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Beutelspacher's parallelism construction

Norman L. Johnson

University of Iowa
Iowa City, Iowa 52242
njohnson@math.uiowa.edu

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Abstract. Beutelspacher's construction of line parallelisms in $PG(2^a - 1, q)$ is generalized to line parallelisms of $PG(2^a - 1, K)$, where K is an arbitrary skewfield admitting a suitable sequence of quadratic skewfield extensions.

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Introduction

Given a projective space Σ, a 'line spread' is a disjoint cover of the points of Σ by a set of mutually skew lines (the spread). A 'parallelism' (or 'line parallelism') is a disjoint cover of the lines of Σ by a set of mutually line disjoint line spreads. In 1974, Buetelspacher [1] gave a construction of parallelisms in $PG(2^r - 1, q)$ for any positive integer r and for any prime power $q = p^t > 2$.

In this article, we generalize Buetelspacher's construction to arbitrary projective spaces over skewfields and construct a variety of new parallelisms. The main difference is where Buetelspacher would use a 'geometric' spread, a so-called 'Desarguesian spread' is used in its place. Hence, it is possible to construct a variety of line parallelisms over infinite skewfields or infinite fields by this method. Basically the only required criterion is to be able to construct parallelisms in three-dimensional projective space that contain at least one Desarguesian spread.

Regarding infinite parallelisms, it might be pointed out that Buetelspacher [2] has also used transfinite induction to construct parallelisms in finite dimensional projective spaces over infinite skewfields. In the cases considered in this article, the constructed infinite parallelisms are somewhat more accessable in that potentially their collineation groups and isomorphism classses may be determined.

1 Preliminaries

We first somewhat generalize the notion of a geometric spread as used by Buetelspacher as follows. It will be assumed that all vector spaces are 'left' vector spaces and skewfield extensions are always considered to be on the left.

1 Definition. Let K_1 be a skewfield and let V be any K_1-vector space. Denote by $PG(V-1,K_1)$ the associated K_1-projective space. We define a 'Desarguesian Sperner space' to be a translation Sperner space given by a spread of lines of $PG(V-1,K_1)$ all of whose associated 2-dimensional vector subspaces over K_1 are 1-dimensional vector subspaces over some skewfield extension K_2 of dimension 2 over K_1. Also, the associated line spread is called a 'Desarguesian line spread.'

2 Proposition. *Let $K_1 \subseteq K_2$ be skewfields, where K_2 is 2-dimensional over K_1 and let V be a K_2-vector space. Let $\mathcal{R}$ denote the set of all 1-dimensional K_2-subspaces of V. Then, $\mathcal{R}$ is a Desarguesian line spread of $PG(V-1,K_1)$.*

Proof. Any non-zero vector in V is contained in a unique 1-dimensional K_2-subspace so any 1-dimensional K_1-subspace is contained in a unique 1-dimensional K_2-subspace. A point P of $PG(V-1,K_1)$ is a 1-dimensional K_1-subspace, generated say by e. So, P is the set $\{\alpha e;\ \alpha \in K_1\}$. Simply take e generated over K_2 (the tensor product with K_2) to construct the unique 1-dimensional K_2 subspace containing P. $\boxed{QED}$

3 Notation. In the following, we shall adopt the following notation: Let $\mathcal{R}$ denote a Desarguesian line spread in $PG(V-1,K_1)$. If g and h are distinct elements of $\mathcal{R}$, let $\langle g,h\rangle$ denote the 2-dimensional K_2-vector subspace generated by g and h, which is also considered a 4-dimensional K_1-vector subspace and projectively as isomorphic to $PG(3,K_1)$.

If L is a subskewfield containing K_1 and $z_1,z_2,\ldots,z_s \in V$, the L-vector subspace generated by $\{z_i;\ i=1,\ldots,s\}$ shall be denoted by $\langle z_i;\ i=1,\ldots,n\rangle_L$.

The following propositions are analogous to similar ones in Buetelspacher [1].

