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# On homogeneous hypersurfaces in the manifold of quaternionic subalgebras of the Cayley algebra

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**Abstract.** In this work we consider a class of real hypersurfaces of $N = G_2/SU(2) \times SU(2)$, the manifold of quaternionic subalgebras of the Cayley algebra. They are the family of tubes centered at the maximal totally geodesic submanifolds of maximal rank of $N$, which are (up to isomorphism) $\mathbb{CP}^2$ and $S^2 \times S^2$. We determine which of those tubes are homogeneous and for them we obtain the spectral decomposition of the shape operator. Moreover we show that the universal covering space of the focal set of $\mathbb{CP}^2$ is the sphere $S^5$.

**Keywords:** manifold of quaternionic algebras, hypersurfaces, tubes, shape operator, Jacobi operator

**MSC 2000 classification:** 53C30, 53C35, 53C42

## Introduction

This article is in some sense a follow up of the paper "On homogeneous hypersurfaces in complex Grassmannians" [8]. We carry out here an analogous study when the ambient space is $G_2/SU(2)\times SU(2)$ which is the manifold of the non-division quaternionic subalgebras of the non-division Cayley algebra [3, p. 316]. On the other hand, this space is the only compact irreducible Riemannian symmetric space of type I obtained from the exceptional Lie group $G_2$ [11, p. 518].

The study of homogeneous real hypersurfaces in compact symmetric spaces has an interesting history which dates back from some years ago.

At present, homogeneous real hypersurfaces in complex projective spaces seem to be very well understood. Information in this direction can be found in the article [10] and references therein.

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Berndt and Suh studied real hypersurfaces in the complex two-plane Grassmannian $G_2(\mathbb{C}^n)$ and found interesting results using deep knowledge of the Grassmanian obtained in [1] and the fact that $G_2(\mathbb{C}^n)$ is the only compact, Kaehler, quaternionic Kaehler manifold with positive scalar curvature. From the Kaehler and quaternionic Kaehler structure they give a very nice and interesting characterization of tubes in $G_2(\mathbb{C}^n)$ centered at the totally geodesic submanifolds $G_2(\mathbb{C}^{n-1})$ and $\mathbb{HP}^{\frac{n-2}{2}}$ for $n$ even (see [2] Theorem 1).

As soon as one considers other grassmannians, $G_k(\mathbb{C}^n)$ with $k > 2$, the quaternionic Kaehler condition is lost and the road, which is difficult in [2] gets rougher. In these cases a very useful tool is the intrinsic structure of the complex simple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$. In [8] using the root structure associated to the Grassmann manifold we studied the homogeneity of the family of tubes in $G_k(\mathbb{C}^n)$ $(k > 2)$ centered at $G_k(\mathbb{C}^m)$ when $m < n$ and centered at the quaternionic Grassmann manifold $G_{\frac{k}{2}}(\mathbb{H}^{\frac{n}{2}})$ when $k$ and $n$ are even. Furthermore, for the homogeneous tubes of this family we found the spectral decomposition of the shape operator and we saw that they are Hopf hypersurfaces.

In the present article we study analogous questions in the ambient space $G_2/SU(2) \times SU(2)$. Since it is not Hermitian, the Hopf condition for a real hypersurface makes no sense.

The symmetric space $G_2/SU(2)\times SU(2)$, has two remarkable totally geodesic submanifolds: $\mathbb{CP}^2$ and $S^2 \times S^2$. In fact, in [5] Chen and Nagano showed that they are (up to isomorphism) the only maximal totally geodesic submanifolds of maximal rank (in their table V appears $AI(3)$ instead of $\mathbb{CP}^2$ but as we see in Section 1 they actually obtained these two submanifolds).

The goal of the present work is to study a certain class of real hypersurfaces of $G_2/SU(2)\times SU(2)$ consisting of the family of tubes around $\mathbb{CP}^2$ and $S^2\times S^2$. Our objetive is to determine which tubes of these families are homogeneous and find for them the spectral decomposition of the shape operator. From a private communication of J. Berndt we understand that the classification of homogeneous hypersurfaces in $G_2/SU(2) \times SU(2)$ can be obtained from results of A. Kollross [12] concerning hyperpolar actions on irreducible simple connected symmetric spaces of compact type. However, our methods are based on the root structure associated to the manifold $G_2/SU(2) \times SU(2)$ and so we think they may be of interest.

In Section 1 we describe inclusions of the Lie algebras of the subgroups $SU(3)$ and $SU(2)\times SU(2)$ into the Lie algebra of $G_2$. This gives us information about the inclusions of $\mathbb{CP}^2$ and $S^2 \times S^2$ into $G_2/SU(2) \times SU(2)$.

In Section 2 we give a complete classification of homogeneous tubes around $\mathbb{CP}^2$ and $S^2\times S^2$ (Theorem 3 and Theorem 6). Since $G_2$ is a simple Lie group and the natural action of $G_2$ on $G_2/SU(2) \times SU(2)$ is effective, $G_2$ is the connected component of the identity of the isometry group of this space. Therefore it seems to be of interest to study the existence of a subgroup of the isometry group of $G_2/SU(2)\times SU(2)$ acting transitively on these tubes. For this purpose we consider the natural action of the isotropy group of the center on the zero-centered sphere of the normal space (see Lemma 1 and Lemma 2) and then we obtain that the tubes around $\mathbb{CP}^2$ are homogeneous real hypersurfaces (Theorem 3). We compute the spectral decomposition of the Jacobi operator $R_Z = R(.,Z)Z$ in the $Z$-direction (for suitable $Z$) where $R$ is the curvature of the Riemannian connection of $N$. These results are summarized in Proposition 4. Using this information and the Riemannian symmetric space structure of the ambient, we finally obtain Theorem 6.

Section 3 contains information about the family of tubes centered at $\mathbb{CP}^2$ and on the focal set of $\mathbb{CP}^2$. The main results are Theorem 7 and Theorem 8. In the first one we give geometric information of that family through the spectral decomposition of their shape operators. The difficulties of the required calculations are related to the fact that $\mathbb{CP}^2$ is not a curvature-adapted submanifold. In the second one we characterize the focal set of $\mathbb{CP}^2$ and we show that its universal covering space is the sphere $S^5$. We feel that these results may be useful and interesting.

## 1 Basic facts

In the present Section we shall deal with the inclusions into the symmetric space $G_2/SU(2)\times SU(2)$ of the maximal totally geodesic submanifolds of maximal rank. From Chen and Nagano we know that these submanifolds are $\mathbb{CP}^2$ and $S^2\times S^2$ (see [5]).

In order to describe how these submanifolds are imbedded into the ambient space we need to introduce some Lie algebraic ingredients.

