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WEAKLY FUNCTIONALLY θ-NORMAL SPACES, θ-SHRINKING OF COVERS AND PARTITION OF UNITY

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Abstract. *Characterizations of weakly functionally θ-normal spaces, similar to that of a normal space, are obtained and used to establish the existence of partition of unity subordinated to certain locally finite open covers.*

Key words: θ-closed set, θ-open set, weakly functionally θ-normal space ($= wf$ θ-normal space), $u\theta$-limit point, θ-continuous function, θ-shrinking, partition of unity.
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1 Introduction

It is of fundamental importance in topology to obtain a factorization of a given topological property in terms of two weaker topological properties. The literature in topology is replete with the results of this nature. Normality is an important topological invariant and hence a decomposition of normality is desirable. First step in this direction was taken by G. Viglino [9], who defined seminormal spaces. Subsequently, Singal and Arya [6] introduced the class of almost normal spaces and proved that a space is normal if and only if it is both a seminormal space and an almost normal space. A search for another decomposition of normality led us to introduce in [4] the class of θ-normal spaces and certain of its variants such as weakly functionally θ-normal (wf θ-normal) spaces. The notion of wf θ-normality serves as a necessary ingredient for a decomposition of normality. In [4], wf θ-normal spaces are defined in terms of the existence of certain continuous real-valued functions. In this paper, in analogy with the normal spaces, we obtain a characterization of wf θ-normal space in terms of separation of certain closed sets by open sets. Moreover, we introduce the notion of a θ-shrinking of an open cover and obtain a characterization of wf θ-normal spaces in terms of θ-shrinking of certain covers. Furthermore, we characterize wf θ-normal spaces in terms of the existence of a partition of unity subordinated to certain locally finite open covers.

Section 2 is devoted to basic definitions and preliminaries. In section 3 we obtain a characterization of wf θ-normal spaces analogous to that of Uryshon Lemma and in section 4 we give a characterization of wf θ-normal spaces in terms of θ-shrinking of θ-open covers and use the same to obtain a characterization of wf θ-normal spaces in terms of the existence of partition of unity subordinated to certain locally finite θ-open covers.

2 Preliminaries and basic definitions

Definition 1 *[8]. Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a* **θ-limit point** *of A if every closed neighbourhood of x intersects A. Let A_θ denote the set of* *all θ-limit points of A. The set A is called* θ-**closed** *if $A = A_\theta$.*

The complement of a θ-closed set will be referred to as a θ-open set.

Lemma 2 *[4]. A subset A in a topological space X is θ-open if and only if for each $x \in A$ there is an open set U containing x such that $\overline{U} \subset A$.*

In general the θ-closure operator is not a Kuratowski closure operator since θ-closure of a set might not be θ-closed (see [3]). However, the following modification in [5] yields a Kuratowski closure operator.

Definition 3 *[5]. Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a $u\theta$-limit point of A if every θ-open set U containing x intersects A. Let $A_{u\theta}$ denote the set of all $u\theta$-limit points of A.*

Lemma 4 *[5]. The correspondence $A \to A_{u\theta}$ is a Kuratowski closure operator.*

It is observed in [5] that the set $A_{u\theta}$ is the smallest θ-closed set containing A.

Definition 5 *[2]. A function $f : X \to Y$ is said to be θ-continuous if for each $x \in X$ and each open set U containing $f(x)$ there exists an open set V containing x such that $f(\overline{V}) \subset \overline{U}$.*

Every continuous function is θ-continuous but the converse is not true in general. However, a θ-continuous function into a regular space is continuous in a somewhat strong sense.

Lemma 6 *[5]. Let $f : X \to Y$ be a θ-continuous function and let U be a θ-open set in Y. Then $\overline{f}^{1}(U)$ is θ-open in X.*

3 Weakly Functionally θ-Normal Spaces

Definition 7 *[4]. A topological space X is said to be weakly functionally θ-normal (wf θ-normal) if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*

The class of wf θ-normal spaces is much larger than the class of normal space. An example of a wf θ-normal spaces which is not normal is given in [4]. Moreover, the cofinite topology on an infinite set is (vacuously) wf θ-normal but not normal. Similarly, the particular point topology [7, p. 44] and the indiscrete rational (irrational) extension of $\mathbb{R}$ [7, p. 88] are wf θ-normal but are not normal. Furthermore, every finite topological space is wf θ-normal which need not be normal.

Theorem 8 *For a topological space X, the following statements are equivalent.*

(a) X is wf θ-normal.

(b) Every pair of disjoint θ-closed sets are contained in disjoint θ-open sets.

(c) For every θ-closed set A and every θ-open set U containing A there exists a θ-open set V such that $A \subset V \subset V_{u\theta} \subset U$.