4 Proposition. *$\mathcal{R}$ as a set of 'points' and $\{\langle g,h\rangle_{K_2};g,\neq h\} \in \mathcal{R}$ as 'lines' is isomorphic to $PG(V-1,K_2)$.*

Let L be a line of $PG(V-1,K_1)-\mathcal{R}$. If L nontrivially intersects $\langle g,h\rangle$ for $g,h \in \mathcal{R}$, then $L \subseteq \langle g,h\rangle_{K_2}$ and there exists a unique $\langle g^*,h^*\rangle_{K_2}$, for $g^*,\neq h^*$ in $\mathcal{R}$ containing L.

Proof. Let $PQ = L$, where P and Q are points of $PG(V-1,K_1)$. Since $\mathcal{R}$ is a spread, there exist unique lines g and h containing P and Q respectively and $g \neq h$ since L is not in $\mathcal{R}$. Let $g = \langle e_g\rangle_{K_2}$ and $h = \langle e_h\rangle_{K_2}$, 1-dimensional K_2-subspaces. P in g means that $P = \langle ae_g\rangle_{K_1}$ where $a \in K_2$. Similarly, $Q =$

$\langle be_h\rangle_{K_1}$, where $b \in K_2$. Then L, as a 2-dimensional K_1-subspace, is $\langle ae_g, be_h\rangle_{K_1}$, which clearly is in $\langle e_g, e_h\rangle_{K_2} = \langle g, h\rangle$.

Let $L = P^*Q^* = PQ$, let $g^* = \langle e_{g^*}\rangle_{K_2}$, $h^* = \langle e_{h^*}\rangle_{K_2}$ contain P^* and Q^* respectively. Then $L = \langle a^*e_{g^*}, b^*e_{h^*}\rangle_{K_1}$, for some a^*, b^* in K_2. Hence,

$$\langle a^*e_{g^*}, b^*e_{h^*}\rangle_{K_1} = \langle ae_g, be_h\rangle_{K_1}.$$

This implies that $a^*e_g^*$ is in $\langle ae_g, be_h\rangle_{K_1}$, implying that e_{g^*} is in $\langle e_g, e_h\rangle_{K_2}$. By symmetry, $\langle g, h\rangle_{K_2} = \langle e_g, e_h\rangle_{K_2} = \langle e_{g^*}, e_{h^*}\rangle_{K_2} = \langle g^*, h^*\rangle_{K_2}$. $\boxed{QED}$

5 Proposition. *$\mathcal{R} \mid \langle g, h\rangle_{K_1}$ is a line spread of $\langle g, h\rangle_{K_1}$ as a projective space isomorphic to $PG(3, K_1)$, for each 'line' $\langle g, h\rangle_{K_2}$ of $\mathcal{R}$.*

Proof. When an element M of $\mathcal{R}$ nontrivially intersects $\langle g, h\rangle_{K_2}$, it is contained in $\langle g, h\rangle_{K_2}$ since M is a 1-dimensional K_2-subspace and $\langle g, h\rangle_{K_2}$ is a 2-dimensional K_2-subspace as well as a 4-dimensional K_1-subspace. Since $\mathcal{R}$ is a spread, each point of $\langle g, h\rangle_{K_2}$ is covered uniquely by some element N of $\mathcal{R}$. $\boxed{QED}$

2 Extension of Beutelspacher's Theorem

Since we are dealing with arbitrary skewfields, we begin with some elementary properties of spreads and parallelisms.

6 Proposition. *Let W be any vector space of finite dimension $d > 3$ over a skewfield L, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be line spreads of $PG(W-1, L)$, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be parallelisms of $PG(W-1, L)$. Then $\operatorname{card}\mathcal{S}_1 = \operatorname{card}\mathcal{S}_2$ and $\operatorname{card}\mathcal{P}_1 = \operatorname{card}\mathcal{P}_2$.*

Proof. Since the proposition is trivial when L is finite, assume that L is infinite.