Inside the complex simple Lie algebra $\mathfrak{g}_2$ we take a Cartan subalgebra $\mathfrak{h}$ and let $\Delta$ be the root system of $\mathfrak{g}_2$ relative to $\mathfrak{h}$. We may write

$$\mathfrak{g}_2 = \mathfrak{h} \oplus \sum_{\gamma\in\Delta^+} (\mathfrak{g}_\gamma + \mathfrak{g}_{-\gamma})$$

where $\Delta^+$ indicates the set of positive roots with respect to some ordering. Set $\pi = \{\alpha_1,\alpha_2\}\subset\Delta^+$ the system of simple roots. Then

$$\Delta^+ = \{\alpha_1,\alpha_2,\beta=\alpha_1+\alpha_2,\gamma=2\alpha_1+\alpha_2,\delta=3\alpha_1+\alpha_2,\mu=3\alpha_1+2\alpha_2\} \quad (1)$$

being $\alpha_1,\beta$ and $\gamma$ the short roots and $\alpha_2,\delta$ and $\mu$ the long ones. We fix their lenghts to be $\sqrt{2}$ and $\sqrt{6}$ respectively.

We take in $\mathfrak{g}_2$ a Weyl basis $\{X_\gamma : \gamma \in \Delta\} \cup \{H_\beta : \beta \in \pi\}$ (see [11] III, 5.5). The following set of vectors provide a basis of the compact real form $\mathfrak{g}_{2_u}$

$$\begin{array}{ll} U_\gamma = \frac{1}{\sqrt{2}}(X_\gamma - X_{-\gamma}) & \gamma \in \Delta^+ \\ U_{-\gamma} = \frac{i}{\sqrt{2}}(X_\gamma + X_{-\gamma}) & \gamma \in \Delta^+ \\ iH_\beta & \beta \in \pi. \end{array} \tag{2}$$

We shall denote by $\mathfrak{h}_u$ the real vector space generated by $\{iH_\beta : \beta \in \pi\}$ and set $\mathfrak{m}_\gamma = \mathbb{R}U_\gamma \oplus \mathbb{R}U_{-\gamma}$. Then

$$\mathfrak{g}_{2_u} = \mathfrak{h}_u \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma$$

and $G_2$ is the compact connected Lie group whose Lie algebra is $\mathfrak{g}_{2_u}$.

We include $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{g}_{2_u}$ in a natural way and so we can write

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \equiv (\mathbb{R}iH_{\alpha_1} \oplus \mathfrak{m}_{\alpha_1}) \oplus (\mathbb{R}iH_\mu \oplus \mathfrak{m}_\mu) \tag{3}$$

As usual we identify

$$T_o(G_2/SU(2) \times SU(2)) \equiv \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\beta \oplus \mathfrak{m}_\gamma \oplus \mathfrak{m}_\delta \tag{4}$$

where $o$ denote the class in the quotient of the identity element of the group.

Due to the natural inclusion of the algebra $\mathfrak{su}(3)$ into $\mathfrak{g}_{2_u}$ we have

$$\mathfrak{su}(3) \equiv \mathfrak{h}_u \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\delta \oplus \mathfrak{m}_\mu \tag{5}$$

and therefore we see $SU(3)$ as the analytic subgroup of $G_2$ corresponding to this subalgebra.

The action of the Weyl group $W(G_2)$ on $\mathfrak{g}_{2_u}$ gives rise to exactly the following copies of the algebra given in (3)

$$(\mathbb{R}iH_\gamma \oplus \mathfrak{m}_\gamma) \oplus (\mathbb{R}iH_{\alpha_2} \oplus \mathfrak{m}_{\alpha_2}) \quad \text{and} \quad (\mathbb{R}iH_\beta \oplus \mathfrak{m}_\beta) \oplus (\mathbb{R}iH_\delta \oplus \mathfrak{m}_\delta).$$

Since for our purposes these two different ways to look at $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ inside $\mathfrak{g}_{2_u}$ are essentially the same, we consider

$$\mathfrak{l} = (\mathbb{R}iH_{\alpha_2} \oplus \mathfrak{m}_{\alpha_2}) \oplus (\mathbb{R}iH_\gamma \oplus \mathfrak{m}_\gamma) \tag{6}$$

and denote by $L$ the corresponding analytic subgroup; so $L \cong SU(2) \times SU(2)$.

Since the ambient space $N = G_2/SU(2) \times SU(2)$ is a Riemannian symmetric space, there exists an involutive automorphism $\sigma$ of $G_2$ such that the connected component of the identity of its fixed point set is $SU(2) \times SU(2)$.

The totally geodesic submanifolds of $N$ are also Riemannian symmetric spaces and, as Chen and Nagano showed in [5, Prop. 6.1], the maximal ones with maximal rank are constructed from the restriction of $\sigma$ to the maximal subgroups of maximal rank in $G_2$. It is known that these subgroups are (up to conjugation) $SU(3)$ and $SU(2) \times SU(2)$ (see for instance [14, p. 281]).

Since $\mathfrak{h}_u \oplus \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_\mu$ and $\mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\beta \oplus \mathfrak{m}_\gamma \oplus \mathfrak{m}_\delta$ are the eigenspaces of $d\sigma_e$ corresponding to eigenvalues 1 and $-1$ respectively, then

$$\mathfrak{su}(3) \equiv (\mathfrak{h}_u \oplus \mathfrak{m}_\mu) \oplus (\mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\delta) \tag{7}$$

and

$$\mathfrak{l} = (\mathbb{R}iH_{\alpha_2} \oplus \mathbb{R}iH_\gamma) \oplus (\mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\gamma)$$

are the decompositions into eigenspaces of the restrictions of $d\sigma_e$ which give rise to the inclusions of the corresponding symmetric spaces as maximal totally geodesic submanifolds of maximal rank in our ambient space. So we have

$$\begin{aligned} SU(3)/S(U(2) \times U(1)) &= \mathbb{CP}^2 \subset G_2/SU(2) \times SU(2) \\ L/T &= S^2 \times S^2 \subset G_2/SU(2) \times SU(2) \end{aligned} \tag{8}$$

where $T$ is the maximal torus of $G_2$ given by $T = \exp \mathfrak{h}_u$.

