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On the behaviour near the boundary of solutions of the Dirichlet problem for elliptic equations

Vahram Zh. Dumanian[i]
Department of Informatics and Applied Mathematics
Yerevan State University, 1, Alex Manoogian, Yerevan, Armenia, 375025
duman@ysu.am

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1 Introduction and statement of the result

In this paper we study the behaviour near the boundary of the solution of the Dirichlet problem in a bounded domain $Q \subset R_n, n \geq 2$, with smooth boundary ∂Q for an elliptic second-order equation

$$-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = f(x) - \operatorname{div} F(x), x \in Q; \tag{1}$$

$$u|_{\partial Q} = u_0, \tag{2}$$

where $u_0 \in L_2(\partial Q)$; the functions f and $F = (f_1, \ldots, f_n)$ belong to $L_{2,\text{loc}}(Q)$, the symmetric matrix $A(x) = (a_{ij}(x))$, whose elements are real measurable functions, satisfies the condition

$$\gamma_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = (\xi, A(x)\xi) \leq \gamma_2|\xi|^2 \tag{3}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R_n$ and $x \in Q$, with positive constants γ_1 and γ_2, the real coefficients $B(x)$ and $c(x)$ are measurable and bounded functions on each strong inner subdomain of the domain Q.

The aim of this paper is to obtain conditions on the coefficients of the lower-order terms of the equation for which the solution of the given problem has the property of $(n-1)$-dimensional continuity. The concept of s-dimensional

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continuity, which is a natural generalization of continuity on several variables, was introduced by A. K. Gushchin in [1] and means the following.

Let μ and ν be probability measures on R_n with supports in $\overline{Q}$ satisfying the condition:

> there exist a constant C such that for all $r > 0$ and $x^0 \in \overline{Q}$ the measure of the ball $\mathcal{B}_{x^0}(r)$ with radius r and centre x^0 is less or equal to Cr^s, where $0 < s < n$; the smallest of such constants C will be called the norm of the measure and denoted by $\|\mu\|$ (or $\|\nu\|$, respectively).

Let ϕ be a measure on R_{2n} with support in $\overline{Q} \times \overline{Q}$ such that $\mu(G) = \phi(G \times R_n), \nu(G) = \phi(R_n \times G)$ for all Borel sets $G \subset \overline{Q}$.

Following [1], a function v will be called s-dimensionally continuous if for any positive number ε there exists a number $\delta > 0$, such that

$$\frac{1}{\|\mu\| + \|\nu\|} \int\limits_{R_{2n}} [v(x) - v(y)]^2 d\phi(x, y) < \varepsilon$$

(the distance between values of the function v on these measures along ϕ is less than ε) as only

$$\int\limits_{R_{2n}} |x - y| d\phi(x, y) < \delta$$

(the distance between the measures μ and ν along ϕ is less than δ).

Note, that if arbitrary measures are taken in the definition, *i.e.* $s = 0$, then one gets the classical defintion of uniform continuity on Q.

The set of all s-dimensionally continuous functions on $\overline{Q}$ forms the Banach space $C_s(\overline{Q})$, which is the completion of the space $C(\overline{Q})$ w.r. to the norm generated by the functional

$$\ell(v) = \int\limits_0^\infty M_s(\{x \in \overline{Q} : |v(x)|^2 > \lambda\}) d\lambda, \quad v \in C(\overline{Q}),$$

where

$$M_s(E) = \inf \left\{ \sum_{i=1}^\infty r_i^s, \ \bigcup_{i=1}^\infty \mathcal{B}(r_i) \supset E \right\},$$

and the infimum is taken over all coverings of the E by means of balls $\mathcal{B}(r_i)$ of radius r_i; for $s = 0$ and $s = n$ we have the special cases $C_0(\overline{Q}) = C(\overline{Q})$ and $C_n(\overline{Q}) = L_2(Q)$, see [1]. The $(n-1)$-dimensional continuity of the solution of the

Dirichlet problem with boundary function u_0 in $L_2(\partial Q)$ for the equation without lower-order terms (*i.e.* $b_i = 0, c = 0$) and with right-hand side $f \in W_2^{-1}(F = 0)$ was established in paper [1]. There it was assumed that the unit inner normal $\overline{\nu}$ to the boundary ∂Q satisfies Dini's condition

$$|\overline{\nu}(x) - \overline{\nu}(y)| \leq w(|x - y|) \tag{4}$$

for all x and y in ∂Q, where w is a monotone function such that

$$\int_0 \frac{w(t)}{t} dt < \infty$$

and the coefficients are continous on the boundary in the sense of Dini:

$$|a_{ij}(x) - a_{ij}(y)| \leq w(|x - y|) \tag{5}$$

for all $x \in \partial Q, y \in Q$ and $i, j = 1, \dots, n$; without loss of generality, of course one can always assume that the function w is the same in (4) and (5). In [2] the above mentioned result was generalized for a wider class of right-hand sides. In this paper it was shown that the theorem holds for right-hand sides with

$$r^{\frac{1}{2}}(x)(1 + |\ln r(x)|)^{\frac{3}{4}} |F(x)| \in L_2(Q) \tag{6}$$

$$r^{\frac{3}{2}}(x)(1 + |\ln r(x)|)^{\frac{3}{4}} |f(x)| \in L_2(Q) \tag{7}$$

where $r(x)$ is the distance of a point $x \in Q$ from the boundary ∂Q. In the sequel we will in the same way assume that the conditions (4)–(7) are satisfied. By a solution of problem (1), (2) we understand a function u in $W^1_{2,\mathrm{loc}}$ satisfying the equation (1) in the sense of generalized functions, *i.e.* for all $\eta \in \overset{\circ}{C}{}^{\infty}(Q)$ the integral identity

$$\int_Q (A(x)\nabla u, \nabla \eta) dx + \int_Q ((B(x), \nabla u) + c(x)u)\eta dx = \int_Q (f\eta + (F, \nabla \eta)) dx \tag{8}$$

is satisfied, and satisfying condition (2) in the following sense:

each point $x^0 \in \partial Q$ has a neighborhood $V_{x^0} \subset \partial Q$ such that

$$\int_{V_{x^0}} (u(x + \delta \overline{\nu}(x^0)) - u_0(x))^2 ds \longrightarrow 0 \text{ as } \delta \longrightarrow +0. \tag{9}$$

