is oscillatory.

2 Preliminary notes

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose that $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ are given numbers and $t_{k+l} = t_k + \sigma$, $k = 0, 1, \ldots$, where $\sigma =$ const > 0 and l is a fixed natural number.

Define $J_{imp} = \{t_k\}_{k=1}^{\infty}$, $\mathbb{R}_+ = [0, +\infty)$, $E^0 = [-\sigma, 0] \times \overline{\Omega}$, $E = (0, +\infty) \times \Omega$, $E^* = \mathbb{R}_+ \times \overline{\Omega}$, $E_{imp} = \{(t,x) \in E : t \in J_{imp}\}$, $E^*_{imp} = \{(t,x) \in E^* : t \in J_{imp}\}$.

Let $C_{imp}[E^0 \cup E^*, \mathbb{R}]$ be the class of all functions u: $E^0 \cup E^* \to \mathbb{R}$ such that:

(i) The restriction of u to the set $E^0 \cup E^* \setminus E^*_{imp}$ is a continuous function.

(ii) For each $(t,x) \in E^*_{imp}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u(q,s) = u(t^-,x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q>t}} u(q,s) = u(t^+,x)$$

and $u(t,x) = u(t^+,x)$ for $(t,x) \in E^*_{imp}$.

The class of functions $C_{imp}[E^*, \mathbb{R}]$ is defined analogously as E^* is written instead of $E^0 \cup E^*$ in the above definition.

Let $C^{\prime}_{imp}[E^0 \cup E^*, \mathbb{R}]$ be the class of all functions $u \in C_{imp}[E^0 \cup E^*, \mathbb{R}]$ such that:

(i) u_t: $E^* \setminus E^*_{imp} \to \mathbb{R}$ and it is a continuous function.

(ii) For each $(t,x) \in E^*_{imp}$ there exist the limits

$$\lim_{\substack{(q,s)\to(t,x)\\ q<t}} u_t(q,s) = u_t(t^-,x), \qquad \lim_{\substack{(q,s)\to(t,x)\\ q>t}} u_t(q,s) = u_t(t^+,x)$$

and $u_t(t,x) = u_t(t^+,x)$ for $(t,x) \in E^*_{imp}$.

Consider the nonlinear hyperbolic differential-difference equation

$$u_{tt}(t,x) - \Delta u(t,x) + p(t,x)f(u(t-\sigma,x)) = H(t,x), \quad (t,x) \in E \setminus E_{imp}, \tag{1}$$

subject to the impulsive conditions

$$u(t,x) - u(t^-,x) = g(t,x,u(t^-,x)), \quad (t,x) \in E^*_{imp}, \tag{2}$$

$$u_t(t,x) - u_t(t^-,x) = h(t,x,u_t(t^-,x)), \quad (t,x) \in E^*_{imp} \tag{3}$$

and the boundary conditions

$$\frac{\partial u}{\partial n}(t,x) + \gamma(t,x)u(t,x) = 0, \qquad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega, \tag{4}$$

or

$$u(t,x) = 0, \qquad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega. \tag{5}$$

The functions $p{:}\,E^* \to \mathbb{R}$, $f{:}\,\mathbb{R} \to \mathbb{R}$, $H{:}\,E^* \to \mathbb{R}$, $g{:}\,E^*_{imp} \times \mathbb{R} \to \mathbb{R}$, $h{:}\,E^*_{imp} \times \mathbb{R} \to \mathbb{R}$, $\gamma{:}\,\mathbb{R}_+ \times \partial\Omega \to \mathbb{R}$ are given.

Definition 1 The function $u{:}\,E^0 \cup E^* \to \mathbb{R}$ is called a solution of problem (1) – (4) ((1)–(3), (5)) if:

(i) $u \in C^t_{imp}[E^0 \cup E^*, \mathbb{R}]$, there exist the derivatives $u_{tt}(t,x)$, $u_{x_ix_i}(t,x)$, $i = 1,\ldots,n$ for $(t,x) \in E \setminus E_{imp}$ and u satisfies (1) on $E \setminus E_{imp}$.

(ii) u satisfies (2)–(4) ((2), (3), (5)).