The preceeding argument corresponds to the proof of Proposition 6.1 given by Chen and Nagano in [5] for the symmetric space $G_2/SU(2) \times SU(2)$, hence they actually obtained these two submanifolds, although in table V appears $AI(3)$ instead of $\mathbb{CP}^2$. As usual we identify

$$\begin{aligned} T_o\mathbb{CP}^2 &= \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\delta \quad \text{and} \quad (T_o\mathbb{CP}^2)^\perp = \mathfrak{m}_\beta \oplus \mathfrak{m}_\gamma \\ T_o(S^2 \times S^2) &= \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_\gamma \quad \text{and} \quad (T_o(S^2 \times S^2))^\perp = \mathfrak{m}_\beta \oplus \mathfrak{m}_\delta \end{aligned} \tag{9}$$

where $o$ denote the class of the identity element of the group in each quotient.

## 2 Homogeneous Tubes in $G_2/SU(2) \times SU(2)$

Our goal in this section is to study the homogeneity of tubes in the symmetric space $G_2/SU(2) \times SU(2)$ centered at $\mathbb{CP}^2$ and $S^2 \times S^2$ which are the maximal totally geodesic submanifolds of maximal rank.

On the ambient space $G_2/SU(2) \times SU(2)$ we consider the $G_2$ -invariant metric induced by the opposite of the Killing form of $\mathfrak{g}_{2_u}$. This metric determines invariant metrics on $\mathbb{CP}^2$ and $S^2 \times S^2$ making the above inclusions isometric imbeddings.

It is well known that the tubes centered at $\mathbb{CP}^2$ and $S^2 \times S^2$ are globally defined for suficiently small radii (see for instance [9]).

Note that $G_2$ acts transitively on $G_2/SU(2)\times SU(2)$ by isometries and that the action of $SU(3)$ on $\mathbb{CP}^2$ is the restriction of this one. Moreover $SU(3)$ acts on the tube of radius $r$ around $\mathbb{CP}^2$ by

$$g.(\exp_p rX) = \exp_{g.p} rg_{*p}X \tag{10}$$

where $p\in\mathbb{CP}^2$ and $X\in(T_p\mathbb{CP}^2)^\perp$ with $\|X\|=1$. This observation is also true when we replace $SU(3)$ by $L\cong SU(2)\times SU(2)$ and $\mathbb{CP}^2$ by $S^2\times S^2$.

Our next objective is to study the natural action of $S(U(2)\times U(1))$ (the isotropy group of $SU(3)$ at the point $o\in\mathbb{CP}^2$ ) on the normal space $(T_o\mathbb{CP}^2)^\perp$.

To this end we need to use the structure of simple Lie algebra of $\mathfrak{g}_2$. Let us introduce some extra notation.

Let $\varepsilon$ and $\rho$ be elements of $\Delta^+$ such that $\varepsilon-\rho\in\Delta$. We define

$$sg(\varepsilon-\rho)=\begin{cases} 1 & \text{if } \varepsilon-\rho\in\Delta^+ \\ -1 & \text{if } \rho-\varepsilon\in\Delta^+ \end{cases}$$

and

$$|\varepsilon-\rho|=\begin{cases} \varepsilon-\rho & \text{if } \varepsilon-\rho\in\Delta^+ \\ \rho-\varepsilon & \text{if } \rho-\varepsilon\in\Delta^+. \end{cases}$$

From [6], for $\varepsilon,\rho\in\Delta^+$ and $\varepsilon\neq\rho$ we have the following formulae

$$\begin{aligned} &[U_\varepsilon,U_{-\varepsilon}]=iH_\varepsilon \\ &[U_\varepsilon,U_\rho]=\tfrac{1}{\sqrt{2}}\{N_{\varepsilon,\rho}U_{\varepsilon+\rho}+sg(\rho-\varepsilon)N_{\varepsilon,-\rho}U_{|\varepsilon-\rho|}\} \\ &[U_\varepsilon,U_{-\rho}]=\tfrac{1}{\sqrt{2}}\{N_{\varepsilon,\rho}U_{-(\varepsilon+\rho)}+N_{\varepsilon,-\rho}U_{-|\varepsilon-\rho|}\}. \end{aligned} \tag{11}$$

Here we understand that if $\varepsilon-\rho$ or $\varepsilon+\rho$ are not roots then the terms $N_{\varepsilon,-\rho}U_{\pm|\varepsilon-\rho|}$ or $N_{\varepsilon,\rho}U_{\pm(\varepsilon+\rho)}$ vanish.

Keeping the notation given in (1) for the positive roots of the algebra $\mathfrak{g}_2$, we compute the constants $N_{\varepsilon,\rho}$ (see [11] III, 5.5) which are written in the following table where the $\varepsilon$-th row and the $\rho$-th column meet.

| | $\alpha_2$ | $\beta$ | $\gamma$ | $\delta$ | $\mu$ | $-\alpha_2$ | $-\beta$ | $-\gamma$ | $-\delta$ | $-\mu$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_1$ | $\sqrt{3}$ | 2 | $-\sqrt{3}$ | 0 | 0 | 0 | $-\sqrt{3}$ | $-2$ | $\sqrt{3}$ | 0 |
| $\alpha_2$ | | 0 | 0 | $-\sqrt{3}$ | 0 | | $\sqrt{3}$ | 0 | 0 | $\sqrt{3}$ |
| $\beta$ | | | $-\sqrt{3}$ | 0 | 0 | | | 2 | 0 | $\sqrt{3}$ |
| $\gamma$ | | | | 0 | 0 | | | | $-\sqrt{3}$ | $-\sqrt{3}$ |
| $\delta$ | | | | | 0 | | | | | $-\sqrt{3}$ |

The relations $N_{\varepsilon,\rho}=-N_{\rho,\varepsilon}$ , $N_{\varepsilon,\rho}=-N_{-\varepsilon,-\rho}$ and $N_{\varepsilon,\pm\varepsilon}=0$ allow to complete the information about the constants $N_{\varepsilon,\rho}$ for $\varepsilon,\rho\in\Delta$.

Now we are in condition to prove the following Lemma.

**1 Lemma.** *The natural action of $S(U(2) \times U(1))$ on the unit sphere of $(T_o\mathbb{CP}^2)^\perp$ is transitive.*

PROOF. Let us consider the element $U_\beta \in (T_o\mathbb{CP}^2)^\perp$ and let $\mathcal{O}$ be the orbit of $U_\beta$ under the natural action of the group $S(U(2) \times U(1))$. Since this group acts by isometries on the ambient space and preserves $T_o\mathbb{CP}^2$ we have

$$\mathcal{O} = Ad(S(U(2) \times U(1)))U_\beta \subset (T_o\mathbb{CP}^2)^\perp \tag{12}$$

and by (7) and (8) we obtain $T_{U_\beta}(\mathcal{O}) \equiv [\mathfrak{h}_u \oplus \mathfrak{m}_\mu, U_\beta]$.