The concept of a solution in $W^1_{2,\mathrm{loc}}$ was introduced by V. P. Mikhailov in [3], [8] for the case of a domain with twice smooth boundary, see also [5], [9], and [10]. Hereby, a solution attains its boundary value in the following sense

$$\int\limits_{\partial Q} (u(\varphi_\delta(x)) - u_0(x))^2 ds \longrightarrow 0 \text{ as } \delta \longrightarrow +0$$

where $\varphi_\delta(x) = x + \delta\overline{\nu}(x)$.

In [3], [8] it was shown that in the case of an equation with smooth coefficients ($a_{ij}(x), b_i(x) \in C^1(\overline{Q}), i, j = 1, \ldots, n, c(x) \in C(\overline{Q})$) the problem (1), (2) in the above mentioned framework is Fredholm and has the same spectrum as the problem in the $W^1_2(Q)$-framework; if the number zero does not belong to the spectrum, then the problem is solvable for any boundary function u_0 in $L_2(\partial Q)$ and for any right-hand side $f(F = 0)$ such that

$$\int\limits_Q r^\Theta(x) f^2(x) dx < \infty \text{ with some } \Theta < 3.$$

A generalization of this result for domains with Lyapunov boundary was obtained in [6] and [7]; in this context, the boundary condition (2) was formulated in local terms - it was required that the condition (9) is satisfied. In this way it could be shown that the map $x \longrightarrow \varphi_\delta(x), x \in \partial Q$, attributing to the points of the boundary points on a “parallel” surface, enables to get away from the chosen before direction (*i.e.* “orthogonal” to the boundary) and take instead the normal at a fixed point in a neighborhood under consideration.

The property of $(n-1)$-dimensional continuity shows that the chosen direction of the normal can be abandoned completely: the values of the boundary function u_0 can be compared to the values of the solution u not only on surfaces “parallel” to the boundary or near to such surfaces, but also on the images of ∂Q under mappings in a fairly large class. In particular, the surface ∂Q can be partitioned into sufficiently small parts, each of which can be moved and turned (without leaving $\overline{Q}$) so that the points are relocated “not too far”; hereby, different points of the boundary may be mapped onto the same point, but it cannot be allowed that there are “too many” such points. Furthermore, this property allows to define a solution of the Dirichlet problem with square summable boundary function, where the smoothness of the boundary is not required (see [1] for more details).

In this paper we shall establish when a solution in $W^1_{2,\mathrm{loc}}$ of the Dirichlet problem for a general second-order equation belongs to $C_{n-1}(\overline{Q})$. We assume, that the coefficients $B(x)$ and $c(x)$ satisfy the conditions there exist a constant $K > 0$ such that

$$|B(x)| \leq \frac{K}{r(x)(1+|\ln r(x)|)^{\frac{3}{4}}}, x \in Q, \tag{10}$$

there exist a monotone function $C(t)$ such that

$$|c(x)| \leq C(r(x)), x \in Q, \text{ and } \int_0 t^3 |\ln t|^{\frac{3}{2}} C^2(t)dt < \infty. \tag{11}$$

Now we exhibit the main result of the article.

Theorem. Assume that the conditons (3) - (7), (10) and (11) are satisfied. Then any solution in $W^1_{2,\mathrm{loc}}$ of the Dirichlet problem (1), (2) belongs to the Gushchin space $C_{n-1}(\overline{Q})$.

2 Proof of the Theorem

The proof of the theorem is based on the following

Lemma. Under the assumptions of the theorem, let u be a solution in $W^1_{2,\mathrm{loc}}$ of the Dirichlet problem (1), (2). Then the function $r(x)|\nabla u(x)|^2$ is integrable over Q, *i.e.*

$$\int_Q r(x)|\nabla u(x)|^2 dx < \infty. \tag{12}$$

This result is well known in the case of an equation with smooth coefficients and Lyapunov domain, see [3]–[10]. In [1] this result was established for an equation without lower-order terms ($b_i = 0, c = 0$) and under the assumption that the conditions (3) - (5) are satisfied. Moreover, the condition (12) is not only necessary but also sufficient for any solution of the equation (1) to be a solution of the Dirichlet problem with some boundary function u_0 in $L_2(\partial Q)$, see [4], [2].

Proof of the Lemma. We will follow the scheme of the proof of lemma 1 of the article [1].

Let $x^0 \in \partial Q$ be an arbitrary point of the boundary ∂Q of the domain Q and (x', x_n) is a local coordinate system with the origin x^0 and the x_n-axis is directed along the inner normal $\nu(x^0)$ to ∂Q at the point x^0. Since ∂Q is of the class C^1, there exist a positive number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R_{n-1})$ with

$$\varphi_{x^0}(0) = 0, \nabla\varphi_{x^0}(0) = 0 \text{ and } |\nabla\varphi_{x^0}(x')| \leq \frac{1}{2} \text{ for all } x' \in R_{n-1}$$

such that the intersection of the domain Q with the ball $U_{x^0}^{(r_{x^0})} = \{x : |x - x^0| < r_{x^0}\}$ of radius r_{x^0} about x^0 has the form

$$Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n > \varphi_{x^0}(x')\}.$$

Then, of course,

$$\partial Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n = \varphi_{x^0}(x')\}.$$

We assume that r_{x^0} be such that $\partial Q \cap U_{x^0}^{(r_{x^0})}$ belongs to the neighbourhood V_{x^0} in condition (9) (this can be achived by decreasing r_{x^0}). Then

$$\int\limits_{\left\{x' \in R_{n-1} : |x'| < \frac{2}{\sqrt{5}} r_{x^0}\right\}} [u(x', \varphi_{x^0}(x') + \delta) - u_0(x', \varphi_{x^0}(x'))]^2 dx' \to 0 \text{ as } \delta \to +0.$$