Proof. To prove the assertion $(a) \Rightarrow (b)$, let X be a wf θ-normal spaces and let A, B be disjoint θ-closed sets in X. By wf θ-normality of X there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Since $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are θ-open sets in $[0,1]$, by Lemma 6 $f^1[0, \frac{1}{2})$ and $f^1(\frac{1}{2}, 1]$ are disjoint θ-open sets in X containing A and B respectively.

To prove $(b) \Rightarrow (c)$, let U be a θ-open set in X containing a θ-closed set A. Then A and $X - U$ are disjoint θ-closed sets in X. So there exist disjoint θ-open sets V and W such that $A \subset V$ and $(X - U) \subset W$. Now, $A \subset V \subset X - W_U$. Since $X - W$ is θ-closed and since $V_{u\theta}$ is the smallest θ-closed set containing $V, A \subset V \subset V_{u\theta} \subset U$.

To prove the implication $(c) \Rightarrow (a)$, let A and B be disjoint θ-closed sets in X. Then $A \subset X - B = U_1$ (say). Since U_1 is θ-open, there exists a θ-open set $U_{1/2}$ such that $A \subset U_{1/2} \subset (U_{1/2})_{u\theta} \subset U_1$. Again, since $(U_{1/2})_{u\theta}$ is a θ-closed set contained in the θ-open set U_1, there exist θ-open set $U_{1/4}$ and $U_{3/4}$ such that $A \subset U_{1/4} \subset (U_{1/4})_{u\theta} \subset U_{1/2}$ and $(U_{1/2})_{u\theta} \subset U_{3/4} \subset (U_{3/4})_{u\theta} \subset U_1$. Continuing the above process, we obtain for each dyadic rational r, a θ-open set U_r satisfying the condition that $r < s$ implies $(U_r)_{u\theta} \subset U_s$. Define a mapping $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r : x \in U_r\}, & \text{if } x \text{ belongs to some } U_r \\ 1, & \text{if } x \text{ does not belong to any } U_r \end{cases}.$$

Clearly f is well defined and $f(A) = 0$, $f(B) = 1$. Now it remains to prove that f is continuous. To this end we first observe that if $x \in U_r$, then $f(x) \leq r$. Similarly, $f(x) \geq r$ if $x \notin (U_r)_{u\theta}$. To prove continuity, let $x \in X$ and (a,b) be an open interval containing $f(x)$. Now choose two dyadic rationals p and q such that $a < p < f(x) < q < b$. Let $U = U_q - (U_p)_{u\theta}$. Then U is an open set containing x. Now for $y \in U$, $y \in U_q$. So $f(y) \leq q$. Also as $y \in U$, $y \notin (U_p)_{u\theta}$. Thus $f(y) \geq p$. And so $f(y) \in [p.q]$. Therefore, $f(U) \subset [p,q] \subset (a,b)$. Hence f is continuous. □

4 θ-Shrinking of Covers and Partition of Unity

Definition 9 *An open cover $u = \{U_\alpha : \alpha \in A\}$ of X is said to be θ-shrinkable if there exists a θ-open cover $v = \{V_\alpha : \alpha \in A\}$ of X such that $(V_\alpha)_{u\theta} \subset U_\alpha$ for each $\alpha \in A$.*

Recall that a covering u of X is said to be *point finite* if every $x \in X$ belongs to only finitely many elements of u.

Theorem 10 *A topological space X is wf θ-normal if and only if every point finite θ-open cover of X is θ-shrinkable.*

Proof. Let X be a wf θ-normal spaces and let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a point finite θ-open cover of X. Well order the set Λ. For convenience we may assume that $\Lambda = \{1, 2, \ldots, \alpha, \ldots\}$. Now construct $\{V_\alpha : \alpha \in \Lambda\}$ by transfinite induction as follows. Let $F_1 = X - \cup_{\alpha>1} U_\alpha$. Then F_1 is a θ-closed set contained in the θ-open set U_1. So by Theorem 8 there exists a θ-open set V_1 such that $F_1 \subset V_1 \subset (V_1)_{u\theta} \subset U_1$. Suppose V_β has been defined for each $\beta < \alpha$. Let $F_\alpha = X - [(\cup_{\beta<\alpha} V_\beta) \cup (\cup_{\gamma>\alpha} U_\gamma)]$. Then F_α is a θ-closed set contained in the θ-open set U_α. So, again, by Theorem 8 there exists a θ-open set V_α such that $F_\alpha \subset V_\alpha \subset (V_\alpha)_{u\theta} \subset U_\alpha$. Now $v = \{V_\alpha : \alpha \in \Lambda\}$ is a θ-shrinking of u provided it cover X. Let $x \in X$. Then x belongs to only finitely many members of u, say $U_{\alpha_1}, \ldots, U_{\alpha_k}$. Suppose $\alpha = \max\{\alpha_1, \ldots, \alpha_k\}$. Now x does not belongs to U_λ for $\lambda > \alpha$ and hence if $x \notin V_\beta$ for $\beta < \alpha$, then $x \in F_\alpha \subset V_\alpha$. So in any case $x \in V_\beta$ for $\beta \leq \alpha$. Thus v is a θ-shrinking of u.