Since

$$\left[iH_{\alpha_j}, U_\beta\right] = \tfrac{i}{\sqrt{2}}\{\beta(H_{\alpha_j})X_\beta - (-\beta(H_{\alpha_j}))X_{-\beta}\} = \beta(H_{\alpha_j})U_{-\beta} \tag{13}$$

it is easy to conclude that

$$[\mathfrak{h}_u, U_\beta] = \mathbb{R}U_{-\beta}. \tag{14}$$

From (11) and the table after formulae (11) we have

$$[U_\mu, U_\beta] = -\sqrt{\tfrac{3}{2}}U_\gamma \quad \text{and} \quad [U_{-\mu}, U_\beta] = -\sqrt{\tfrac{3}{2}}U_{-\gamma}$$

and hence

$$[\mathfrak{m}_\mu, U_\beta] = \mathfrak{m}_\gamma. \tag{15}$$

Then we get

$$T_{U_\beta}(\mathcal{O}) \equiv \mathbb{R}U_{-\beta} \oplus \mathfrak{m}_\gamma \tag{16}$$

and so $\dim \mathcal{O} = 3$. Further, since $\mathcal{O}$ is compact and hence closed in the unitary sphere $S^3 \subset (T_o\mathbb{CP}^2)^\perp$ it follows that $\mathcal{O} = S^3$. $\boxed{QED}$

For the natural action of the isotropy group of $L$ at the point $o \in L/T = S^2 \times S^2$ on the normal space $(T_o(S^2 \times S^2))^\perp$ we have the following result.

**2 Lemma.** *The natural action of a maximal torus $T$ in $SU(2) \times SU(2)$ on the unit sphere of $(T_o(S^2 \times S^2))^\perp$ is not transitive.*

PROOF. It follows because $\dim(T_o(S^2 \times S^2))^\perp = 4$ and the $T$ -orbit of an element in the unit sphere of $(T_o(S^2 \times S^2))^\perp$ has dimension at most two. $\boxed{QED}$

Denote by $(M)_r$ the tube of radius $r > 0$ in $G_2/SU(2) \times SU(2)$ around the submanifold $M$.

**3 Theorem.** *For small enough positive radii $r$ the tubes $(\mathbb{CP}^2)_r$ in $G_2/SU(2) \times SU(2)$ around $\mathbb{CP}^2$ are homogeneous real hypersurfaces. Moreover, there exist an element $Y$ in the Lie algebra $\mathfrak{h}_u$ of a maximal torus in $SU(3)$, such that*

$$(\mathbb{CP}^2)_r = SU(3)/S^1$$

*where $S^1 = \{\exp tY : t \in \mathbb{R}\}$.*

Proof. From Lemma 1 and formula (10) it follows that, for each radius $r$, the natural action of the group $SU(3)$ on the tube $(\mathbb{CP}^2)_r$ is transitive. Let us denote by $H$ the isotropy group of this action at the point $\exp_o rU_\beta$.

If $g \in SU(3)$ then

$$g \in H \Longleftrightarrow g.o = o \quad \text{and} \quad g_{*o}U_\beta = Ad(g)U_\beta = U_\beta.$$

Consequently, we can write the orbit $\mathcal{O}$ given in (12) as $\mathcal{O} \cong S(U(2)\times U(1))/H$. Since $\mathcal{O} \cong S^3$, $H$ is connected and its Lie algebra is given by $\{X \in \mathfrak{h}_u \oplus \mathfrak{m}_\mu : [X, U_\beta] = 0\}$. From (15) $adU_\beta$ is an isomorphism of $\mathfrak{m}_\mu$ onto $\mathfrak{m}_\gamma$ and by (14) $adU_\beta : \mathfrak{h}_u \to \mathbb{R}U_{-\beta}$ is onto. Then there exist $Y \in \mathfrak{h}_u$ such that $\mathbb{R}Y$ is the Lie algebra of $H$ and so $H$ is included into $S(U(2) \times U(1))$ as the one dimensional torus $\{\exp tY : t \in \mathbb{R}\}$. $\boxed{QED}$

Our next purpose is to compute the spectral decomposition of the Jacobi operator $R_Z$ for suitable $Z \in T_o(G_2/SU(2) \times SU(2))$ in order to study the homogeneity of the tube $(S^2 \times S^2)_r$.

If $R$ is the curvature tensor of a Riemannian manifold $N$, the Jacobi operator in the $Z$-direction is defined by $R_Z := R(., Z)Z$.

It is known and easy to see that for a Riemannian symmetric space $N = G/K$ the Jacobi operator $R_Z$, for $Z \in T_oN$, is given by

$$R_Z = -(adZ)^2$$

where $ad$ is the adjoint representation of the Lie algebra of $G$.

Set $Z = U_\beta$. From (11) and the table after formulae (11) we have

$$\begin{array}{ll}
[U_\beta, U_{\alpha_1}] = -\sqrt{2}U_\gamma + \sqrt{\tfrac{3}{2}}U_{\alpha_2} & [U_\beta, U_{\alpha_2}] = -\sqrt{\tfrac{3}{2}}U_{\alpha_1} \\
[U_\beta, U_\gamma] = -\sqrt{\tfrac{3}{2}}U_\mu + \sqrt{2}U_{\alpha_1} & [U_\beta, U_\delta] = 0 \\
[U_\beta, U_\mu] = \sqrt{\tfrac{3}{2}}U_\gamma &
\end{array}$$

Then

$$\begin{array}{l}
[U_\beta, [U_\beta, U_{\alpha_2}]] = -\tfrac{3}{2}U_{\alpha_2} + \sqrt{3}U_\gamma \\
[U_\beta, [U_\beta, U_\gamma]] = \sqrt{3}U_{\alpha_2} - \tfrac{7}{2}U_\gamma \\
[U_\beta, [U_\beta, U_\delta]] = 0
\end{array}$$

In an analogous way we obtain

$$\begin{array}{l}
[U_\beta, [U_\beta, U_{-\alpha_2}]] = -\tfrac{3}{2}U_{-\alpha_2} - \sqrt{3}U_{-\gamma} \\
[U_\beta, [U_\beta, U_{-\gamma}]] = -\sqrt{3}U_{-\alpha_2} - \tfrac{7}{2}U_{-\gamma} \\
[U_\beta, [U_\beta, U_{-\delta}]] = 0 \\
[U_\beta, [U_\beta, U_{-\beta}]] = [U_\beta, iH_\beta] = -2U_{-\beta}
\end{array}$$

(the last equality is obtained in a similar way as (13) and using that $\beta$ has length $\sqrt{2}$).

The above calculations are summarized in the following proposition.