Let $\ell_{x^0} = r_{x^0}/\sqrt{2}$; from the covering $\{U_{x^0}^{(\ell_{x^0})}, x^0 \in \partial Q\}$ of the boundary ∂Q select a finite subcovering $U_{x^m}^{(\ell_{x^m})}, m = 1, \dots, p$; following [1], for brevity denote the balls $U_{x^m}^{(r_{x^m})}, m = 1, \dots, p$, by U_m, r_{x^m} by r_m, ℓ_{x^m} by ℓ_m, and φ_{x^m} by φ_m. Set

$$h = \frac{1}{3}\Big(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\Big) \min(1, r_1, \dots, r_p);$$

Then, each of the curvilinear cylinders

$$\Pi_m^{\ell_m + h, h} = \{(x', x_n) : |x'| < \ell_m + h, \varphi_m(x') < x_n < \varphi_m(x') + h\}$$

lies in the corresponding ball U_m, and also in $U_m \cap Q$ (recall that (x', x_n) are here the coordinates of a point in a local system of coordinates with origin at x^m). Let $\ell_0 \in (0, h/4)$ be such that the complement of the domain $Q_{3\ell_0} = \{x \in Q : r(x) = \operatorname{dist}(x, \partial Q) > 3\ell_0\}$ in Q lies in the union of the "cylinders"

$$\Pi_m^{\ell_m, h} = \{(x', x_n) : |x'| < \ell_m, \varphi_m(x') < x_n < \varphi_m(x') + h\}, m = 1, \dots, p;$$
$$Q^{3\ell_0} = \{x \in Q : r(x) = \operatorname{dist}(x, \partial Q) \le 3\ell_0\} \subset \bigcup_{m=1}^{p} \Pi^{\ell_m, h}.$$

Put

$$\Pi_m^h = \Pi_m^{\ell_m + l_0, h} \subset \Pi_m^{\ell_m + h, h} \subset U_m \cap Q,\ Q_m = (Q \setminus Q^{2\ell_0}) \cup \Pi_m^h,$$
$$Q'_m = (Q \setminus Q^{3l_0}) \cup \Pi_m^{\ell_m, h}.$$

It is easily seen that for all $x = (x', x_n) \in \Pi_m^h, m = 1, \dots, p$

$$r(x) \le x_n - \varphi_m(x') \le \frac{\sqrt{5}}{2} r(x) < \frac{4}{3} r(x). \tag{13}$$

We fix an index m, $1 \le m \le p$, and take a local coordinate system with origin at x^m; in the sequel the dependence of the function φ_m on the number m will not be indicated: $\varphi = \varphi_m$.

We define a mapping $\mathcal{L}$ of the space R_n onto itself by the relation $\mathcal{L}(x) = (x', x_n - \varphi(x'))$, where $x = (x', x_n)$; $\mathcal{L}_{-1}(y) = (y', y_n + \varphi(y'))$.

The image of a set under the mapping $\mathcal{L}$ will be denoted by the same letter with $\sim$ on top; in particular $\mathcal{L}(Q) = \tilde{Q}$, $\mathcal{L}(Q_m) = \tilde{Q}_m$, $\mathcal{L}(\Pi_m^h) = \tilde{\Pi}_m^h$, $\mathcal{L}(\Pi_m^{\ell_m,h}) = \tilde{\Pi}_m^{\ell_m,h}$.

Let $u(x)$ be a solution in $W_{2,\text{loc}}^1$ of the problem (1), (2). We take an arbitrary function $\tilde{\eta}$ in $W_2^1(\tilde{Q})$ with support in $\tilde{Q}$. Then, the function $\eta(x) = \tilde{\eta}(x', x_n - \varphi(x'))$, $x = (x', x_n) \in Q$, belongs to $W_2^1(Q)$ and its support is contained in Q.

Denoting $u(y', y_n + \varphi(y'))$ by $\tilde{u}(y)$, $f(y', y_n + \varphi(y'))$ by $\tilde{f}(y)$ and $c(y', y_n + \varphi(y'))$ by $\tilde{c}(y)$, we get from the integral identity (8)

$$\int\limits_{\tilde{Q}} \sum_{i,j=1}^{n} \tilde{a}_{ij}(y) \tilde{u}_{y_i}(y) \tilde{\eta}_{y_j}(y) dy + \int\limits_{\tilde{Q}} \Big(\sum_{i=1}^{n} \tilde{b}_i(y) \tilde{u}_{y_i}(y) + \tilde{c}(y)\tilde{u}(y) \Big) \tilde{\eta}(y) dy =$$

$$\int\limits_{\tilde{Q}} \tilde{f}(y)\tilde{\eta}(y) dy + \int\limits_{\tilde{Q}} \sum_{i=1}^{n} \tilde{f}_i(y) \tilde{\eta}_{y_i}(y) dy, \tag{$\tilde{8}$}$$

where the matrix $\tilde{A}(y) = (\tilde{a}_{ij}(y))$ and the vectors $\tilde{B}(y) = (\tilde{b}_1(y), \dots, \tilde{b}_n(y))$, $\tilde{F}(y) = (\tilde{f}_1(y), \dots, \tilde{f}_n(y))$ have the form:

$$\tilde{a}_{ij}(y) = a_{ij}(y', y_n + \varphi(y')) \text{ for } i < n, j < n,$$

$$\tilde{a}_{ni}(y) = \tilde{a}_{in}(y) = a_{ni}(y', y_n + \varphi(y')) - \sum_{k=1}^{n-1} a_{ki}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k} \text{ for } i < n,$$

$$\tilde{a}_{nn}(y) = \sum_{k,m=1}^{n-1} \frac{\partial \varphi(y')}{\partial y_k} a_{km}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_m}$$

$$- 2 \sum_{k=1}^{n-1} a_{nk}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k} + a_{nn}(y', y_n + \varphi(y')),$$

$$\tilde{b}_i(y) = b_i(y', y_n + \varphi(y')) \text{ for } i < n,$$

$$\tilde{b}_n(y) = b_n(y', y_n + \varphi(y')) - \sum_{k=1}^{n-1} b_k(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k},$$

$$\tilde{f}_i(y) = f_i(y', y_n + \varphi(y')) \text{ for } i < n,$$
$$\tilde{f}_n(y) = f_n(y', y_n + \varphi(y')) - \sum_{k=1}^{n-1} f_k(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k}.$$