Note that a line spread of $PG(3, L)$ corresponds to a translation plane and the spread is in bijective correspondence with the line at infinity of the associated translation plane. But, when L is infinite, a translation plane with spread in $PG(3, L)$ has cardinality $\operatorname{card} L$ since this is the cardinality of a set of points on a projective line. Let $\{e_i;\ i = 1, \ldots, d\}$ be a basis for W. Then, $\langle \alpha e_1 + e_2\rangle_L$ for fixed non-zero α determines a set of mutually distinct 1-dimensional L-subspaces of cardinality $\operatorname{card} L$. The number of non-zero vectors is $(\operatorname{card} L)^d = \operatorname{card} L$, implying that $\operatorname{card}\mathcal{S}_1 \leq \operatorname{card} L$. Clearly, there are also $\operatorname{card} L$ 2-dimensional L-subspaces and not all can belong to a line spread in the projective space. Given any 2-dimensional L-subspace Z that does not belong to $\mathcal{S}_1$, there are $\operatorname{card} L$ lines of $\mathcal{S}_1$ required to cover Z. Hence, $\operatorname{card} L \leq \operatorname{card}\mathcal{S}_1 \leq \operatorname{card} L$.

In order to cover the set of 2-dimensional subspaces with line disjoint spreads, a given point P is clearly in $\operatorname{card} L$ 2-dimensional L-subspaces, implying that at least $\operatorname{card} L$ spreads. It then similarly follows that $\operatorname{card}\mathcal{P}_1 = \operatorname{card} L$. $\boxed{QED}$

7 Theorem. *Let $K_1 \subseteq K_2$ be skewfields, where K_2 is 2-dimensional over K_1 and let V be a K_2-vector space. Let $\mathcal{R}$ denote the set of all 1-dimensional K_2-subspaces; an associated Desarguesian spread of $PG(V-1,K_1)$. Let $\mathcal{P}$ be a line parallelism of $PG(V-1,K_2)$ and for each $PG(3,K_1)$, $\langle g,h\rangle$, $g \neq h$ of $\mathcal{R}$, let $\mathcal{M}_{\langle g,h\rangle}$ denote a parallelism of $\langle g,h\rangle$ containing $\mathcal{R} \mid \langle g,h\rangle$.*

More generally, index the spreads of the parallelism $\mathcal{P}$ by Ω, so that

$$\mathcal{P} = \cup\{\mathcal{M}^s \ ; \ s \in \Omega\}.$$

Let Λ be a set of cardinality the cardinality minus 1 of any parallelism of line spreads of any $PG(3,K_1)$ and let Δ be a set of cardinality the cardinality of lines of any spread of lines of $PG(V-1,K_2)$.

For each spread $\mathcal{M}^s$ of $\mathcal{P}$ in $PG(V-1,K_2)$; (the points are those of $\mathcal{R}$), for $s \in \Omega$, index the spread lines by $m^{s,k}$ by $k \in \Delta$.

Assume that each $m^{s,k}$ as a projective space isomorphic to $PG(3,K_1)$ admits a parallelism containing the parallelism induced by $\mathcal{R}$ and let $\mathcal{P}^{m^{s,k}}$ denote the set of spreads not equal to the induced spread. Denote by $S_\Lambda^{\mathcal{P}^{m^{s,k}}}$ the symmetric group on $\mathcal{P}^{m^{s,k}}$ permuting the spreads, a set of cardinality Λ and let $m_t^{s,k}$, for $t \in \Lambda$ denote the spreads of $\mathcal{P}^{m^{s,k}}$.

For $i \in \Lambda$, denote by $\lambda^{\mathcal{P}^{m^{s,k}}}(i)$ the action of the associated permutation on the element i. Hence, we are considering a family of groups isomorphic to S_Λ and acting on Λ considered as $\mathcal{P}^{m^{s,k}}$.