**4 Proposition.** *The spectral decomposition of the Jacobi operator $R_{U_\beta}$ of the symmetric space $G_2/SU(2)\times SU(2)$ is given by the following table in which $c$ denote its eigenvalues and $V_c$ the corresponding eigenspaces.*

| $c$ | $V_c$ |
|---|---|
| $0$ | $\mathbb{R}U_\beta \oplus \mathbb{R}U_\delta \oplus \mathbb{R}U_{-\delta}$ |
| $2$ | $\mathbb{R}U_{-\beta}$ |
| $\frac{1}{2}$ | $\mathbb{R}(\frac{\sqrt{3}}{2}U_{\alpha_2}+\frac{1}{2}U_\gamma)\oplus\mathbb{R}(\frac{\sqrt{3}}{2}U_{-\alpha_2}-\frac{1}{2}U_{-\gamma})$ |
| $\frac{9}{2}$ | $\mathbb{R}(-\frac{1}{2}U_{\alpha_2}+\frac{\sqrt{3}}{2}U_\gamma)\oplus\mathbb{R}(\frac{1}{2}U_{-\alpha_2}+\frac{\sqrt{3}}{2}U_{-\gamma})$. |

**5 Remark.** Note that from (9) and the eigenspaces decomposition in the preceeding proposition it follows that $T_o(S^2\times S^2)$ is invariant by $R_{U_\beta}$ whereas $T_o\mathbb{CP}^2$ is not. Consequently the submanifold $S^2\times S^2$ is curvature adapted and $\mathbb{CP}^2$ is not curvature adapted.

**6 Theorem.** *The tubes $(S^2\times S^2)_r$ in $G_2/SU(2)\times SU(2)$ around $S^2\times S^2$ are non-homogeneous real hypersurfaces.*

Proof. From (9) $U_\beta$ and $U_\delta$ are in $(T_o(S^2\times S^2))^\perp$. By (11) and the table below (11) we obtain

$$[U_\delta,[U_\delta,U_\gamma]]=\sqrt{\tfrac{3}{2}}\,[U_\delta,U_{\alpha_1}]=-\tfrac{3}{2}U_\gamma$$

and so

$$R_{U_\delta}(U_\gamma)=\tfrac{3}{2}U_\gamma. \tag{17}$$

We consider now $\varphi$ and $\psi$ geodesics in $G_2/SU(2)\times SU(2)$ emanating from $o$ with velocities $U_\beta$ and $U_\delta$ respectively.

Since $G_2/SU(2)\times SU(2)$ is a Riemannian symmetric space, the Jacobi operators $R_{\dot\varphi(t)}$ and $R_{\dot\psi(t)}$ have constant eigenvalues along $\varphi$ and $\psi$ respectively. Thus, by Proposition 4 the spectrum of $R_{\dot\varphi(r)}$ is $\{0,2,\frac{1}{2},\frac{9}{2}\}$ and formula (17) shows that $\frac{3}{2}$ belongs to the spectrum of $R_{\dot\psi(r)}$. This says that both spectra are different.

It is known that if a subgroup of the isometry group of $G_2/SU(2)\times SU(2)$ acts transitively on the tube $(S^2\times S^2)_r$ then $(R_{Z_1})_{p_1}$ and $(R_{Z_2})_{p_2}$ have the same eigenvalues for $p_i\in(S^2\times S^2)_r$ and $Z_i\in(T_{p_i}(S^2\times S^2)_r)^\perp, \|Z_i\|=1, i=1,2$. This fact completes the proof. $\boxed{QED}$

## 3 $\mathbb{CP}^2$-centered tubes in $G_2/SU(2) \times SU(2)$

In this paragraph we shall make use of the spectral decomposition of the Jacobi operator $R_{U_\beta}$ obtained in Proposition 4 to determine the focal set of $\mathbb{CP}^2$ in $G_2/SU(2) \times SU(2)$ and to obtain information of the geometry of the family of $\mathbb{CP}^2$ -centered tubes.

Let $M$ be a submanifold of a complete Riemannian manifold $N$ and let $Z \in T_pM^\perp$ a unit normal vector at some point $p \in M$. Let $\varphi_Z$ be the geodesic in $N$ with $\varphi_Z(0) = p$ and $\dot{\varphi}_Z(0) = Z$. We recall from [7] that the point $\varphi_Z(t_0)$ $(t_0 > 0)$ is said to be a *focal point* of $M$ along $\varphi_Z$ if the differential of the normal exponential map of $M$ is singular at $t_0 Z$. Equivalently $\varphi_Z(t_0)$ is a focal point of $M$ along $\varphi_Z$ if there exist a Jacobi vector field $J(t)$ along $\varphi_Z$ satisfying

$$\begin{array}{ll} (i) & J(0) \in T_pM \\ (ii) & J'(0) + A_Z(J(0)) \in T_pM^\perp \\ (iii) & J(t_0) = 0 \end{array} \tag{18}$$

where $A_Z$ is the shape operator of $M$ in the direction of $Z$. If there are focal points along $\varphi_Z$, define

$$t_Z := \min\{t_0 > 0 : \varphi_Z(t_0) \text{ is a focal point of } M \text{ along } \varphi_Z\}$$

and call $\varphi_Z(t_Z)$ the first focal point of $M$ along $\varphi_Z$. By the *focal set* of $M$ we mean the set $FM$ consisting of first focal points of $M$ along all the geodesics $\varphi_Z$ departing from $M$ with $Z$ normal unit vectors to $M$.

In our situation $M = \mathbb{CP}^2 \subset G_2/SU(2) \times SU(2)$ and from (9) we know that $U_\beta \in (T_o\mathbb{CP}^2)^\perp$. Proposition 4 provides the following ordered orthonormal basis of eigenvectors of $R_{U_\beta}$

$$\{E_1 = U_\beta, E_2 = U_{-\beta}, E_3 = U_\delta, E_4 = U_{-\delta}, F_1, F_3,\ F_2, F_4\} \tag{19}$$

where

$$\begin{array}{ll} F_1 = \frac{\sqrt{3}}{2}U_{\alpha_2} + \frac{1}{2}U_\gamma & F_2 = \frac{\sqrt{3}}{2}U_{-\alpha_2} - \frac{1}{2}U_{-\gamma} \\ F_3 = -\frac{1}{2}U_{\alpha_2} + \frac{\sqrt{3}}{2}U_\gamma & F_4 = \frac{1}{2}U_{-\alpha_2} + \frac{\sqrt{3}}{2}U_{-\gamma} \end{array}$$

Note that $E_1,\ E_2 \in (T_o\mathbb{CP}^2)^\perp,\ E_3,\ E_4 \in T_o\mathbb{CP}^2$ and the vectors $F_i$, $1 \le i \le 4$ are not tangent neither normal to $\mathbb{CP}^2$. In particular this shows that the totally geodesic submanifold $\mathbb{CP}^2$ is not curvature-adapted, as we already mentioned in Remark 5.

