Note di Matematica Vol. 19 - n. 2, 285-291 (1999)

A NOTE ON EMBEDDINGS OF PROJECTIVE SPACES

HIROAKI TANIGUCHI

Abstract. *Let* $\mathbf{k}$ *and* $\mathbf{K}$ *be commutative fields, and* l, m *integers with* $l \geq 1$, $m \geq 2$. *Suppose that there exists an embedding* ψ *of* $PG(m+l,\mathbf{k})$ *to* $PG(m,\mathbf{K})$, *then we have* $r = \dim_{\mathbf{k}} \mathbf{K} \geq 4$ *and* $m \geq \lceil \frac{3l}{r-3} \rceil - 1$. *Conversely, there exists an embedding* ψ *of* $PG(l+m,\mathbf{k})$ *to* $PG(m,\mathbf{K})$ *if* $m \geq \lceil \frac{3l}{r-3} \rceil - 1$ *and if (1)* $\dim_{\mathbf{k}} \mathbf{K} = 4$, *or (2)* $\dim_{\mathbf{k}} \mathbf{K} > 4$ *and* $\mathbf{K}$ *is a cyclic extension of* $\mathbf{k}$ *with some additional conditions on* l *and* r.

1 Introduction

Let $\mathbf{k}$ and $\mathbf{K}$ be commutative fields and let n and m be integers not less than 2. An embedding ψ of an n-dimensional projective space $PG(n,\mathbf{k})$ defined over $\mathbf{k}$ into an m-dimensional projective space $PG(m,\mathbf{K})$ defined over $\mathbf{K}$ is a mapping which satisfies the following conditions:

1. ψ is an injective mapping from $PG(n,\mathbf{k})$ to $PG(m,\mathbf{K})$.
2. Let S be a subset of $PG(n,\mathbf{k})$.
   (a) If $S \subset l$ for some line l of $PG(n,\mathbf{k})$, then $\psi(S) \subset l'$ for some line l' of $PG(m,\mathbf{K})$, that is, ψ maps collinear points to collinear points.
   (b) If $S \not\subset l$ for any line l of $PG(n,\mathbf{k})$, then $\psi(S) \not\subset l'$ for any line l' of $PG(m,\mathbf{K})$, that is, ψ maps non-collinear points to non-collinear points.

It is well known that if there exists an embedding ψ of $PG(n,\mathbf{k})$ into $PG(m,\mathbf{K})$, then $\mathbf{k}$ is a subfield of $\mathbf{K}$. The first non-trivial example of an embedding of affine spaces was given by J. A. Thas [4]. In [1], M. Limbos characterized the embeddings of projective spaces in case $\mathbf{k}$ and $\mathbf{K}$ are finite fields, using a projection $\pi : PG(n,\mathbf{K}) \to PG(m,\mathbf{K})$.

In this note, we study the conditions on n, m and $\dim_{\mathbf{k}} \mathbf{K}$ for existence of an embedding of $PG(n,\mathbf{k})$ into $PG(m,\mathbf{K})$.

2 A characterization of embeddings

In this section, we give a characterization of embeddings of projective spaces in Theorem 1 and Theorem 2.

Considering a point in a projective space as an equivalence class of points in a vector space, we denote by $[(x_0,x_1,\ldots,x_n)] \in PG(n,\mathbf{k})$ the equivalence class containing a non-zero element $(x_0,x_1,\ldots,x_n)$ of $\mathbf{k}^{n+1}$. Thus, for a non-zero $a \in \mathbf{k}^{n+1}$, we denote by $[a] \in PG(n,\mathbf{k})$ the equivalence class containing a.

Theorem 1 is an immediate consequense of Proposition 1 of [3].

Theorem 1 *Let n and m be integers not less than 2, and assume that ψ is an embedding of $PG(n,\mathbf{k})$ into $PG(m,\mathbf{K})$. Then there exists an isomorphism θ from $\mathbf{k}$ into $\mathbf{K}$, and there exists an $(n+1)\times(m+1)$ matrix B with entries in $\mathbf{K}$, such that ψ can be expressed as follows:*

$$\psi([(x_0,x_1,\ldots,x_n)]) = [(x_0^\theta,x_1^\theta,\ldots,x_n^\theta)B]. \qquad (*)$$

Moreover, θ in $(*)$ is uniquely determined, and B is uniquely determined up to a multiplication of non-zero element of $\mathbf{K}$.