(1) For each $m^{s,k}$ choose a permutation $\lambda^{\mathcal{P}^{m^{s,k}}} \in S_\Lambda^{\mathcal{P}^{m^{s,k}}}$. Denote by the symbol $m^{s,k}_{\lambda^{\mathcal{P}^{m^{s,k}}}(i)}$, the spread $m_j^{s,k}$in $\mathcal{P}^{m^{s,k}}$ chosen by $j = \lambda^{\mathcal{P}^{m^{s,k}}}(i)$.

Then, for each $s \in \Omega$ and spread $\mathcal{M}^s$ of $\mathcal{P}$ in $PG(F-1,K_2)$; for each $\lambda^{\mathcal{P}^{m^{s,k}}} \in S_\Lambda^{\mathcal{P}^{m^{s,k}}}$, and for each $i \in \Lambda$,

$$\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}} = \left\{ m^{s,k}_{\lambda^{\mathcal{P}^{m^{s,k}}}(i)} ; m^{s,k} \in \mathcal{M}^s \right\}$$

is a line spread of $PG(V-1,K_1)$.

(2) $\Gamma_{\{\lambda^{\mathcal{P}^{m^{s,k}}}\}} = \mathcal{R} \cup_{\ i\in\Lambda} \left\{ \Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}} ; \right\}$ is a line parallelism of $PG(V-1,K_1)$.

(3) From each parallelism of $PG(V-1,K_2)$, and for each parallelism of each line as a projective space isomorphic to $PG(3,K_1)$ containing the induced spread of each spread not equal to $\mathcal{R}$, there is a corresponding line parallelism of $PG(V-1,K_1)$. The cardinality of line parallelisms obtained is this way is

$$((\mathrm{card}\, S_\Lambda)^{\mathrm{card}\,\Delta})^{\mathrm{card}\,\Omega}.$$

PROOF. To prove (1), we need to show that every point P of $PG(V-1,K_1)$ is incident with a unique line of $\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}}$. Every point P is incident with a unique line g_P of $\mathcal{R}$ which is, as a point, on a unique line m of $\mathcal{M}$. Now m is isomorphic to $PG(3,K_1)$ and contains g_P. Furthermore, m_i is a spread in m and each point P is incident with exactly one line of each spread m_i for each $i\in\lambda$. Hence, $\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}}$ is a line spread. This proves (1).

We note, by the index assumption, that no line of m_i is in $\mathcal{R}$.

To show that $\Gamma_{\{\lambda^{\mathcal{P}^{m^{s,k}}}\}} = \mathcal{R}\cup_{\,i\in\Lambda}\left\{\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}};\right\}$ is a line parallelism, we need to show that every line L which is not in $\mathcal{R}$ is in a unique spread of the form $\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}}$. We know that L is in a unique $\langle g,h\rangle_{K_2}$ for $g,\neq h$ of $\mathcal{R}$. Moreover, since $\langle g,h\rangle_{K_2}$ is a line of $\mathcal{R}$, and $\mathcal{P}$ is a parallelism of $\mathcal{R}$ where $\mathcal{R}$ is the set of points of $PG(V-1,K_2)$, it follows that there is a unique spread $\mathcal{M}$ of $\mathcal{P}$ containing $\langle g,h\rangle_{K_2}=m$ as a line. But, m is isomorphic to $PG(3,K_1)$ and the assumed parallelism of m contains $\mathcal{R}\mid m$. Since L is a line not in $\mathcal{R}$, it follows that there is a unique spread m_j of m containing L. Thus, every line L is in exactly one spread of the form $\Gamma^{\mathcal{M}^s}_{\{\lambda^{\mathcal{P}^{m^{s,k}}}(i)\}}$. This completes the proof of part (2).

To obtain a line parallelism, we choose a fixed permutation from each symmetric group of each parallelism of $PG(3,K_1)$ arising from each line of each spread not equal to $\mathcal{R}$ of the parallelism $\mathcal{P}$. This proves (3). $\boxed{QED}$

The following corollary show how parallelisms 'grow' using the construction.