Definition 2 The nonzero solution $u(t,x)$ of equation (1) is said to be nonoscillating if there exists a number $\mu \geq 0$ such that $u(t,x)$ has a constant sign for $(t,x) \in [\mu,+\infty) \times \Omega$. Otherwise, the solution is said to oscillate.

For the function sign we have adopted the following definition

$$\operatorname{sign} x = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

Introduce the following assumptions:

H1. $p \in C_{imp}[E^*, \mathbb{R}_+]$.

H2. $g \in C(E^*_{imp} \times \mathbb{R}, \mathbb{R})$.

H3. $\gamma \in C_{imp}[\mathbb{R}_+ \times \partial\Omega, \mathbb{R}_+]$.

H4. $f \in C(\mathbb{R},\mathbb{R})$, $f(u) = -f(-u)$ for $u \geq 0$, f is a positive and convex function in the interval $(0,+\infty)$.

H5. $H \in C_{imp}[E^*, \mathbb{R}]$.

In the sequel the following notations will be used:

$$P(t) = \min\{p(t,x)\colon x \in \overline{\Omega}\},$$

$$V(t) = \int\limits_{\Omega} u(t,x)dx \left(\int\limits_{\Omega} dx \right)^{-1},$$

$$H_0(t) = \int\limits_{\Omega} H(t,x)dx \left(\int\limits_{\Omega} dx \right)^{-1}.$$

3 Main results

We give sufficient conditions for oscillation of the solutions of the problem (1)–(4).

Lemma 1 *Let the following conditions hold:*

1. *Assumptions* H1–H5 *are fulfilled.*

2. $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *is a positive solution of the problem* (1) – (4) *in the domain* E.

3. $g(t_k,x,\xi) \le L_k\xi$, $h(t_k,x,\eta) = L_k\eta$, $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $x \in \overline{\Omega}$, $k = 1,2,\ldots$, $L_k \ge 0$ *are constants.*

Then the function $V(t)$ *satisfies for* $t \ge \sigma$ *the impulsive differential inequality*

$$V''(t) + P(t)f(V(t-\sigma)) \le H_0(t), \quad t \ne t_k, \tag{6}$$

$$V(t_k) \le (1+L_k)V(t_k^-), \tag{7}$$

$$V'(t_k) = (1+L_k)V'(t_k^-). \tag{8}$$

Proof. Let $t \ge \sigma$. Integrating equation (1) with respect to x over the domain Ω, we obtain

$$\begin{aligned} \frac{d^2}{dt^2}\int\limits_{\Omega} u(t,x)dx \quad & - \int\limits_{\Omega} \Delta u(t,x)dx + \\ & + \int\limits_{\Omega} p(t,x)f(u(t-\sigma,x))dx = \int\limits_{\Omega} H(t,x)dx, \quad t \ne t_k. \end{aligned} \tag{9}$$

From the Green formula and H3 it follows that

$$\int\limits_{\Omega} \Delta u(t,x)dx = \int\limits_{\partial\Omega} \frac{\partial u}{\partial n}dS = -\int\limits_{\partial\Omega} \gamma(t,x)u(t,x)dS \le 0, \quad t \ne t_k. \tag{10}$$

Moreover, for $t \ne t_k$, the Jensen inequality enables us to get

$$\begin{aligned} \int\limits_{\Omega} p(t,x) \quad & f(u(t-\sigma,x))dx \ge P(t)\int\limits_{\Omega} f(u(t-\sigma,x))dx \ge \\ & \ge P(t)f\left(\int\limits_{\Omega} u(t-\sigma,x)dx \left(\int\limits_{\Omega} dx \right)^{-1} \right) \int\limits_{\Omega} dx = P(t)f(V(t-\sigma))\int\limits_{\Omega} dx. \end{aligned} \tag{11}$$

By virtue of (10) and (11) we obtain from (9) that

$$V''(t) + P(t)f(V(t-\sigma)) \le H_0(t), \qquad t \ne t_k.$$

For $t = t_k$ we have that

$$V(t_k) - V(t_k^-) \le L_k \left(\int_\Omega dx \right)^{-1} \int_\Omega u(t_k^-, x)dx = L_k V(t_k^-),$$

that is,

$$V(t_k) \le (1 + L_k)V(t_k^-),$$

and analogously,

$$V'(t_k) = (1 + L_k)V'(t_k^-).$$

□

Definition 3 The solution $V \in C^t_{imp}[[-\sigma, 0] \cup \mathbb{R}_+, \mathbb{R}] \cap C^2(\cup_{k=0}^{\infty}(t_k, t_{k+1}), \mathbb{R})$ of the differential inequality (6)–(8) is called eventually positive (negative) if there exists a number $t^* \ge 0$ such that $V(t) > 0$ ($V(t) < 0$) for $t \ge t^*$.