This means, that the function $\tilde{u}(y)$ (in $W^1_{2,\text{loc}}(\tilde{Q})$) is a solution of the equation

$$-\operatorname{div}(\tilde{A}(y), \nabla \tilde{u}(y)) + (\tilde{B}(y), \nabla \tilde{u}(y)) + \tilde{c}(y)\tilde{u}(y) = \tilde{f}(y) - \operatorname{div} \tilde{F}(y). \quad (\tilde{1})$$

The matrix $\tilde{A}(y)$ is positive-definite uniformly with respect to $y \in \tilde{Q}$ and the coefficient $\tilde{a}_{nn}(y)$ satisfies the inequalities

$$\gamma_1 \leq \gamma_1(1 + |\nabla \varphi(y')|^2) \leq \tilde{a}_{nn}(y) \leq \gamma_2(1 + |\nabla \varphi(y')|^2) \leq \frac{5}{4}\gamma_2.$$

Denote by $A_0(y) = (a^0_{ij}(y))$ the matrix, the elements of which are defined on $\tilde{\Pi}^h_m$ and have the following form:

$$a^0_{ij}(y) = \tilde{a}_{ij}(y) \text{ for } i < n, j < n,$$
$$a^0_{ni}(y) = a^0_{in}(y) = a^0_{in}(y', y_n)$$
$$= \frac{1}{\operatorname{mes}_{n-1}\{\xi \in R_{n-1} : |\xi| < y_n\}} \int\limits_{\{\xi \in R_{n-1} : |\xi - y'| < y_n\}} \tilde{a}_{in}(\xi, 0) d\xi \text{ for } i < n,$$
$$a^0_{nn}(y) = \tilde{a}_{nn}(y', 0).$$

It was established in [1] that in $\tilde{\Pi}^h_m$

$$\left[\sum_{i=1}^{n} |a^0_{in}(y) - \tilde{a}_{in}(y)|^2\right]^{\frac{1}{2}} \leq \tilde{w}(y_n), \quad (14)$$

and

$$\left|\frac{\partial a^0_{in}(y)}{\partial y_i}\right| \leq \frac{\tilde{w}(y_n)}{y_n}, i = 1, \ldots, n-1, \quad (15)$$

where $\tilde{w}(t) = Cw(2\sqrt{2}t)$ ($w(t)$ comes from the conditions (4) and (5)); the constant C depends only on n and γ_2.

Let $\delta_0 < \frac{\ell_0}{2}$ be a fixed positive number; in the sequel the dependence on the chosen and fixed numbers p, r_m, ℓ_m, $m = 1, \ldots, p$, ℓ_0, n, γ_1, γ_2, δ_0 will not be indicated in the notation.

For an arbitrary $\delta \in (0, \delta_0)$ we define the function $\varrho_\delta(y)$ on the domain $\tilde{Q}_m$ by

$$\varrho_\delta(y) = \begin{cases} 0 & \text{for } |y'| < \ell_m + \ell_0, 0 < y_n < \delta, \\ y_n - \delta & \text{for } |y'| < \ell_m + \ell_0, \delta \le y_n \le 4\delta_0, \\ 4\delta_0 - \delta & \text{for the remaining points } y \text{ in } \tilde{Q}_m. \end{cases}$$

The function ϱ_δ satisfies the inequalities

$$r_\delta(x) \le \varrho_\delta(\mathcal{L}(x)) \le \frac{4}{3} r_{\frac{3}{4}\delta}(x) \text{ for all } x \in Q_m, \tag{16}$$

where $r_\delta(x) = \min\{3\delta_0, \max\{0, r(x) - \delta\}\}$, see [1]. Moreover $\||\nabla \varrho_\delta|\|_{L_\infty(\tilde{Q}_m)} \le 1$. We fix a function $\psi \in C^1(\overline{Q})$ such that $\psi(x) = 1$ for $x \in Q'_m, \psi = 0$ for $x \in Q^{\frac{5}{2}\ell_0} \setminus \Pi_m^{\ell_m + \frac{1}{2}\ell_0, h}$, and $0 \le \psi(x) \le 1$ for all $x \in Q$; it will also be assumed that for $|y'| < \ell_m + \ell_0$ and $0 < y_n < 2\ell_0$ the function $\tilde{\psi}(y) = \psi(\mathcal{L}_{-1}(y))$ does not depend on y_n.

Taking in the integral identity $(\tilde{8})$ the function $\tilde{\eta}(y)$ as $\varrho_\delta(y)\tilde{\psi}(y)\tilde{u}(y)$ we get

$$\begin{aligned} &\int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{\psi} (\nabla \tilde{u}, \tilde{A} \nabla \tilde{u}) dy + \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{u} (\nabla \tilde{\psi}, \tilde{A} \nabla \tilde{u}) dy \\ &\quad + \int\limits_{\tilde{Q}_m} \tilde{\psi} \tilde{u} (\nabla \varrho_\delta, \tilde{A} \nabla \tilde{u}) dy + \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{\psi} \tilde{u} (\tilde{B}, \nabla \tilde{u}) dy + \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{\psi} \tilde{c} \tilde{u}^2 dy \\ &\quad = \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{\psi} \tilde{u} \tilde{f} dy + \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{\psi} (\tilde{F}, \nabla \tilde{u}) dy \\ &\quad + \int\limits_{\tilde{Q}_m} \varrho_\delta \tilde{u} (\tilde{F}, \nabla \tilde{\psi}) dy + \int\limits_{\tilde{Q}_m} \tilde{\psi} \tilde{u} (\tilde{F}, \nabla \varrho_\delta) dy \end{aligned} \tag{17}$$