8 Corollary. *In the finite case and K_1 isomorphic to $GF(q)$, $\Lambda=q+q^2$ (the number of spreads minus 1 in a parallelism of $PG(3,q)$) so that S_{q+q^2} has cardinality $(q+q^2)!$. With K_2 isomorphic to $GF(q^2)$ and V a $2d$-dimensional $GF(q^2)$-space, a line spread in $PG(d-1,q^2)$ has $(q^{2d}-1)/(q^4-1)$ lines and a parallelism has $((q^{4d}-1)(q^{4d}-q^2))/((q^4-1)(q^4-q^2))$ line spreads. Hence,*

$$\operatorname{card}\Delta=(q^{4d}-1)/(q^4-1)$$

and

$$\operatorname{card}\Omega=((q^{4d}-1)(q^{4d}-q^2))/((q^4-1)(q^4-q^2)).$$

Hence,

(1) the number of possible line parallelisms of $PG(4d-1,q)$ constructed as above is

$$(((q+q^2)!)^{\left(\frac{q^{4d}-1}{q^4-1}\right)\left(\frac{q^{2(2d-1)}-1}{q^2-1}\right)}.$$

(2) Assume that for $d=2$, there is a parallelism of $PG(3,q^2)$ containing a regular spread $\mathcal{R}$ and there is a parallelism of $PG(3,q)$ containing a regular

spread. Then there are

$$(((q+q^2)!)^{\left(\frac{q^8-1}{q^4-1}\right)\left(\frac{q^{2(3)}-1}{q^2-1}\right)} = (((q+q^2)!)^{\left(q^4+1\right)\left(1+q^2+q^4\right)}.$$

parallelisms in $PG(2^3-1,q)$ *each of which contains the regular spread induced from* $\mathcal{R}$. *Let* $q=h^2$ *and reapply the process constructing in* $PG(2^4-1,\sqrt{q})$ *then from each of*

$$((q+q^2)!)^{\left(\frac{q^8-1}{q^4-1}\right)\left(\frac{q^{2(3)}-1}{q^2-1}\right)}$$

parallelisms of $PG(2^3-1,q)$ *there are*

$$((h+h^2)!)^{\left(\frac{h^{4d}-1}{h^4-1}\right)\left(\frac{h^{2(2d-1)}-1}{h^2-1}\right)} = (((\sqrt{q}+q)!)^{\left(q^2+1\right)\left(1+q+q^2\right)}$$

line parallelisms of $PG(2^4-1,\sqrt{q})$.

Hence, there are

$$((\sqrt{q}+q)!)^{\left(q^2+1\right)\left(1+q+q^2\right)}{}^{((q+q^2)!)^{\left(q^4+1\right)\left(1+q^2+q^4\right)}}$$

possible line parallelisms of $PG(2^4-1,\sqrt{q})$.

3 Some Parallelisms Containing Desarguesian Spreads

In this section, we mention some of the results and constructions on parallelisms that may be used to construct a vast number of new infinite examples of line parallelisms in higher dimensional spaces; *i.e.* applications of the generalization of Beutelspacher's construction.

We note that if K_2 and K_1 are fields, and V is a 2-dimensional K_2-vector space, then $\mathcal{R}$ is a Pappian spread in $PG(3,K_1)$ and within this spread K_1 corresponds to a regulus. Hence, when the skewfields are fields, we are considering those (line) spreads of $PG(3,K_1)$ that contain a given regular spread Σ.

There are many new examples of parallelisms in $PG(3,q)$ admitting a given regular parallelism, and we shall review a few of the various constructions.

The examples considered in Beutelspacher [1] basically involve a particular example due to Denniston [3] using the Klein quadric. In particular, there is a unique Desarguesian spread $\mathcal{R}$ in the parallelisms. Beutelspacher constructs these parallelisms using a different method. In this case, the remaining spreads of the parallelisms in $PG(3,q)-\mathcal{R}$ are taken to be to be Hall spreads, and it is more or less implicit in Beutelspacher that the parallelisms are isomorphic, although Beutelspacher does not exclude the possibility that the parallelisms are of the same basic type but still not isomorphic.