Theorem 1 *Let the following conditions hold:*

1. *Assumptions* H1–H5 *are fulfilled.*

2. $g(t_k, x, \xi) \le L_k \xi$, $h(t_k, x, \eta) = L_k \eta$, $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $x \in \overline{\Omega}$, $k = 1, 2, \ldots$, $L_k \ge 0$ *are constants and* $g(t_k, x, \xi) = -g(t_k, x, -\xi)$.

3. *The differential inequality* (6)–(8) *and the differential inequality*

$$V''(t) + P(t)f(V(t-\sigma)) \le -H_0(t), \quad t \ne t_k, \tag{12}$$

$$V(t_k) \le (1 + L_k)V(t_k^-), \tag{13}$$

$$V'(t_k) = (1 + L_k)V'(t_k^-) \tag{14}$$

have no eventually positive solutions.

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of problem* (1) – (4) *oscillates in the domain E.*

Proof. Suppose the conclusion of the theorem is not true, i.e., $u(t,x)$ is a nonzero solution of problem (1)–(4) which is of the class $C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ and it has a constant sign in the domain $E_\mu = [\mu, +\infty) \times \Omega$, $\mu \ge 0$. If $u(t,x) > 0$ for $(t,x) \in E_\mu$, then from Lemma 1 it follows that the function $V(t)$ is a positive solution of the differential inequality (6)–(8) for $t \ge \mu + \sigma$ which contradicts condition 3 of the theorem.

If $u(t,x) < 0$ for $(t,x) \in E_\mu$, then the function $-u(t,x)$ is a solution of the problem

$$\begin{aligned}
&u_{tt}(t,x) - \Delta u(t,x) + p(t,x)f(u(t-\sigma, x)) = -H(t,x), \quad (t,x) \in E \setminus E_{imp},\\
&u(t,x) - u(t^-, x) = g(t, x, u(t^-, x)), \quad (t,x) \in E^*_{imp},\\
&u_t(t,x) - u_t(t^-, x) = h(t, x, u_t(t^-, x)), \quad (t,x) \in E^*_{imp},\\
&\tfrac{\partial u}{\partial n}(t,x) + \gamma(t,x)u(t,x) = 0, \quad (t,x) \in (\mathbb{R}_+ \setminus J_{imp}) \times \partial\Omega,
\end{aligned}$$

which is positive in E_μ. From Lemma 1 it follows that the function

$$\int_\Omega [-u(t,x)]dx \left(\int_\Omega dx \right)^{-1}$$

is a positive solution of the differential inequality (12)–(14) for $t \geq \mu + \sigma$, which also contradicts condition 3 of the theorem. □

Theorem 2 *Let the following conditions hold:*

1. $P \in C_{imp}[\mathbb{R}_+, \mathbb{R}_+]$, $H_0 \in C_{imp}[\mathbb{R}_+, \mathbb{R}]$.
2. $f(u) \geq 0$ *for* $u \geq 0$.
3. $\sum_{k=1}^{\infty} L_k < +\infty$, $L_k \geq 0$, $k = 1, 2, \ldots$, *are constants.*
4. *For any number* $\tilde{t}_0 \geq \sigma$ *we have*

$$\liminf_{t\to\infty} \frac{1}{t - \tilde{t}_0} \int_{\tilde{t}_0}^{t} \prod_{s<t_k\leq t} (1 + L_k)(t - s) H_0(s) ds = -\infty.$$