In view of (13)

$$\begin{aligned} \tilde{I}_1^{(m)}(\delta) &= \int\limits_{\tilde{Q}_m} \varrho_\delta(y) \tilde{\psi}(y) (\nabla \tilde{u}(y), \tilde{A}(y) \nabla \tilde{u}(y)) dy \\ &\ge \int\limits_{Q'_m} r_\delta(x) (\nabla u(x), A(x) \nabla u(x)) dx \ge \gamma_1 \int\limits_{Q'_m} r_\delta(x) |\nabla u(x)|^2 dx. \end{aligned}$$

we are going to obtain upper estimates for the remaining terms of equality (17).

The estimation of the integral

$$\tilde{I}_2^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \varrho_\delta(y)\tilde{u}(y)(\nabla\tilde{\psi}(y), \tilde{A}(y)\nabla\tilde{u}(y))dy.$$

Again in view of (13)

$$\begin{aligned}
|\tilde{I}_2^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)|u(x)||(\nabla\psi(x), A(x)\nabla u(x))|dx \\
&\le \frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2 \int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)|u(x)||\nabla u(x)|dx \\
&\le \frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2 \Big\{\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)u^2(x)dx\Big\}^{\frac{1}{2}} \cdot \\
&\qquad \cdot \Big\{\int\limits_{\left(Q\setminus Q^{\frac{5\ell_0}{2}}\right)\cup\Pi_m^{\ell_m+\frac{\ell_0}{2},h}} r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx\Big\}^{\frac{1}{2}} \\
&\le \frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2 \Big\{\int\limits_{Q} r(x)u^2(x)dx\Big\}^{\frac{1}{2}} \cdot \\
&\qquad \cdot \Big\{\frac{\delta}{2}\int\limits_{\left(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}}\right)\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2dx + 2\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2dx\Big\}^{\frac{1}{2}} \\
&\le \epsilon\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2dx + \frac{\epsilon\delta}{4}\int\limits_{\left(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}}\right)\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2dx \\
&\qquad + \frac{C_2'}{\epsilon}\int\limits_{Q} r(x)u^2(x)dx,
\end{aligned}$$

where $0 < \epsilon < 1$ is to be chosen later.

Since the estimate is valid for solutions of the elliptic equation (1) (see [11])

$$\int\limits_{G'} |\nabla u(x)|^2dx \le C_0(\gamma_1,\gamma_2)\left[\left(\frac{1}{\sigma^2} + \frac{\|B\|_{L_\infty(G)}}{\sigma} + \|B\|^2_{L_\infty(G)}\right)\int\limits_{G} u^2(x)dx\right.$$

$$+\sigma^2 \int\limits_G f^2(x)dx + \int\limits_G |F(x)|^2 dx + \int\limits_G |c(x)|u^2(x)dx\Bigg] \quad (18)$$

where $G' \subset G$ and $\sigma = \operatorname{dist}(G', \partial G)$, then in view of (10) and (11) it follows that

$$\delta \int\limits_{\left(\Pi_m^{\ell_m+\frac{\ell_0}{2},h} \cap Q^{\frac{5\delta}{4}}\right)\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2 dx$$

$$\le C_0\delta\Bigg[\left(\frac{16}{\delta^2} + \frac{4}{\delta}\|B\|_{L_\infty\left(Q_{\frac{\delta}{2}}\setminus \overline{Q}_{\frac{3}{2}\delta}\right)} + \|B\|^2_{L_\infty\left(Q_{\frac{\delta}{2}}\setminus \overline{Q}_{\frac{3}{2}\delta}\right)}\right)\cdot$$

$$\cdot \int\limits_{\left(\Pi_m^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} u^2(x)dx + \frac{\delta^2}{16} \int\limits_{\left(\Pi_m^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} f^2(x)dx$$

$$+ \int\limits_{\left(\Pi_m^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} |F(x)|^2 dx + \int\limits_{\left(\Pi_m^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} |c(x)|u^2(x)dx\Bigg]$$

$$\le C_2''\Bigg[\left(1 + \frac{1}{(1+|\ln\delta|)^{\frac{3}{4}}} + \frac{1}{(1+|\ln\delta|)^{\frac{3}{2}}} + \delta\int\limits_{\frac{3\delta}{8}}^{\frac{3\delta}{2}} C(t)dt\right)\cdot$$

$$\cdot \max_{\frac{\delta}{2}\le y_n\le 2\delta} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{u}^2(y', y_n)dy'$$

$$+ \frac{1}{(1+|\ln\frac{3}{2}\delta|)^{\frac{3}{2}}} \int\limits_{\left(\Pi_m^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} r^3(x)(1+|\ln r(x)|)^{\frac{3}{2}} f^2(x)dx$$

$$+ \frac{1}{(1+|\ln\frac{3}{2}\delta|)^{\frac{3}{2}}} \int\limits_{\left(\Pi^{\ell_m+\ell_0,h} \cap Q^{\frac{3\delta}{2}}\right)\setminus Q^{\frac{\delta}{2}}} r(x)(1+|\ln r(x)|)^{\frac{3}{2}} |F(x)|^2 dx\Bigg].$$

We introduce the notation

$$M = \max_{0\le y_n\le h} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{u}^2(y', y_n)dy',$$

$$\|f\|^2 = \int\limits_Q r^3(x)(1+|\ell n r(x)|)^{\frac{3}{2}} f^2(x)dx,$$

$$\|F\|^2 = \int\limits_Q r(x)(1+|\ell n r(x)|)^{\frac{3}{2}}|F(x)|^2 dx.$$

Since by (11) $\delta\int_{\frac{3}{8}\delta}^{\frac{3}{2}\delta} C(t)dt \le \frac{C'''}{(1+|ln\delta|)^{\frac{3}{4}}}$, then we have

$$\delta \int\limits_{\left(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}}\right)\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2 dx \le \tilde{C}_0[M+\|f\|^2+\|F\|^2] \tag{19}$$

Consequently, the estimate is valid

$$|\tilde{I}_2^{(m)}(\delta)| \le \epsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2 dx + I_2^{(m)}(\epsilon),$$

where $I_2^{(m)}(\epsilon) = C_2(\frac{1}{\epsilon}\int_Q r(x)u^2(x)dx + \epsilon[M+\|f\|^2+\|F\|^2])$.