Conversely, suppose A and B are disjoint θ-closed subsets of X. Then $\{X-A, X-B\}$ is a point finite θ-open cover of X. So, by hypothesis there exists a θ-shrinking $\{U,V\}$ of $\{X-A, X-B\}$. Now $X-(U)_{u\theta}$ and $X-(V)_{u\theta}$ are disjoint θ-open sets containing A and B, respectively. Again, in view of Theorem 8 X is wf θ-normal. □

Recall that for a continuous real-valued function f defined on X, the support of f is the closed set $\overline{\{x \in X : f(x) \neq 0\}}$.

Definition 11 *[1]. A family $\{f_\alpha :\in \Lambda\}$ of continuous functions from a space X to the closed unit interval $[0,1]$ is called a partition of unity on X if the collection $\{support\ f_\alpha : \alpha \in \Lambda\}$ forms a locally finite closed cover of X and $\sum_{\alpha\in\Lambda} f_\alpha(x) = 1$ for every $x \in X$.*

A partition of unity $\{f_\alpha : \alpha \in \Lambda\}$ on a space X is said to be *subordinated* to a cover $u = \{U_\alpha : \alpha \in \Lambda\}$ of X if support $f_\alpha \subset U_\alpha$ for each $\alpha \in \Lambda$.

Theorem 12 *A space X is wf θ-normal if and only if for every locally finite θ-open cover u of X there exists a partition of unity subordinated to u.*

Proof. Let X be a wf θ-normal space and let $u = \{U_\alpha : \alpha \in \Lambda\}$ be a locally finite θ-open cover of X. Since every locally finite collection is point finite, by Theorem 10 choose a θ-shrinking $v = \{V_\alpha : \alpha \in \Lambda\}$ of u, i.e. $(V_\alpha)_{u\theta} \subset U_\alpha$ for each $\alpha \in \Lambda$. Since the collection u is locally finite, so is the collection v and thus v is point finite. Again by Theorem 10 choose a θ-shrinking $w = \{W_\alpha : \alpha \in \Lambda\}$ of v. The cover w is locally finite, since v is locally finite. Since X is wf θ-normal, for each $\alpha \in \Lambda$ there exists a continuous function $\phi_\alpha : X \to [0,1]$ such that $\phi_\alpha((W_\alpha)_{u\theta} = 1$ and $\phi_\alpha(X - V_\alpha) = 0$. Since $\phi_\alpha^{-1}(0,1]$ is contained in V_α and since $V_\alpha \subset (V_\alpha)_{u\theta} \subset U_\alpha$, support $\phi_\alpha \subset U_\alpha$. Now let $x \in X$. Again, since w is locally finite, there exists a neighbourhood U_x of x and a finite subset $\Lambda_0 = \{\alpha_1, \ldots, \alpha_n\}$ of Λ such that $\phi_\alpha(x) = 0$ for all $\alpha \in \Lambda - \Lambda_0$. Thus for each $x \in X$, $\varphi = \sum_{i=1}^n \phi_{\alpha_i}(x)$ is positive. Therefore, we may define, for each α, $f_\alpha(x) = \phi_\alpha\ (x)/\varphi(x)$. Then the collection $\{f_\alpha : \alpha \in \Lambda\}$ is the desired partition of unity subordinated to u.

Conversely, suppose that every locally finite θ-open cover of X has a partition of unity subordinaterd to it and let A and B be any two disjoint θ-closed sets in X. Then $\{X-A, X-B\}$ constitutes a finite (and hence locally finite) θ-open cover of X and so there exists a partition of unity $\{f_1, f_2\}$ subordinated to it. Suppose that support $f_1 \subset X - A$. Then support $f_2 \subset X - B$. Therefore $A \subset X - \overline{f_1^{-1}(0,1]} \subset X - f_1^{-1}(0,1]$ and $B \subset X - \overline{f_2^{-1}(0,1]} \subset X - f_2^{-1}(0,1]$. Now define $h : X \to [0,1]$ by $h(x) = \frac{f_1(x)}{f_1(x)+f_2(x)}$. Clearly h is continuous, $h(A) = 0$ and $h(B) = 1$. Thus X is a wf θ-normal spaces . □

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