Then the differential inequality (6)–(8) *has no eventually positive solutions.*

Proof. Suppose that the conclusion of the theorem is not true and let $V(t)$ be a positive solution of differential inequality (6)–(8) in the interval $[t^*, +\infty)$, $t^* \geq 0$. Then it follows from condition 2 of the theorem that

$$V''(t) \leq H_0(t), \qquad t \geq t^* + \sigma, \quad t \neq t_k.$$

Integrating twice over the interval $[\tilde{t}_1, t]$, $t^* + \sigma \leq \tilde{t}_1 < t$, we obtain

$$\begin{aligned} V(t) \leq & \textstyle\prod_{\tilde{t}_1 < t_k \leq t} (1 + L_k) \left[V(\tilde{t}_1) + V'(\tilde{t}_1)(t - \tilde{t}_1) \right] + \\ & + \int_{\tilde{t}_1}^{t} \textstyle\prod_{s<t_k\leq t} (1 + L_k)(t - s) H_0(s) ds. \end{aligned} \tag{15}$$

Dividing both sides of (15) by $t - \tilde{t}_1 > 0$, we get

$$\frac{V(t)}{t - \tilde{t}_1} \leq \frac{c_1}{t - \tilde{t}_1} + c_2 + \frac{1}{t - \tilde{t}_1} \int_{\tilde{t}_1}^{t} \prod_{s<t_k\leq t} (1 + L_k)(t - s) H_0(s) ds. \tag{16}$$

Then for $t \to \infty$ it follows from (16) that

$$\liminf_{t\to\infty} \frac{V(t)}{t - \tilde{t}_1} = -\infty. \tag{17}$$

On the other hand, since $V(t) > 0$ for $t \geq \tilde{t}_1$ we obtain that

$$\liminf_{t\to\infty} \frac{V(t)}{t - \tilde{t}_1} \geq 0,$$

which contradicts (17). This completes the proof. □

Corollary 2 *Let the following conditions hold:*

1. *Assumptions* H1–H5 *are fulfilled.*

2. $g(t_k,x,\xi) \le L_k\xi$, $h(t_k,x,\eta) = L_k\eta$, $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $x \in \overline{\Omega}$, $k = 1,2,\dots$, $L_k \ge 0$ *are constants such that* $\sum\limits_{k=1}^{\infty} L_k < +\infty$ *and* $g(t_k,x,\xi) = -g(t_k,x,-\xi)$.

3. *For any number* $\tilde{t}_0 \ge \sigma$ *we have*

$$\liminf_{t\to\infty} \frac{1}{t-\tilde{t}_0} \int\limits_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1+L_k)(t-s)H_0(s)ds = -\infty$$

and

$$\limsup_{t\to\infty} \frac{1}{t-\tilde{t}_0} \int\limits_{\tilde{t}_0}^{t} \prod_{s<t_k\le t} (1+L_k)(t-s)H_0(s)ds = +\infty.$$

Then each nonzero solution $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *of the problem* (1) – (4) *oscillates in the domain* E.

Corollary 1 follows from Theorem 1 and Theorem 2.

Now we give sufficient conditions for oscillation of the solutions of problem (1)–(3), (5). Consider the following Dirichlet problem

$$\begin{aligned} &\Delta\varphi + \alpha\varphi = 0 \qquad \text{in} \quad \Omega, \\ &\varphi_{|\partial\Omega} = 0, \end{aligned} \tag{18}$$

where $\alpha =$ const. It is known that the smallest eigenvalue α_0 of the problem (18) is positive and the corresponding eigenfunction $\varphi_0(x) > 0$ for $x \in \Omega$. Without loss of generality we may assume that φ_0 is normalized, i.e., $\int\limits_{\Omega} \varphi_0(x)dx = 1$.

Introduce the notations

$$W(t) = \int\limits_{\Omega} u(t,x)\varphi_0(x)dx,$$

$$H_1(t) = \int\limits_{\Omega} H(t,x)\varphi_0(x)dx.$$

Lemma 2 *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4, H5 *are fulfilled.*

2. $u \in C^2(E \setminus E_{imp}) \cap C^1(E^* \setminus E^*_{imp})$ *is a positive solution of the problem* (1)–(3), (5) *in the domain* E.