Next, let θ be an isomorphism from $\mathbf{k}$ into $\mathbf{K}$ and B an $(n+1)\times(m+1)$ matrix with entries in $\mathbf{K}$. Now, we give a condition that the mapping ψ defined by $\psi([(x_0,x_1,\ldots,x_n)]) = [(x_0^\theta,x_1^\theta,\ldots,x_n^\theta)B]$ is an embedding in Theorem 2.

We denote by $V(u_0,u_1,\ldots,u_n)$, or by $V(u)$ for short, the $\mathbf{k}^\theta$-vector subspace of $\mathbf{K}^{n+1}$ generated by $\{u_0,u_1,\ldots,u_n\}$.

Theorem 2 *Let n and m be integers not less than 2. Let θ be an isomorphism from $\mathbf{k}$ into $\mathbf{K}$, and B an $(n+1)\times(m+1)$ matrix B with entries in $\mathbf{K}$. We define a mapping ψ from $PG(n,\mathbf{k})$ to $PG(m,\mathbf{K})$ by $\psi([(x_0,x_1,\ldots,x_n)]) = [(x_0^\theta,x_1^\theta,\ldots,x_n^\theta)B]$.*

If U is the subset $\{x\in\mathbf{K}^{n+1}|xB=\mathbf{0}\}$ of $\mathbf{K}^{n+1}$, then ψ is an embedding if and only if $\dim_{\mathbf{k}^\theta} V(u) \geq 4$ for any non-zero element $u=(u_0,u_1,\ldots,u_n)\in U$.

It is easy to see that the mapping ψ in Theorem 2 is well defined if and only if $\dim_{\mathbf{k}^\theta} V(u) \geq 2$ for any non-zero element $u=(u_0,\ldots,u_n)\in U$. Because, for a non-zero u in U, if $\dim_{\mathbf{k}^\theta} V(u) = 1$ then $u=\lambda x^\theta \in U$ for some non-zero $\lambda\in\mathbf{K}$ and for some non-zero $x\in\mathbf{k}^{n+1}$, hence $x^\theta\in U$, therefore we can not determine $\psi([x])$. The converse is easy.

To prove Theorem 2, we need the following lemma. We regard $\mathbf{k}^{n+1}\subset\mathbf{K}^{n+1}$ by the isomorphism θ from $\mathbf{k}$ into $\mathbf{K}$.

Lemma 3 *Assume that the mapping ψ in Theorem 2 is well defined. Then ψ is an embedding if and only if $U\cap V=\{0\}$ for any 3-dimensional $\mathbf{K}$-vector subspace V of $\mathbf{K}^{n+1}$ spanned by three elements of $\mathbf{k}^{n+1}$.*

Proof. Let x, y and z in $\mathbf{k}^{n+1}$ be linearly independent elements over $\mathbf{k}$. Then $[x]$, $[y]$ and $[z]$ in $PG(n,\mathbf{k})$ are not on any line. Let V be the vector subspace in $\mathbf{K}^{n+1}$ spanned by x^θ, y^θ and z^θ. If $U\cap V\neq\{0\}$, then there exist elements λ, μ and ν of $\mathbf{K}$ such that not all elements are zero and $(\lambda x^\theta+\mu y^\theta+\nu z^\theta)B=0$. We may assume that $\lambda\neq 0$. If we put $\mu'=\mu/\lambda$ and $\nu'=\nu/\lambda$, then we have $(x^\theta+\mu' y^\theta+\nu' z^\theta)B=0$, which implies that $[x^\theta B]=[\mu' y^\theta B+\nu' z^\theta B]$. Thus we see that $\psi([x])$, $\psi([y])$ and $\psi([z])$ are collinear. Therefore we conclude that ψ is not an embedding.