The estimation of the integral

$$\tilde{I}_3^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \tilde{\psi}(y)\tilde{u}(y)(\nabla\varrho_\delta(y), \tilde{A}(y)\nabla\tilde{u}(y))dy.$$

$$\begin{aligned}
\tilde{I}_3^{(m)}(\delta) = & \int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')\tilde{u}(y',y_n) \\
& \qquad (\nabla\varrho_\delta(y_n), (\tilde{A}(y',y_n)-A_0(y',y_n))\nabla\tilde{u}(y',y_n))dy'dy_n \\
& - \frac{1}{2}\int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')\tilde{u}^2(y',y_n)\sum_{i=1}^{n-1}\frac{\partial a_{in}^0(y',y_n)}{\partial y_i}dy'dy_n \\
& - \frac{1}{2}\int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{u}^2(y',y_n)\sum_{i=1}^{n-1}a_{in}^0(y',y_n)\frac{\partial\tilde{\psi}(y')}{\partial y_i}dy'dy_n \\
& + \frac{1}{2}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{a}_{nn}(y',0)\tilde{\psi}(y')\tilde{u}^2(y',4\delta_0)dy' \\
& - \frac{1}{2}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{a}_{nn}(y',0)\tilde{\psi}(y')\tilde{u}^2(y',\delta)dy' \\
& = \tilde{I}_{31}^{(m)}(\delta)+\tilde{I}_{32}^{(m)}(\delta)+\tilde{I}_{33}^{(m)}(\delta)+\tilde{I}_{34}^{(m)}(\delta_0)+\tilde{I}_{35}^{(m)}(\delta).
\end{aligned}$$

In view of (13) and (14)

$$|\tilde{I}_{31}^{(m)}(\delta)| \le \int\limits_{\delta}^{4\delta_0} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|\tilde{\omega}(y_n)dy'dy_n \le I_{31}^{(m)'}(\delta)+$$

$$\tilde{\omega}(4\delta_0)\int\limits_{\delta_0}^{4\delta_0} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|dy'dy_n,$$

where

$$\begin{aligned} I_{31}^{(m)'}(\delta) &= \Bigg(\int\limits_{\delta}^{\delta_0} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')y_n|\nabla\tilde{u}(y)|^2dy'dy_n\Bigg)^{\frac{1}{2}} \Bigg(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\Bigg)^{\frac{1}{2}} \\ &\le \Bigg(\frac{\sqrt{5}}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q_{\frac{2\delta}{\sqrt{5}}}} r(x)|\nabla u(x)|^2dx\Bigg)^{\frac{1}{2}} \Bigg(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\Bigg)^{\frac{1}{2}} \\ &\le \Bigg(4\sqrt{5}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}} r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx\Bigg)^{\frac{1}{2}} \Bigg(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\Bigg)^{\frac{1}{2}} \\ &\le \frac{\epsilon}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}} r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx + \frac{8\sqrt{5}}{\epsilon}M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n. \end{aligned}$$

Next, in view of (19)

$$\begin{aligned} &\frac{\epsilon}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}} r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx \\ &\quad\le \frac{\epsilon}{2}\Bigg[\frac{\delta}{2}\int\limits_{\left(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}}\right)\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2dx + 2\int\limits_Q r_\delta(x)|\nabla u(x)|^2dx\Bigg] \\ &\qquad\le \epsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2dx + \frac{\epsilon}{4}\tilde{C}_0[M+\|f\|^2+\|F\|^2]. \end{aligned}$$

Thus
$$|\tilde{I}_{31}^{(m)}(\delta)| \le \epsilon \int\limits_Q r_\delta(x)|\nabla u(x)|^2 dx + I_{31}^{(m)}(\delta_0,\epsilon),$$

where

$$I_{31}^{(m)}(\delta_0,\epsilon) = \frac{\epsilon}{4}\tilde{C}_0[M + \|f\|^2 + \|F\|^2] + \frac{8\sqrt{5}}{\epsilon} M \int\limits_0^{\delta_0} \frac{\tilde{\omega}^2(y_n)}{y_n} dy_n +$$
$$\tilde{\omega}(4\delta_0)\int\limits_{\delta_0}^{4\delta_0} \int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|dy'dy_n.$$

In view of (15)

$$|\tilde{I}_{32}^{(m)}(\delta)| \le \frac{n-1}{2}\int\limits_0^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{\psi}(y')\tilde{u}^2(y',y_n)\frac{\tilde{\omega}(y_n)}{y_n}dy'dy_n$$
$$\le M\frac{n-1}{2}\int\limits_0^{4\delta_0}\frac{\tilde{\omega}(y_n)}{y_n}dy_n = I_{32}^{(m)}(\delta_0)$$
$$|\tilde{I}_{33}^{(m)}(\delta)| \le \frac{1}{2}\int\limits_0^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0} \tilde{u}^2(y',y_n)\Big|\sum_{i=1}^{n-1} a_{in}^0(y',y_n)\frac{\partial\tilde{\psi}(y')}{\partial y_i}\Big|dy'dy_n = I_{33}^{(m)}(\delta_0)$$
$$|\tilde{I}_{35}^{(m)}(\delta)| \le \frac{5}{8}\gamma_2 M = I_{35}^{(m)}.$$