3. $g(t_k,x,\xi) \le L_k\xi$, $h(t_k,x,\eta) = L_k\eta$, $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $x \in \overline{\Omega}$, $k = 1,2,\dots$, $L_k \ge 0$ *are constants.*

Then the function $W(t)$ satisfies for $t \geq \sigma$ the impulsive differential inequality

$$W''(t) + \alpha_0 W(t) + P(t) f(W(t-\sigma)) \leq H_1(t), \quad t \neq t_k, \tag{19}$$

$$W(t_k) \leq (1+L_k) W(t_k^-), \tag{20}$$

$$W'(t_k) = (1+L_k) W'(t_k^-). \tag{21}$$

Proof. Let $t \geq \sigma$. We multiply both sides of equation (1) by the eigenfunction $\varphi_0(x)$ and integrating with respect to x over Ω, we obtain

$$\begin{aligned} \frac{d^2}{dt^2} \int_\Omega u(t, x)\varphi_0(x)dx - \int_\Omega \Delta u(t,x)\varphi_0(x)dx + \\ + \int_\Omega p(t,x) f(u(t-\sigma,x))\varphi_0(x)dx = \int_\Omega H(t,x)\varphi_0(x)dx, \quad t \neq t_k. \end{aligned} \tag{22}$$

From the Green formula it follows that

$$\begin{aligned} \int_\Omega \Delta u(t,x)\varphi_0(x)dx &= \int_\Omega u(t,x)\Delta\varphi_0(x)dx = \\ &= -\alpha_0 \int_\Omega u(t,x)\varphi_0(x)dx = -\alpha_0 W(t), \quad t \neq t_k, \end{aligned} \tag{23}$$

where $\alpha_0 > 0$ is the smallest eigenvalue of the problem (18).

Moreover, from the Jensen inequality

$$\begin{aligned} \int_\Omega p(t,x) \quad f(u(t-\sigma,x))\varphi_0(x)dx \geq P(t) \int_\Omega f(u(t-\sigma,x))\varphi_0(x)dx \geq \\ \geq P(t) f\left(\int_\Omega u(t-\sigma,x)\varphi_0(x)dx \right) = P(t) f(W(t-\sigma)), \quad t \neq t_k. \end{aligned} \tag{24}$$

Making use of (23) and (24), we obtain from (22) that

$$W''(t) + \alpha_0 W(t) + P(t) f(W(t-\sigma)) \leq H_1(t), \qquad t \neq t_k.$$

For $t = t_k$ we have that

$$W(t_k) - W(t_k^-) \leq L_k \int_\Omega u(t_k^-, x)\varphi_0(x)dx = L_k W(t_k^-),$$

that is,

$$W(t_k) \leq (1+L_k) W(t_k^-),$$

and analogously, □

$$W'(t_k) = (1+L_k) W'(t_k^-).$$

Analogously to Theorem 1 we can prove the following theorem.

Theorem 3 *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4, H5 *are fulfilled.*

2. $g(t_k,x,\xi) \leq L_k\xi$, $h(t_k,x,\eta) = L_k\eta$, $\xi \in \mathbb{R}_+$, $\eta \in \mathbb{R}$, $x \in \overline{\Omega}$, $k = 1,2,\ldots$, $L_k \geq 0$ *are constants and* $g(t_k,x,\xi) = -g(t_k,x,-\xi)$.

3. *The differential inequality* (19)–(21) *and the differential inequality*

$$\begin{aligned} &W''(t)+\alpha_0 W(t)+P(t)f(W(t-\sigma)) \le -H_1(t), \quad t\ne t_k,\\ &W(t_k)\le (1+L_k)W(t_k^-),\\ &W'(t_k)=(1+L_k)W'(t_k^-), \end{aligned}$$

have no eventually positive solutions.

Then each nonzero solution $u \in C^2(E\setminus E_{imp})\cap C^1(E^*\setminus E^*_{imp})$ *of problem* (1)–(3), (5) *oscillates in the domain* E.

Theorem 4 *Let the following conditions hold:*

1. $P\in C_{imp}[\mathbb{R}_+,\mathbb{R}_+]$, $H_1\in C_{imp}[\mathbb{R}_+,\mathbb{R}]$.
2. $f(u)\ge 0$ *for* $u\ge 0$.
3. $\sum_{k=1}^{\infty} L_k < +\infty$, $L_k\ge 0$, $k=1,2,\ldots$, *are constants.*
4. *For any number* $\tilde t_0\ge\sigma$ *we have*

$$\liminf_{t\to\infty}\frac{1}{t-\tilde t_0}\int_{\tilde t_0}^{t}\prod_{s<t_k\le t}(1+L_k)(t-s)H_1(s)ds=-\infty.$$

Then the differential inequality (19)–(21) *has no eventually positive solutions.*

The proof of Theorem 4 is analogous to the proof of Theorem 2. It is omitted here.