If $E_k(t)$ and $F_k(t)$ denote the parallel transport along the geodesic $\varphi(t) = \varphi_{U_\beta}(t)$ of $E_k$ and $F_k$ respectively, then it is not difficult to see that the most general Jacobi vector field along $\varphi(t)$ satisfying conditions (i) and (ii) of (18) is given by

$$\begin{aligned} J(t) = {} & p_1 t E_1(t) + p_2 \sin(\sqrt{2}t)E_2(t) + p_3 E_3(t) + p_4 E_4(t) + \\ & + \sqrt{3}(-p_5 \cos(\tfrac{t}{\sqrt{2}}) + p_6 \sin(\tfrac{t}{\sqrt{2}}))F_1(t) + \sqrt{3}(p_7 \cos(\tfrac{t}{\sqrt{2}}) - p_8 \sin(\tfrac{t}{\sqrt{2}}))F_2(t) + \\ & + (p_5 \cos(\tfrac{3t}{\sqrt{2}}) + p_6 \sin(\tfrac{3t}{\sqrt{2}}))F_3(t) + (p_7 \cos(\tfrac{3t}{\sqrt{2}}) + p_8 \sin(\tfrac{3t}{\sqrt{2}}))F_4(t). \end{aligned} \tag{20}$$

where $p_i$ are arbitrary real constants (note that since $\mathbb{CP}^2$ is totally geodesic in $G_2/SU(2) \times SU(2)$ condition (ii) of (18) becomes $J'(0) \in (T_o\mathbb{CP}^2)^{\perp}$).

After some calculations we conclude that the smallest $t_o > 0$ for which there exist a Jacobi field $J$ of the form (20) with $J(t_o) = 0$ is $\frac{\pi}{2\sqrt{2}}$ and so

$$t_{U_\beta} = \frac{\pi}{2\sqrt{2}}.$$

At the same time we obtain that the most general Jacobi vector field satisfying (18) for $t_o = \frac{\pi}{2\sqrt{2}}$ is given by

$$\begin{aligned} J(t) = {} & \sqrt{3}p_6(-\cos(\tfrac{t}{\sqrt{2}}) + \sin(\tfrac{t}{\sqrt{2}}))F_1(t) + \sqrt{3}p_8(\cos(\tfrac{t}{\sqrt{2}}) - \sin(\tfrac{t}{\sqrt{2}}))F_2(t) + \\ & + p_6(\cos(\tfrac{3t}{\sqrt{2}}) + \sin(\tfrac{3t}{\sqrt{2}}))F_3(t) + p_8(\cos(\tfrac{3t}{\sqrt{2}}) + \sin(\tfrac{3t}{\sqrt{2}}))F_4(t) \end{aligned} \tag{21}$$

where $p_6$ and $p_8$ are arbitrary real constants.

Hence the first focal point of $\mathbb{CP}^2$ along $\varphi = \varphi_{U_\beta}$ is precisely $\varphi(\frac{\pi}{2\sqrt{2}})$.

**7 Theorem.** *Any tube $(\mathbb{CP}^2)_r$ of radius $0 < r < \frac{\pi}{2\sqrt{2}}$ around $\mathbb{CP}^2$ is an embedded isoparametric real hypersurface of $G_2/SU(2) \times SU(2)$ with four distinct principal curvatures. The principal curvatures $k_i$ of $(\mathbb{CP}^2)_r$, with respect to the outward unit normal field $Z$, and the corresponding space of principal directions $V_i$ are listed in the following table*

| $k_i$ | $V_i$ |
|---|---|
| $-\sqrt{2}\cot(\sqrt{2}r)$ | $\mathbb{R}E_2(r)$ |
| $0$ | $\mathbb{R}E_3(r) \oplus \mathbb{R}E_4(r)$ |
| $-\sqrt{2}\cot(2\sqrt{2}r) + \sqrt{2\cot^2(2\sqrt{2}r) + \frac{3}{2}}$ | $\mathbb{R}F^-_{1,3}(r) \oplus \mathbb{R}F^-_{2,4}(r)$ |
| $-\sqrt{2}\cot(2\sqrt{2}r) - \sqrt{2\cot^2(2\sqrt{2}r) + \frac{3}{2}}$ | $\mathbb{R}F^+_{1,3}(r) \oplus \mathbb{R}F^+_{2,4}(r)$ |

*where $F^{\pm}_{j,j+2}(r) = F_j(r) + (-1)^{j+1}(\frac{\cos(2\sqrt{2}r) \pm \sqrt{\cos^2(2\sqrt{2}r)+3}}{\sqrt{3}})F_{j+2}(r)$ , $E_j(r)$ and $F_j(r)$ are the parallel transport along the geodesic $\varphi = \varphi_{U_\beta}$ of the vectors $E_k$ and $F_k$ defined in (19).*

Proof. Let $A^r$ denote the shape operator of the tube $(\mathbb{CP}^2)_r$ for $0 < r < \frac{\pi}{2\sqrt{2}}$ with respect to the outward unit normal field. Since the action (by isometries of ambient space) of $SU(3)$ on $(\mathbb{CP}^2)_r$ is transitive, $A^r$ has the same eigenvalues every point of $(\mathbb{CP}^2)_r$.

Fix the geodesic $\varphi = \varphi_{U_\beta}$ in $G_2/SU(2)\times SU(2)$ and denote by $J_X$ the Jacobi field along $\varphi$ with initial conditions $J_X(0) = 0$ and $J'_X(0) = X$ if $X \in (T_o\mathbb{CP}^2)^\perp$ and with $J_X(0) = X$ and $J'_X(0) = 0$ if $X \in T_o\mathbb{CP}^2$.

Recall that $E_1 = U_\beta$, $E_2 = U_{-\beta}$, $E_3 = U_\delta$, $E_4 = U_{-\delta}$ and set $E_5 = U_{\alpha_2}$, $E_6 = U_{-\alpha_2}$, $E_7 = U_\gamma$, $E_8 = U_{-\gamma}$. From (9) these vectors form a basis of $T_o(G_2/SU(2) \times SU(2))$ such that $E_3, E_4, E_5, E_6 \in T_o\mathbb{CP}^2$ and $E_1, E_2, E_7, E_8 \in (T_o\mathbb{CP}^2)^\perp$. Let $E_k(t)$ denote the parallel transport along $\varphi(t)$ of $E_k$.