Thus, we get
$$|\tilde{I}_3^{(m)}(\delta)| \le \epsilon \int\limits_Q r_\delta(x)|\nabla u(x)|^2 dx + I_3^{(m)}(\epsilon),$$

where

$$I_3^{(m)}(\epsilon) = I_{31}^{(m)}(\delta_0,\epsilon) + I_{32}^{(m)}(\delta_0) + I_{33}^{(m)}(\delta_0) + \tilde{I}_{34}^{(m)}(\delta_0) + I_{35}^{(m)}.$$

The estimation of the integral

$$\tilde{I}_4^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \varrho_\delta(y)\tilde{\psi}(y)\tilde{u}(y)(\tilde{B}(y),\nabla\tilde{u}(y))dy.$$

In view of (16)

$$
\begin{aligned}
|\tilde{I}_4^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)\psi(x)|u(x)||B(x)||\nabla u(x)|dx \\
&\le \Bigg(2\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}} r(x)\psi(x)u^2(x)|B(x)|^2dx\Bigg)^{\frac{1}{2}}\Bigg(\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}} \psi(x)r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx\Bigg)^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2}\int\limits_{\left(Q\setminus Q^{\frac{5\ell_0}{2}}\right)\cup\Pi_m^{\ell_m+\frac{\ell_0}{2},h}} r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2dx + \frac{1}{\epsilon}\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}}\frac{K^2u^2(x)}{r(x)(1+|\ln r(x)|)^{\frac{3}{2}}}dx \\
&\le \epsilon\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2dx + \frac{\epsilon}{4}\tilde{C}_0[M+\|f\|^2+\|F\|^2]+ \\
&\quad \frac{K^2}{\epsilon}\Bigg(\int\limits_{Q\setminus Q^{2\ell_0}}\frac{u^2(x)}{r(x)(1+|\ln r(x)|)^{\frac{3}{2}}}dx + \int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q_{\frac{3\delta}{4}}}\frac{u^2(x)}{r(x)(1+|\ln r(x)|)^{\frac{3}{2}}}dx\Bigg) \\
&\le \epsilon\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2dx + \frac{\epsilon}{4}\tilde{C}_0[M+\|f\|^2+\|F\|^2] + \frac{K^2}{2\epsilon\ell_0}\|u\|^2_{L_2(Q)}+ \\
&\quad \frac{\sqrt{5}}{2}\frac{K^2}{\epsilon}\int\limits_{\frac{3\delta}{4}}^{h}\int\limits_{|y'|<\ell_m+\ell_0}\frac{\tilde{u}^2(y)}{y_n(1+|\ln y_n|)^{\frac{3}{2}}}dy'dy_n \\
&\le \epsilon\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2dx + I_4^{(m)}(\epsilon),
\end{aligned}
$$

where

$$
I_4^{(m)}(\epsilon) = \frac{\epsilon}{4}\tilde{C}_0[M+\|f\|^2+\|F\|^2] + \frac{K^2}{2\epsilon\ell_0}\|u\|^2_{L_2(Q)} + \frac{\sqrt{5}}{2}\frac{K^2}{\epsilon}M\int\limits_0^h\frac{dy_n}{y_n(1+|\ell n y_n|)^{\frac{3}{2}}}.
$$

The estimation of the integral

$$
\tilde{I}_5^{(m)}(\delta) = \int\limits_{\tilde{Q}_m}\varrho_\delta(y)\tilde{\psi}(y)\tilde{c}(y)\tilde{u}^2(y)dy.
$$

In view of (11)

$$
\begin{aligned}
|\tilde{I}_5^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)\psi(x)|c(x)|u^2(x)dx \\
&\le \frac{4}{3}\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}} r(x)C(r(x))\psi(x)u^2(x)dx \\
&\le \frac{4}{3}\Bigg(\int\limits_{Q\setminus Q^{2\ell_0}} r(x)C(2\ell_0)u^2(x)dx + \int\limits_{\Pi_m^{\ell_m+\ell_0,h}\cap Q_{\frac{3\delta}{4}}} r(x)C(r(x))u^2(x)dx\Bigg) \\
&\le \frac{4}{3}\Bigg(C(2\ell_0)\int\limits_{Q} r(x)u^2(x)dx + \int\limits_{\frac{3\delta}{4}}^{h}\int\limits_{|y'|<\ell_m+\ell_0} y_n C\Big(\frac{2}{\sqrt{5}}y_n\Big)\tilde{u}^2(y',y_n)dy'dy_n\Bigg) \\
&\le C_5\Bigg(\int\limits_{Q} r(x)u^2(x)dx + M\int\limits_0^h y_n C\Big(\frac{2}{\sqrt{5}}y_n\Big)dy_n\Bigg) = I_5^{(m)}.
\end{aligned}
$$

The estimation of the integral

$$
\tilde{I}_6^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \varrho_\delta(y)\tilde{\psi}(y)\tilde{u}(y)\tilde{f}(y)dy.
$$

$$
\begin{aligned}
|\tilde{I}_6^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r(x)u(x)f(x)dx \\
&\le C_6\Bigg(\|u\|^2_{L_2(Q)} + \|f\|^2 + M\int\limits_0^h \frac{dy_n}{y_n(1+|\ln y_n|)^{\frac{3}{2}}}\Bigg) = I_6^{(m)}
\end{aligned}
$$

The estimation of the integral

$$
\tilde{I}_7^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \varrho_\delta(y)\tilde{\psi}(y)(\tilde{F}(y),\nabla\tilde{u}(y))dy.
$$

Analogously to the estimations of $\tilde{I}_2^{(m)}(\delta)$ and $\tilde{I}_4^{(m)}(\delta)$

$$
\begin{aligned}
|\tilde{I}_7^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)|F(x)||\nabla u(x)|dx \\
&\le \frac{\epsilon}{2}\int\limits_{Q_m} \psi(x) r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2 dx + \frac{16}{9\epsilon}\|F\|^2 \\
&\le \epsilon \int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx + \frac{\epsilon}{4}\tilde{C}_0[M + \|f\|^2 + \|F\|^2] + \frac{16}{9\epsilon}\|F\|^2 \\
&= \epsilon \int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx + I_7^{(m)}(\epsilon).
\end{aligned}
$$