Conversely, assume that ψ is not an embedding. Then there exist $[x]$, $[y]$ and $[z]$ in $PG(n,\mathbf{k})$, such that, although $[x]$, $[y]$ and $[z]$ are not on any line, $\psi([x])$, $\psi([y])$ and $\psi([z])$ are on a line of $PG(m,\mathbf{K})$. Note that x, y and z are linearly independent over $\mathbf{k}$. We may assume that $\psi([x])$ is either on the line $\overline{\psi([y])\psi([z])}$ with $\psi([y])\neq\psi([z])$, or $\psi([x])=\psi([y])=\psi([z])$. This implies that there exist λ, μ and ν in $\mathbf{K}$ such that not all elements are zero and $(\lambda x^\theta)B=(\mu y^\theta+\nu z^\theta)B$. Let V be the $\mathbf{K}$-vector subspace of dimension 3 spanned by x^θ, y^θ and z^θ. Then we have $U\cap V\ni\lambda x^\theta-\mu y^\theta-\nu z^\theta\neq 0$. □

Proof. [Proof of Theorem 2] We may assume that ψ is well defined. Now, assume that there exists a non-zero $u \in U$ such that $\dim_{\mathbf{k}^\theta} V(u) \leq 3$. Then there exist a, b and c in $\mathbf{K}$, and α_i, β_i and γ_i in $\mathbf{k}$ for $0 \leq i \leq n$, such that $u_i = a\alpha_i^\theta + b\beta_i^\theta + c\gamma_i^\theta$. If we put $\alpha = (\alpha_0, \ldots, \alpha_n), \beta = (\beta_0, \ldots, \beta_n)$ and $\gamma = (\gamma_0, \ldots, \gamma_n)$, then we have $u = a\alpha^\theta + b\beta^\theta + c\gamma^\theta$. Let $V \subset \mathbf{K}^{n+1}$ be a 3-dimensional $\mathbf{K}$-vector subspace spanned by three elements of $\mathbf{k}^{n+1}$ such that V contains α^θ, β^θ and γ^θ. Then we have $U \cap V \ni u \neq 0$. Thus we conclude that ψ is not an embedding by Lemma 3.

Conversely, assume that ψ is not an embedding. Then by Lemma 3, there exists a 3 dimensional vector subspace $V \subset \mathbf{K}^{n+1}$ spanned by three elements α, β and γ of $\mathbf{k}^{n+1}$, such that $U \cap V \neq \{0\}$. Hence there exists a non-zero $u = a\alpha^\theta + b\beta^\theta + c\gamma^\theta \in U$ with a, b and c in $\mathbf{K}$. Thus we have $\dim_{\mathbf{k}^\theta} V(u) \leq 3$, since $V(u) \subset \mathbf{k}^\theta a + \mathbf{k}^\theta b + \mathbf{k}^\theta c$. □

Corollary 4 is a consequense of Theorem 1 and Theorem 2.

Corollary 4 *Let ψ be an embedding of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$ for $m \geq 2$ and $l \geq 1$, and θ an isomorphism from $\mathbf{k}$ into $\mathbf{K}$ as in Theorem 1. Then we have* $\dim_{\mathbf{k}^\theta} \mathbf{K} \geq 4$.

3 A numerical bound

As for the relations among n, m and $\dim_{\mathbf{k}^\theta} \mathbf{K}$, we show the following result.

Theorem 5 *Let ψ be an embedding of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$ for $m \geq 2$ and $l \geq 1$, and θ an isomorphism from $\mathbf{k}$ into $\mathbf{K}$ as in Theorem 1. Then we have $m \geq \lceil \frac{3l}{r-3} \rceil - 1$, where* $r = \dim_{\mathbf{k}^\theta} \mathbf{K}$.