Since the fields $J_X$ satisfy the two first conditions of (18), they are of the form given in (20) for adecuate constants. Then, from that expression and some calculation it is not difficult to conclude that

$$
\begin{aligned}
&J_{E_1}(t) = tE_1(t) && J_{E_5}(t) = \tfrac{\sqrt{3}}{2}\cos(\tfrac{t}{\sqrt{2}})F_1(t) - \tfrac{1}{2}\cos(\tfrac{3t}{\sqrt{2}})F_3(t)\\
&J_{E_2}(t) = \tfrac{1}{\sqrt{2}}\sin(\sqrt{2}t)E_2(t) && J_{E_6}(t) = \tfrac{\sqrt{3}}{2}\cos(\tfrac{t}{\sqrt{2}})F_2(t) + \tfrac{1}{2}\cos(\tfrac{3t}{\sqrt{2}})F_4(t)\\
&J_{E_3}(t) = E_3(t) && J_{E_7}(t) = \tfrac{1}{\sqrt{2}}\sin(\tfrac{t}{\sqrt{2}})F_1(t) + \tfrac{1}{\sqrt{6}}\sin(\tfrac{3t}{\sqrt{2}})F_3(t)\\
&J_{E_4}(t) = E_4(t) && J_{E_8}(t) = -\tfrac{1}{\sqrt{2}}\sin(\tfrac{t}{\sqrt{2}})F_2(t) + \tfrac{1}{\sqrt{6}}\sin(\tfrac{3t}{\sqrt{2}})F_4(t).
\end{aligned}
\tag{22}
$$

It is well known that, for $J_X(r)$ tangent to the tube $(\mathbb{CP}^2)_r$, the operator $A^r$ is given by the following formula

$$A^r J_X(r) = -[J'_X(r)]^T$$

where $W^T$ indicates the projection of $W$ onto the tangent space of $(\mathbb{CP}^2)_r$. Then

$$
\begin{aligned}
&A^r J_{E_2}(r) = -\cos(\sqrt{2}r)E_2(r)\\
&A^r J_{E_3}(r) = A^r J_{E_4}(r) = 0\\
&A^r J_{E_5}(r) = \tfrac{\sqrt{3}}{2\sqrt{2}}\sin(\tfrac{r}{\sqrt{2}})F_1(r) - \tfrac{3}{2\sqrt{2}}\sin(\tfrac{3r}{\sqrt{2}})F_3(r)\\
&A^r J_{E_6}(r) = \tfrac{\sqrt{3}}{2\sqrt{2}}\sin(\tfrac{r}{\sqrt{2}})F_2(r) + \tfrac{3}{2\sqrt{2}}\sin(\tfrac{3r}{\sqrt{2}})F_4(r)\\
&A^r J_{E_7}(r) = -\tfrac{1}{2}\cos(\tfrac{r}{\sqrt{2}})F_1(r) - \tfrac{\sqrt{3}}{2}\cos(\tfrac{3r}{\sqrt{2}})F_3(r)\\
&A^r J_{E_8}(r) = \tfrac{1}{2}\cos(\tfrac{r}{\sqrt{2}})F_2(r) - \tfrac{\sqrt{3}}{2}\cos(\tfrac{3r}{\sqrt{2}})F_4(r).
\end{aligned}
$$

From (22) we obtain

$$
\begin{aligned}
&F_1(r) = \tfrac{1}{\sin(2\sqrt{2}r)}\{\tfrac{2}{\sqrt{3}}\sin(\tfrac{3r}{\sqrt{2}})J_{E_5}(r) + \sqrt{2}\cos(\tfrac{3r}{\sqrt{2}})J_{E_7}(r)\}\\
&F_3(r) = \tfrac{1}{\sin(2\sqrt{2}r)}\{-2\sin(\tfrac{r}{\sqrt{2}})J_{E_5}(r) + \sqrt{6}\cos(\tfrac{r}{\sqrt{2}})J_{E_7}(r)\}\\
&F_2(r) = \tfrac{1}{\sin(2\sqrt{2}r)}\{\tfrac{2}{\sqrt{3}}\sin(\tfrac{3r}{\sqrt{2}})J_{E_6}(r) - \sqrt{2}\cos(\tfrac{3r}{\sqrt{2}})J_{E_8}(r)\}\\
&F_4(r) = \tfrac{1}{\sin(2\sqrt{2}r)}\{2\sin(\tfrac{r}{\sqrt{2}})J_{E_6}(r) + \sqrt{6}\cos(\tfrac{r}{\sqrt{2}})J_{E_8}(r)\}
\end{aligned}
$$

and thus, in the ordered basis $\{E_2(r), E_3(r), E_4(r), F_1(r), F_3(r), F_2(r), F_4(r)\}$ of $T_{\varphi(r)}(\mathbb{CP}^2)_r$, $A^r$ is given by the matrix in block form

$$\begin{bmatrix} -\sqrt{2}\cot(\sqrt{2}r)I_1 & & & \\ & 0I_2 & & \\ & & M_1 & \\ & & & M_2 \end{bmatrix}$$

where $I_j$ denotes the identity $j \times j$-matrix and

$$M_j = \begin{bmatrix} -\frac{1}{\sqrt{2}}\cot(2\sqrt{2}r) & (-1)^j\sqrt{\frac{3}{2}}(\sin(2\sqrt{2}r))^{-1} \\ (-1)^j\sqrt{\frac{3}{2}}(\sin(2\sqrt{2}r))^{-1} & -\frac{3}{\sqrt{2}}\cot(2\sqrt{2}r) \end{bmatrix}.$$

From this the proof easily follows. $\boxed{QED}$

It is difficult in general to determine the nature of the focal set of a submanifold. However, in our case, by using the root space structure associated to $G_2/SU(2) \times SU(2)$ and geometric facts, we obtain the following result which we feel that may be found interesting.

**8 Theorem.** *(i) The focal set* $F\mathbb{CP}^2$ *of* $\mathbb{CP}^2$ *consists of those points in* $G_2/SU(2) \times SU(2)$ *which are at distance* $\frac{\pi}{2\sqrt{2}}$ *from* $\mathbb{CP}^2$*. Furthermore,* $SU(3)$ *acts on* $F\mathbb{CP}^2$ *transitively by* $g.\varphi_Z(\frac{\pi}{2\sqrt{2}}) = \varphi_{g_*Z}(\frac{\pi}{2\sqrt{2}})$ *where* $\varphi_Z$ *is the geodesic starting at* $p \in \mathbb{CP}^2$ *with initial unitary speed* $Z \in (T_p\mathbb{CP}^2)^\perp$*.*

*(ii) The universal covering space of* $F\mathbb{CP}^2$ *is the sphere* $S^5$*.*

Proof. We already know that the first focal point of $\mathbb{CP}^2$ along $\varphi = \varphi_{U_\beta}$ is $\varphi(\frac{\pi}{2\sqrt{2}})$. Let $J(t)$ be a Jacobi vector field of the form (21) then for any $g \in SU(3)$, $g_*J(t)$ is a Jacobi vector field along the geodesic $\varphi_{g_*U_\beta}(t) = g.\varphi_{U_\beta}(t)$ which satisfies also conditions (i), (ii) and (iii) of (18) with $t_0 = \frac{\pi}{2\sqrt{2}}$. Then $\varphi_{g_*U_\beta}(\frac{\pi}{2\sqrt{2}})$ is a focal point of $\mathbb{CP}^2$ along $\varphi_{g_*U_\beta}$ and by an analogous argument we have that $\varphi_{g_*U_\beta}(\frac{\pi}{2\sqrt{2}})$ is the first one.