In Johnson [4], there is a construction of a class of parallelisms in $PG(3, K_1)$, where K_1 is a field that admits a quadratic extension field K_2, using the full central collineation group with fixed affine axis of a corresponding Pappian affine plane coordinatized by K_2 with spread Σ within $PG(3, K_1)$. Although the parallelisms obtained are of the same general type as the Denniston/Beutelspacher parallelisms, the group construction allows a enumeration of isomorphism types. The following constuctions are in [4].

9 Theorem. *Let Σ be any Pappian spread in $PG(3, K)$, where K is a field. Let R be any regulus in the spread and L any line of R. Assume there exists a second Pappian spread Σ' in $PG(3, K)$ containing R (*i.e. *K not $GF(2)$).*

Let L' denote a line of Σ' which is not in the spread of Σ.

Let G denote the full central collineation group of Σ with axis L.

Then, $\{\Sigma' g; g \in G\}$ is a set of Desarguesian partial spreads which cover the lines which are skew to L and not in Σ.

10 Theorem. *Let Σ be a Pappian spread in $PG(3, K)$ for K a field. Assume that there exists a regulus $\mathcal{R}$ which is contained in at least two distinct Pappian spreads Σ and Σ'. Let ℓ be a fixed component of Σ and let G denote the full group of central collineations of the affine translation plane $\mathcal{A}$ associated with Σ with axis ℓ.*

Consider the set of spreads $\{\Sigma' g;\ g \in G\}$ and form the Hall spreads $\overline{\Sigma' g}$ by derivation of each Rg.

(1) $\overline{\Sigma' g} = \overline{\Sigma'} g$; there is a group of Σ acting transitively on the set of Hall spreads.

(2) $\Sigma \cup \{\overline{\Sigma' g};\ g \in G\}$ is a parallelism consisting of one Pappian spread and the remaining Hall spreads.

Using the previous construction, we may obtain another parallelism by the derivation of Σ and $\overline{\Sigma'}$.

11 Theorem. *Under the assumptions of the previous theorem, let $\overline{\Sigma}$ denote the Hall spread obtained by the derivation of $\mathcal{R}$.*

Then $\overline{\Sigma} \cup \Sigma' \cup \{\overline{\Sigma'} g$ for $g \neq 1$ of $G\}$ is a parallelism of $PG(3, K)$.

We have mentioned previously the Denniston and Beutelspacher constructions of parallelisms with one Desarguesian spread and the remaining spreads of the parallelism are Hall. We shall see that there are a great variety of such parallelisms. One benefit of the construction technique given in the previous subsection is that we may determine the isomorphism classes of the parallelisms.

The results of the rest of this section are in Johnson and Pomareda [8].

The isomorphism results basically rely on the following connections between the collineation groups of Desarguesian and Hall spreads.

12 Theorem. *Let π be a Pappian plane with spread in $PG(3,K)$, for K a field. If $|K|>3$ then the full collineation group of a Hall plane constructed from π by derivation of a regulus net R is the group inherited from π; the full collineation group leaves the regulus net R invariant.*

13 Corollary. *Let π^* and ρ^* denote two Hall affine planes with spreads in $PG(3,K)$, for K a field, and let π and ρ denote the associated Pappian planes which construct the indicated planes, respectively, by derivation of the regulus nets $\mathcal{R}_\pi$ and $\mathcal{R}_\rho$ with opposite regulus nets $\mathcal{R}_\pi^*$ and $\mathcal{R}_\rho^*$.*

If $|K|>3$ and π^* is isomorphic to ρ^* by a mapping σ then σ maps the regulus net $\mathcal{R}_\pi^*$ onto the regulus net $\mathcal{R}_\rho^*$.

14 Remark. We note that we shall distinguish notationally between a spread $\mathcal{R}$ and the affine plane defined by the spread by using $\pi_{\mathcal{R}}$ to denote the plane. More generally, the translation net defined by a partial spread $\mathcal{Z}$ shall be denoted by $\pi_{\mathcal{Z}}$.