Remark 6 *By Corollary 4, we have $r \geq 4$.*

Proof. Assume that $m < \lceil \frac{3l}{r-3} \rceil - 1$ and that $\psi(PG(m+l, \mathbf{k}))$ spans $PG(m, \mathbf{K})$ as $\mathbf{K}$-vector spaces. Let B be the matrix given in the expression $(*)$ of ψ in Theorem 1. Then rank $B = m+1$. Hence if we put $U = \{x \in \mathbf{K}^{m+l+1} | xB = \mathbf{0}\}$, then we have $\dim_{\mathbf{K}} U = l$, and therefore, $\dim_{\mathbf{k}^\theta} U = rl$. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis of $\mathbf{K}$ over $\mathbf{k}^\theta$ and $\{g_1, g_2, \ldots, g_{rl}\}$ a basis of U over $\mathbf{k}^\theta$, and let us express g_k for $1 \leq k \leq rl$ as

$$g_k = (e_1, \ldots, e_r) \begin{pmatrix} a_{1,k}^\theta & \cdots & a_{m+l+1,k}^\theta \\ a_{m+l+2,k}^\theta & \cdots & a_{2(m+l+1),k}^\theta \\ \vdots & \ddots & \vdots \\ a_{(r-1)(m+l+1)+1,k}^\theta & \cdots & a_{r(m+l+1),k}^\theta \end{pmatrix},$$

where $a_{1,k}, a_{2,k}, \ldots, a_{r(m+l+1),k}$ are elements of $\mathbf{k}$. We define $\mathbf{k}^\theta$-vector spaces W_i for $1 \leq i \leq r(m+l+1)$ as

$$W_i = \{\, x_1 g_1 + x_2 g_2 + \cdots + x_{rl} g_{rl} \mid x_k \in \mathbf{k}^\theta \text{ for} 1 \leq k \leq rl \\ \text{with } x_1 a_{i,1}^\theta + x_2 a_{i,2}^\theta + \cdots + x_{rl} a_{i,rl}^\theta = 0 \,\},$$

where $a_{i,1}, a_{i,2}, \ldots, a_{i,rl}$ are elements of $\mathbf{k}$ which appear in the expression of g_k for $1 \leq k \leq rl$. It is obvious that $W_i \subset U$ and $\dim_{\mathbf{k}^\theta} W_i \geq \dim_{\mathbf{k}^\theta} U - 1$. We define a $\mathbf{k}^\theta$-vector space W' as

$W' = W_1 \cap W_2 \cap \cdots \cap W_{rl-2} \cap W_{rl-1} \subset U$. Then, since $m < \lceil \frac{3l}{r-3} \rceil - 1$, we have $m < \frac{3l}{r-3} - 1$, which implies $(r-3)(m+l+1) \leq rl - 1$. Since $\dim_{\mathbf{k}^\theta} U = rl$, it is easy to see that there exists a non-zero element $u \in W' \subset U$. If we express u as $u = x_1 g_1 + x_2 g_2 + \cdots + x_{rl} g_{rl}$, then, since $x_1 a^\theta_{i,1} + x_2 a^\theta_{i,2} + \cdots + x_{rl} a^\theta_{i,rl} = 0$ for $1 \leq i \leq rl-1$, we have

$$u = (e_1, \ldots, e_r) \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ b^\theta_{(r-3)(m+l+1)+1} & \cdots & b^\theta_{(r-2)(m+l+1)} \\ b^\theta_{(r-2)(m+l+1)+1} & \cdots & b^\theta_{(r-1)(m+l+1)} \\ b^\theta_{(r-1)(m+l+1)+1} & \cdots & b^\theta_{r(m+l+1)} \end{pmatrix},$$

where $b_j \in \mathbf{k}$ for $(r-3)(m+l+1)+1 \leq j \leq r(m+l+1)$. Thus we have $u = (u_0, u_1, \ldots, u_{m+l}) \in U \subset \mathbf{K}^{m+l+1}$ with

$$u_{j-1} = b^\theta_{(r-3)(m+l+1)+j} e_{r-2} + b^\theta_{(r-2)(m+l+1)+j} e_{r-1} + b^\theta_{(r-1)(m+l+1)+j} e_r$$

for $1 \leq j \leq m+l+1$. Hence $\dim_{\mathbf{k}^\theta} V(u) \leq 3$, which contradicts that ψ is an embedding by Theorem 2. □

4 Existence of an embedding in case $\dim_{\mathbf{k}^\theta} \mathbf{K} = 4$

In this section, we prove that the converse of Theorem 3 is true in case $\dim_{\mathbf{k}^\theta} \mathbf{K} = 4$.