We conclude the proof of (i) considering that $SU(3)$ acts transitively on $\mathbb{CP}^2$ by isometries of $G_2/SU(2)\times SU(2)$ and that the natural action of $S(U(2)\times U(1))$ on the unit sphere of $(T_o\mathbb{CP}^2)^\perp$ is transitive (see Lemma 1).

Fix $r^* = \frac{\pi}{4\sqrt{2}}$ and consider the displacement map $f$ of the tube $(\mathbb{CP}^2)_{r^*}$ in direction of the outward normal unit vector field $Z$ given by

$$f : (\mathbb{CP}^2)_{r^*} \to F\mathbb{CP}^2, \quad q \mapsto \psi_q(r^*)$$

where $\psi_q$ denotes the geodesic in $G_2/SU(2)\times SU(2)$ with $\psi_q(0) = q$ and $\dot{\psi}_q(0) = Z_q$. This map is $SU(3)$ -equivariant, *i.e.*, $f(g.q) = g.f(q)$ for $q \in (\mathbb{CP}^2)_{r^*}$ and $g \in SU(3)$.

It is known that $df_q(X) = \tilde{J}_X(r^*)$ where $\tilde{J}_X$ is the Jacobi vector field along $\psi_q$ determined by $\tilde{J}_X(0) = X$ and $\tilde{J}'_X(0) = -A^{r^*}X$.

Denote by $p^* = \varphi_{U_\beta}(r^*) \in (\mathbb{CP}^2)_{r^*}$. It easy to see that if $2 \leq k \leq 8$

$$df_{p^*}(J_{E_k}(r^*)) = J_{E_k}(2r^*)$$

where the vectors $J_{E_k}(r^*)$ are given by (22). Since $\{J_{E_k}(r^*) : 2 \leq k \leq 8\}$ is a basis of $T_{p^*}(\mathbb{CP}^2)_{r^*}$ we can conclude that the $\mathrm{Ker}(df_{p^*})$ is generated by $J_{E_5}(r^*) - \sqrt{\frac{3}{2}}J_{E_7}(r^*)$ and $J_{E_6}(r^*) + \sqrt{\frac{3}{2}}J_{E_8}(r^*)$. Then it has dimension 2.

On the other hand, we know that the focal set $F\mathbb{CP}^2$ is a quotient space $SU(3)/K$ where $K$ is the isotropy group at $p_o = \varphi_{U_\beta}(\frac{\pi}{2\sqrt{2}}) = \exp(\frac{\pi}{2\sqrt{2}}U_\beta).o$ and that the canonical projection $\pi$ of $SU(3)$ onto $SU(3)/K$ is a Riemannian submersion. Then, given a smooth curve $\beta$ in $F\mathbb{CP}^2$ with $\beta(0) = p_o$ there is a unique smooth horizontal lift $\alpha$ such that $\alpha(0)$ is the identity of $SU(3)$ (see [4, p. 65-66]). If $\pi_1$ denotes the canonical projection of $SU(3)$ onto $SU(3)/S^1 = (\mathbb{CP}^2)_{r^*}$ we have the smooth curve $\delta(t) = \pi_1 \circ \alpha(t) = \alpha(t).p^*$. Then $f(\delta(t)) = f(\alpha(t).p^*) = \alpha(t).f(p^*) = \alpha(t).p_o = \pi(\alpha(t)) = \beta(t)$ and it follows that $df_{p^*}$ is onto. Hence $F\mathbb{CP}^2 = SU(3)/K$ is a five dimensional compact homogeneous manifold and so, $\dim K = 3$.

Our next purpose is to determine the connected component of the identity of the compact group $K$. It easy to see that $K = g_o(SU(2)\times SU(2))g_o^{-1} \cap SU(3)$ where $g_o = \exp \frac{\pi}{2\sqrt{2}}U_\beta$ and therefore the Lie algebra of $K$ is $\mathfrak{k} = Ad(g_o)(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \cap \mathfrak{su}(3)$.

Using formulae (11) we can see that there exist real constants $a_i, b_i, c_i$ and $d_i$ satisfying

$$\begin{aligned}
&Ad(g_o)iH_{\alpha_1} = (iH_{\alpha_1} + \tfrac{1}{2}iH_\beta) + \tfrac{1}{\sqrt{2}}U_{-\beta}\\
&Ad(g_o)iH_\beta = -\sqrt{2}U_{-\beta}\\
&Ad(g_o)U_{\alpha_1} = a_1U_{\alpha_1} + b_1U_{\alpha_2} + c_1U_\gamma + d_1U_\mu\\
&Ad(g_o)U_\mu = a_2U_{\alpha_1} + b_2U_{\alpha_2} + c_2U_\gamma + d_2U_\mu\\
&Ad(g_o)U_{-\alpha_1} = a_3U_{-\alpha_1} + b_3U_{-\alpha_2} + c_3U_{-\gamma} + d_3U_{-\mu}\\
&Ad(g_o)U_{-\mu} = a_4U_{-\alpha_1} + b_4U_{-\alpha_2} + c_4U_{-\gamma} + d_4U_{-\mu}
\end{aligned}$$

From this, (3), (5) and the fact that $\dim \mathfrak{k} = 3$ it is not difficult to conclude that there is a real number $b \neq 0$ such that $\mathfrak{k}$ is the algebra generated by $\{Z_1 = iH_{\alpha_1} + \frac{1}{2}iH_\beta, Z_2 = bU_{\alpha_2} + U_\mu, Z_3 = [Z_1, Z_2]\}$. Now, with some extra calculations we can obtain that $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. It is well known that (up to isomorphism) the only compact three dimensional real Lie algebra with this property is $\mathfrak{su}(2)$. Then the connected component of the identity of the compact group $K$ is $SU(2)$ and so $SU(3)/SU(2) = S^5$ is the universal covering space of $F\mathbb{CP}^2$. $\boxed{QED}$

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