15 Theorem. *Let K be a field of cardinality >3 and let Σ, Σ', and Σ'' denote Pappian spreads containing a regulus R where Σ' and Σ'' are distinct from Σ and let ℓ denote the axis of the central collineation group G of π_Σ.*

Let $\mathcal{P}_{\Sigma,\Sigma'}=\Sigma\cup\{\overline{\Sigma' g};\ g\in G\}$ and $\mathcal{P}_{\Sigma,\Sigma''}=\Sigma\cup\{\ \overline{\Sigma''} g;\ g\in G\}$ be parallelisms in $PG(3,K)$.

(1) If σ is an isomorphism from $\mathcal{P}_{\Sigma,\Sigma'}$ onto $\mathcal{P}_{\Sigma,\Sigma''}$ then σ is a collineation of the Pappian plane π_Σ which leaves invariant ℓ and may be assumed to leave π_R invariant and maps Σ' onto Σ''.

(2) Furthermore, we may assume that σ leaves at least three parallel classes of π_R invariant which implies that σ is an element of a group isomorphic to $\Gamma L(1,K^2)/GL(1,K)$ where K^2 denotes the quadratic extension of K corresponding to the Pappian plane π_Σ.

16 Corollary. *Let K be any field for which there exists a quadratic field extension K^2. Let $\mathcal{Q}$ denote the set of all quadratic field extensions of K.*

Then the group $\Gamma L(1,K^2)/GL(1,K)$ acts as a (not necessarily faithful) permutation group on $\mathcal{Q}$ and the orbits not equal to K^2 define the isomorphism classes of the parallelisms $\mathcal{P}_{\Sigma,\Sigma'}$ where Σ is the Pappian spread defined by K^2.

In the finite case and, for example, when K is the field of real numbers, all quadratic field extensions are isomorphic. Since any such Pappian spread may be embedded into $PG(3,K)$, we see that the isomorphisms may be taken within $\Gamma L(4,K)$. However, when there exist non-isomorphic quadratic extensions, we obtain other non-isomorphic parallelisms.

17 Corollary. *Let Σ and Δ denote Pappian spreads in $PG(3,K)$, for K a field of cardinality >3. Let R_Σ and R_Δ denote reguli in Σ and Δ respectively.*

Let Σ' and Δ' denote Pappian spreads distinct from Σ and Δ respectively and containing R_Σ and R_Δ respectively. Form the parallelisms $\mathcal{P}_{\Sigma,\Sigma'}$ and $\mathcal{P}_{\Delta,\Delta'}$.

Then the two parallelisms are not isomorphic in any of the following situations:

(1) The field K^2_Σ coordinatizing π_Σ is not isomorphic to the field K^2_Δ coordinatizing π_Δ.

(2) Assuming that Σ and Δ are isomorphic, the field $K^2_{\Sigma'}$ coordinatizing $\pi_{\Sigma'}$ is not isomorphic to the field $K^2_{\Delta'}$ coordinatizing $\pi_{\Delta'}$.

(3) Assuming that Σ and Δ are isomorphic, identify Σ and Δ under the isomorphism. The field $K^2_{\Sigma'}$ coordinatizing $\pi_{\Sigma'}$ and the field $K^2_{\Delta'}$ coordinatizing $\pi_{\Delta'}$ are in distinct $\Gamma L(1,K^2)/GL(1,K)$ orbits.

Using the above result in the finite case, the following may be obtained:

18 Theorem. *If $K \simeq GF(q)$ then the number of mutually non-isomorphic parallelisms is at least*

$$1 + [(q-3)/2r] \text{ for } q \text{ odd or}$$
$$[(q/2-1)/2r] \text{ for } q \text{ even}$$

where $q = p^r$ for r a positive integer and p a prime.

If $q = p$, an odd prime, then the number of mutually non-isomorphic parallelisms is exactly $(p-1)/2$.

Any parallelism obtained via the group construction given above will be called a 'Johnson parallelism'.