Theorem 7 *Let m be an integer not less than 2 and l an integer not less than 1. Let θ be an isomorphism from $\mathbf{k}$ into $\mathbf{K}$ such that $\dim_{\mathbf{k}^\theta} \mathbf{K} = 4$, and assume that $m \geq 3l - 1$. Then there exists an embedding ψ of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$ defined as in Theorem 2.*

Note that, if $r = \dim_{\mathbf{k}^\theta} \mathbf{K} = 4$, then $\lceil \frac{3l}{r-3} \rceil - 1 = 3l - 1$.

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be a basis of $\mathbf{K}$ over $\mathbf{k}^\theta$. For $1 \leq i \leq l$, let f_i be an element of $\mathbf{K}^{m+l+1}$ defined by

$$f_i = (e_1, e_2, e_3, e_4) \begin{pmatrix} & & \overset{4i-3}{} & \overset{4i-2}{} & \overset{4i-1}{} & \overset{4i}{} & & \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & & 0 & 1 & 0 & 0 & & 0 \\ 0 & & 0 & 0 & 1 & 0 & & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \end{pmatrix}.$$

Since $m \geq 3l - 1$, we have $m + l + 1 \geq 4l$, hence we can define $f_1, f_2, \ldots, f_l$. Notice that, for any non-zero $\lambda \in \mathbf{K}$, there exists a 4×4 matrix $M_{i\lambda}$ of rank 4 with entries in $\mathbf{k}^\theta$ such that

$$\lambda f_i = (e_1, e_2, e_3, e_4)(0, \ldots, 0, M_{i\lambda}, 0, \ldots, 0).$$

Therefore, it is easy to see that f_1, f_2, $\ldots$, f_l are linearly independent over $\mathbf{K}$. Let U be the l-dimensional $\mathbf{K}$-vector subspace of $\mathbf{K}^{m+l+1}$ spanned by $\{f_1, f_2, \ldots, f_l\}$. Then for any non-zero element u of U, it is also easy to see that $\dim_{\mathbf{k}^\theta} V(u) = 4$. We define an $(m+l+1) \times (m+1)$ matrix $B = (b_{ij})$ with $b_{ij} \in \mathbf{K}$ by $U = \{x \in \mathbf{K}^{m+l+1} | xB = \mathbf{0}\}$. Then by Theorem 2, the mapping ψ from $PG(m+l,\mathbf{k})$ to $PG(m,\mathbf{K})$ defined by $\psi([(x_0,x_1,\dots,x_{m+l})]) = [(x_0^\theta, x_1^\theta, \dots, x_{m+l}^\theta)B]$ is an embedding. □

5 Existence of an embedding in case K be a cyclic extension with $\dim_{\mathbf{k}^\theta} \mathbf{K} > 4$

In connection with the converse of Theorem 3 in case $\dim_{\mathbf{k}^\theta} \mathbf{K} > 4$, we prove the following Theorem 8.

Theorem 8 *Let m be an integer not less than 2 and l an integer not less than 1. Let θ be an isomorphism from $\mathbf{k}$ into $\mathbf{K}$ such that $\mathbf{K}$ is a cyclic extension of $\mathbf{k}^\theta$, and that $\dim_{\mathbf{k}^\theta} \mathbf{K} = r > 4$. Let $m_0 = \lceil \frac{3l}{r-3} \rceil - 1$, and assume that $l \equiv t-3 \pmod{r-3}$, where $r > t \geq 3$. Then there exists an embedding ψ from $PG(m+l,\mathbf{k})$ into $PG(m,\mathbf{K})$ if one of the following conditions is satisfied.*

1. *$t-3 = 0$ or $(2/3)(r-3) < t-3$, and $m \geq m_0$.*
2. *$(1/3)(r-3) < t-3 \leq (2/3)(r-3)$, and $m \geq m_0 + 1$.*
3. *$0 < t-3 \leq (1/3)(r-3)$, and $m \geq m_0 + 2$.*

The following two lemmas are some variations of Theorem 1 of [2]. Thus we omit the proofs.