Corollary 4 *Let the following conditions hold:*

1. *Assumptions* H1, H2, H4, H5 *are fulfilled.*
2. $g(t_k,x,\xi)\le L_k\xi$, $h(t_k,x,\eta)=L_k\eta$, $\xi\in\mathbb{R}_+$, $\eta\in\mathbb{R}$, $x\in\overline{\Omega}$, $k=1,2,\ldots$, $L_k\ge 0$ *are constants such that* $\sum_{k=1}^{\infty}L_k<+\infty$ *and* $g(t_k,x,\xi)=-g(t_k,x,-\xi)$.
3. *For any number* $\tilde t_0\ge\sigma$ *we have*

$$\liminf_{t\to\infty}\frac{1}{t-\tilde t_0}\int_{\tilde t_0}^{t}\prod_{s<t_k\le t}(1+L_k)(t-s)H_1(s)ds=-\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{t-\tilde t_0}\int_{\tilde t_0}^{t}\prod_{s<t_k\le t}(1+L_k)(t-s)H_1(s)ds=+\infty.$$

Then each nonzero solution $u \in C^2(E\setminus E_{imp})\cap C^1(E^*\setminus E^*_{imp})$ *of problem* (1)–(3), (5) *oscillates in the domain* E.

Corollary 2 follows from Theorem 3 and Theorem 4.

Acknowledgements

The present investigation was partially supported by the Bulgarian Ministry of Education and Science under grant MM–702.

References

[1] D. BAINOV, Z. KAMONT AND E. MINCHEV, First order impulsive partial differential inequalities, *Intern. J. of Theoret. Physics,* **33** (1994), No. 6, 1341–1358.

[2] D. BAINOV, Z. KAMONT AND E. MINCHEV, Periodic boundary value problem for impulsive hyperbolic partial differential equations of first order, *Appl. Math. and Comput.,* **68** (1995), 95–104.

[3] L. BYSZEWSKI, Impulsive degenerate nonlinear parabolic functional-differential inequalities, *Journ. Math. Anal. Appl.,* **164** (1992), 549–559.

[4] C.Y. CHAN AND L. KE, Remarks on impulsive quenching problems, *Proceedings of Dynamic Systems and Applications,* **1** (1994), 59–62.

[5] C.Y. CHAN, L. KE AND A. VATSALA, Impulsive quenching for reaction-diffusion equations, *Nonlinear Analysis, TMA,* **22** (1994), No. 11, 1323–1328.

[6] L.H. ERBE, H.I. FREEDMAN, X.Z. LIU AND J.H. WU, Comparison principles for impulsive parabolic equations with applications to models of single species growth, *J. Austral. Math. Soc., Ser. B,* **32** (1991), 382–400.

[7] V. GUPTA, *Parabolic Equations with Impulse Effect: a Semigroup Approach,* Ph.D. Thesis, Kanpur, India, 1994.

[8] K. KREITH, Oscillation properties of weakly time dependent hyperbolic equations, *Canad. Math. Bull.,* **26** (1983), No. 3, 368–373.

[9] G. PETROV, Impulsive moving mirror model in a Schrödinger picture with impulse effect in a Banach space, *Communications of the Joint Institute for Nuclear Research,* Dubna, Russia, preprint E2–92–272, 1992.

[10] N. YOSHIDA, Forced oscillations of extensible beams, *SIAM J. Math. Anal.,* **16** (1985), No. 2, 211–220.

[11] N. YOSHIDA, Oscillation of nonlinear parabolic equations with functional arguments, *Hiroshima Math. J.,* **16** (1986), No. 2, 305–314.

Received November 20, 1998
DRUMI BAINOV
EMIL MINCHEV
Medical University of Sofia
Sofia 1504
P.O. Box 45
BULGARIA
Fax: +35929879874
e-mail: dbainov@mbox.pharmfac.acad.bg