The estimation of the integral

$$
\tilde{I}_8^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \varrho_\delta(y)\tilde{u}(y)(\tilde{F}(y), \nabla\tilde{\psi}(y))dy.
$$

$$
\begin{aligned}
|\tilde{I}_8^{(m)}(\delta)| &\le \frac{4}{3}\int\limits_{Q_m} r_{\frac{3}{4}\delta}(x)|u(x)||F(x)||\nabla\psi(x)|dx \\
&\le \frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\left(\int\limits_{Q} r(x)u^2(x)dx + \|F\|^2\right) = I_8^{(m)}.
\end{aligned}
$$

And finally, the estimation of the integral

$$
\tilde{I}_9^{(m)}(\delta) = \int\limits_{\tilde{Q}_m} \tilde{\psi}(y)\tilde{u}(y)(\tilde{F}(y), \nabla\varrho_\delta(y))dy.
$$

$$
\begin{aligned}
|\tilde{I}_9^{(m)}(\delta)| &\le \int\limits_{Q_m} |u(x)||F(x)|dx \\
&\le \int\limits_{Q_m} \frac{u^2(x)dx}{r(x)(1+|\ln r(x)|)^{\frac{3}{2}}} + \|F\|^2 \\
&\le \frac{1}{2\ell_0}\int\limits_{Q} u^2(x)dx + \frac{\sqrt{5}}{2}M\int\limits_0^h \frac{dy_n}{y_n(1+|\ln y_n|)^{\frac{3}{2}}} + \|F\|^2 = I_9^{(m)}.
\end{aligned}
$$

Substituting the above obtained estimates in the equality (17) we get

$$\gamma_1 \int\limits_{Q'_m} r_\delta(x)|\nabla u(x)|^2 dx \le \tilde{I}_1^{(m)}(\delta) \le \sum_{k=2}^{9} |\tilde{I}_k^{(m)}(\delta)|$$
$$\le 4\epsilon \int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx + I^{(m)}(\epsilon),$$

where $I^{(m)}(\epsilon) = \sum_{k=2}^{9} I_k^{(m)}$.

Next, summing over all m with $1 \le m \le p$ we get

$$\gamma_1 \int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx \le \gamma_1 \sum_{m=1}^{p} \int\limits_{Q'_m} r_\delta(x)|\nabla u(x)|^2 dx$$
$$\le 4\epsilon p \int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx + \sum_{m=1}^{p} I^{(m)}(\epsilon).$$

Choosing $\epsilon < \frac{\gamma_1}{8p}$ we get

$$\int\limits_{Q} r_\delta(x)|\nabla u(x)|^2 dx \le \frac{2}{\gamma_1} \sum_{m=1}^{p} I^{(m)}(\epsilon). \tag{20}$$

Since the right-hand side of the last inequality (20) does not depend on δ, it obviously follows that the function $r(x)|\nabla u(x)|^2$ is integrable over Q. The lemma is proved. $\boxed{QED}$

Proof of the Theorem. Let $u(x)$ be a solution in $W^1_{2,\mathrm{loc}}$ of the problem (1), (2). Then, by lemma, the integral (10) is bounded. On the other hand, it is clear, that the function $u(x)$ will be also a solution in $W^1_{2,\mathrm{loc}}$ of the Dirichlet problem:

$$-\operatorname{div}(A(x), \nabla v(x)) = f(x) - (B(x), \nabla u(x)) - c(x)u(x) - \operatorname{div} F(x),$$
$$v|_{\partial Q} = u_0, \tag{1'}$$

Therefore, as follows from the results of the article [2], for obtaining $(n-1)$-dimentional continuity (*i.e.* the belonging to $C_{n-1}(Q)$) of the solution $v(x) = u(x)$ of the problem $(1')$, it is sufficient to show that the function $g(x) = f(x) - (B(x), \nabla u(x)) - c(x)u(x)$ satisfies an analogous condition to (7), that is

$$r^{\frac{3}{2}}(x)(1 + |\ln r(x)|)^{\frac{3}{4}} g(x) \in L_2(Q) \quad (7')$$

(the function $F(x)$ satisfies the conditon (6)).

In view of the lemma and conditions (7), (10) it immediately follows that

$$r^{\frac{3}{2}}(x)(1+|\ln r(x)|)^{\frac{3}{4}}(f(x)-(B(x),\nabla u(x)))\in L_2(Q).$$

In view of (11)

$$\begin{aligned}
&\int\limits_Q r^3(x)(1+|\ln r(x)|)^{\frac{3}{2}}c^2(x)u^2(x)dx\\
&\quad\le C^2(2\ell_0)\int\limits_{Q_{2\ell_0}} r^3(x)(1+|\ln r(x)|)^{\frac{3}{2}}u^2(x)dx\\
&\qquad\qquad+\sum_{m=1}^{p}\int\limits_{\Pi_m^{\ell_m,h}} r^3(x)(1+|\ln r(x)|)^{\frac{3}{2}}C^2(r(x))u^2(x)dx\\
&\quad\le C'\|u\|^2_{L_2(Q)}+\sum_{m=1}^{p}\int\limits_{\tilde\Pi_m^{\ell_m,h}} y_n^3(1+|\ln\frac{2}{\sqrt5}y_n|)^{\frac{3}{2}}C^2(\frac{2}{\sqrt5}y_n)\tilde u^2(y',y_n)dy'dy_n\\
&\quad<\infty.
\end{aligned}$$

Thus, the function $g(x)$ satisfies the condition (7′) and consequently $u\in C_{n-1}(\overline{Q})$.

The theorem is proved. $\boxed{QED}$

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