Lemma 9 *Let $\{e_1, e_2, \dots, e_r\}$ be a basis of $\mathbf{K}$ over $\mathbf{k}^\theta$. Let σ be a generator of the Galois group of $\mathbf{K}$ over $\mathbf{k}^\theta$, and let $s \geq 1$. We define a mapping ψ from $PG(rs-1,\mathbf{k})$ to $PG(3s-1,\mathbf{K})$ as follows:*

$$\begin{aligned}&\psi([(x_1^1,\dots,x_1^r,x_2^1,\dots,x_2^r,\dots,x_s^1,\dots,x_s^r)])\\ &= [(a_1,a_1^\sigma,a_1^{\sigma^2},a_2,a_2^\sigma,a_2^{\sigma^2},\dots,a_s,a_s^\sigma,a_s^{\sigma^2})],\end{aligned}$$

where $a_i = (x_i^1)^\theta e_1 + \cdots + (x_i^r)^\theta e_r$ for $1 \leq i \leq s$. Then ψ is an embedding.

Lemma 10 *We assume the same conditions of Lemma 9. Let t be $4 \leq t < r$ and $s \geq 0$. If we define a mapping ψ from $PG(rs+t-1,\mathbf{k})$ to $PG(3s+2,\mathbf{K})$ as*

$$\begin{aligned}&\psi([(x_1^1,\dots,x_1^r,x_2^1,\dots,x_2^r,\dots,x_s^1,\dots,x_s^r,x_{s+1}^1,\dots,x_{s+1}^t)])\\ &= [(a_1,a_1^\sigma,a_1^{\sigma^2},a_2,a_2^\sigma,a_2^{\sigma^2},\dots,a_s,a_s^\sigma,a_s^{\sigma^2},a_{s+1},a_{s+1}^\sigma,a_{s+1}^{\sigma^2})],\end{aligned}$$

where $a_i = (x_i^1)^\theta e_1 + \cdots + (x_i^r)^\theta e_r$ for $1 \leq i \leq s$, and $a_{s+1} = (x_{s+1}^1)^\theta e_1 + \cdots + (x_{s+1}^t)^\theta e_t$, then ψ is an embedding.

Lemma 11 *If there exists an embedding ψ of $PG(m_0+l,\mathbf{k})$ into $PG(m_0,\mathbf{K})$, then, for any $m \geq m_0$, there exists an embedding ψ' of $PG(m+l,\mathbf{k})$ into $PG(m,\mathbf{K})$.*

Proof. By Theorem 1, there exists an isomorphism θ from $\mathbf{k}$ into $\mathbf{K}$ and an $(m_0+l+1) \times (m_0+1)$ matrix $B = (b_{ij})$ with $b_{ij} \in \mathbf{K}$, such that ψ can be expressed as: $\psi([(x_0,x_1,\dots,x_{m_0+l})])$

$= [(x_0^\theta, x_1^\theta, \ldots, x_{m_0+l}^\theta)B]$. If we put $U = \{x \in \mathbf{K}^{m_0+l+1} | xB = \mathbf{0}\}$, then, by Theorem 2, we have $\dim_{\mathbf{k}^\theta} V(u) \geq 4$ for any non-zero element $u \in U$. Let E be the identity matrix of order $m - m_0$, i.e., $E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, and let B' be the $(m+l+1) \times (m+1)$ matrix defined by $B' = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix}$. Let $U' = \{x \in \mathbf{K}^{n+m+1} | xB' = \mathbf{0}\}$. Then we have a $\mathbf{k}^\theta$-isomorphism $U \simeq U'$, and hence, for any non-zero element $u' \in U'$, we have $\dim_{\mathbf{k}^\theta} V(u') \geq 4$. Thus, by Theorem 2, the mapping ψ' from $PG(m+l, \mathbf{k})$ to $PG(m, \mathbf{K})$ defined by $\psi'([(x_0, x_1, \ldots, x_{m+l})]) = [(x_0^\theta, x_1^\theta, \ldots, x_{m+l}^\theta)B']$ is an embedding. □

Proof. [Proof of Theorem 8] Since $l = s(r-3) + t - 3$ for some $s \geq 0$, we have $m_0 + l + 1 = rs + t + \lceil \frac{3(t-3)}{r-3} \rceil - 3$ and $m_0 + 1 = 3s + \lceil \frac{3(t-3)}{r-3} \rceil$.
(1) The case that $t - 3 = 0$ or $(2/3)(r-3) < t - 3$.
If $(2/3)(r-3) < t-3$, then we have $r > t > 3$ and hence $\lceil \frac{3(t-3)}{r-3} \rceil = 3$. Thus, we have $m_0 + l = rs + t - 1$ and $m_0 = 3s + 2$. If $t - 3 = 0$, then we have $\lceil \frac{3(t-3)}{r-3} \rceil = 0$, which induces that $m_0 + l = rs - 1$ and $m_0 = 3s - 1$. Note that, if $t - 3 = 0$, then $s \geq 1$ by $l \geq 1$. Hence, by Lemma 10 and Lemma 9, there exists an embedding ψ of $PG(m_0 + l, \mathbf{k})$ into $PG(m_0, \mathbf{K})$. Therefore, by Lemma 11, $m \geq m_0 = \lceil \frac{3l}{r-3} \rceil - 1$ implies that there exists an embedding ψ' of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$.
(2) The case that $(1/3)(r-3) < t - 3 \leq (2/3)(r-3)$.
In this case, we have $\lceil \frac{3(t-3)}{r-3} \rceil = 2$, and therefore $m_0 + l + 1 = rs + t - 1$ and $m_0 + 1 = 3s + 2$. Consequently, by Lemma 10, there exists an embedding ψ of $PG(m_0 + 1 + l, \mathbf{k})$ into $PG(m_0 + 1, \mathbf{K})$, and hence by Lemma 11, if $m \geq m_0 + 1 = \lceil \frac{3l}{r-3} \rceil$, there exists an embedding ψ' of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$.
(3) The case that $0 < t - 3 \leq (1/3)(r-3)$.
In this case, we have $\lceil \frac{3(t-3)}{r-3} \rceil = 1$, which implies that $m_0 + 2 + l = rs + t - 1$ and $m_0 + 2 = 3s + 2$. Hence by Lemma 10, there exists an embedding $\psi : PG(m_0 + 2 + l, \mathbf{k}) \to PG(m_0 + 2, \mathbf{K})$, and therefore, by Lemma 11, if $m \geq m_0 + 2 = \lceil \frac{3l}{r-3} \rceil + 1$, there exists an embedding ψ' of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$. Thus we complete the proof of Theorem 8. □

In relation to the above results, the author conjectures that there always exists an embedding ψ of $PG(m+l, \mathbf{k})$ into $PG(m, \mathbf{K})$ if $m \geq \lceil \frac{3l}{r-3} \rceil - 1$ and if $\mathbf{K}$ is a cyclic extension of $\mathbf{k}^\theta$ with $\dim_{\mathbf{k}^\theta} \mathbf{K} > 4$.

References

[1] Limbos, M.: "A characterization of the embeddings of $PG(m,q)$ into $PG(n,q^r)$", *Journal of Geometry*, Vol.**16** (1981), 50–55.

[2] Taniguchi, H.: "On embeddings of projective spaces", *Utilitas Mathematica*, Vol.**54** (1998), 25–31

[3] Taniguchi, H.: "On the embedding of an affine space into a projective space", *Geometriae Dedicata*, Vol. **80** (2000), 99–123

[4] Thas, J. A.: "Connection between the n-dimensional affine space $A_{n,q}$ and the curve C, with equation $y = x^q$, of the affine plane A_{2,q^n}", *Rend. ist. di Matem.* Univ. di Trieste, Vol. **II**, fasc. **II** (1970), 146–151.

Received September 29, 1998 and in revised form September 9, 1999
HIROAKI TANIGUCHI
72-3-202, Kounami
Wakayama City
649-6315
JAPAN
E-mail address: NBF03137@nifty.ne.jp