We now consider parallelisms of the second class of parallelism constructed. That is, let $\overline{\Sigma}$ denote the Hall spread obtained by the derivation of $\mathcal{R}$ and let $\mathcal{P}$ denote the previously constructed parallelism.

Then $\overline{\Sigma} \cup \Sigma' \cup \{\mathcal{P}-\{\Sigma, \overline{\Sigma'}\}\}$ is a parallelism of $PG(3,K)$. We shall call this a parallelism $\mathcal{P}^*$ 'derived' from $\mathcal{P}$.

We first note that any such parallelism $\mathcal{P}^*$ admits a collineation group isomorphic to $G_\mathcal{R}$.

We now ask if there is any enumeration process for the derived Johnson parallelisms. We first note the following theorem:

19 Theorem. *Any parallelism $\overline{\Sigma} \cup \Sigma' \cup \{\mathcal{P}-\{\Sigma, \overline{\Sigma'}\}\} = \mathcal{P}^*$ admits $G_\mathcal{R}$ as a collineation group that fixes $\overline{\Sigma}$ and Σ' for $\mathcal{R}$ a regulus contained in Pappian spreads Σ and Σ'*

Furthermore, if $|K| > 3$ then the full collineation group must leave $\overline{\Sigma}$ and Σ' invariant and permute the set of reguli $\{\mathcal{R}g;\ g \in G\}$.

20 Corollary. *Assume that $|K| > 3$.*

Let Σ' and Σ'' be Pappian spreads distinct from Σ and containing $\mathcal{R}$.

Form the two parallelisms $\overline{\Sigma} \cup \Sigma' \cup \{\mathcal{P}-\{\Sigma, \overline{\Sigma'}\}\} = \mathcal{P}^*_{\Sigma'}$ *and parallelism* $\overline{\Sigma} \cup \Sigma'' \cup \{\mathcal{P}-\{\Sigma, \overline{\Sigma''}\}\} = \mathcal{P}^*_{\Sigma''}$.

Then any isomorphism from $\mathcal{P}^*_{\Sigma'}$ *onto* $\mathcal{P}^*_{\Sigma''}$ *must map* Σ' *onto* Σ'' *maps* Σ *onto* Σ*, permutes the set* $\{\mathcal{R}g\ ;\ g \in G\}$ *and leaves the axis of* G *invariant.*

21 Corollary. *Assume that* $|K| > 3$*. Two derived parallelisms are isomorphic if and only if the two original parallelisms are isomorphic.*

That is, $\mathcal{P}^*_{\Sigma'}$ *is isomorphic to* $\mathcal{P}^*_{\Sigma'}$ *if and only if the parallelism* $\mathcal{P}_{\Sigma'}$ *is isomorphic to* $\mathcal{P}_{\Sigma''}$.

22 Theorem. *A derived parallelism* $\mathcal{P}^*$ *cannot be isomorphic to* $\mathcal{P}$.

We now have:

23 Theorem. *Let* $\mathcal{P}^*$ *be a derived Johnson parallelism in* $PG(3,q)$ *for* $q > 3$.

(1) Then the full projective collineation group $H_{\mathcal{P}^*}$ *of* $\mathcal{P}^*$ *is a collineation subgroup of* $\Gamma L(2, GF(q^2))$ *which fixes a derivable net* $\mathcal{R}$ *and a component* ℓ *of* $\mathcal{R}$*. This group contains a central collineation group* C *of order* $q(q-1)$ *with axis* ℓ *that acts* 2*-transitively on the components of* $\mathcal{R} - \ell$.

(2) $H_{\mathcal{P}^*}$ *is a subgroup of* $C \cdot GalGF(q^2)$.

Hence, we have the following conclusions:

24 Conclusion. If q is odd p^r for p an odd prime then there are at least $2(1+[(q-3)/2r]$ mutually non-isomorphic parallelisms in $PG(3,q)$ constructed as above.

If q is an odd prime then there are exactly $p-1$ mutually non-isomorphic parallelisms in $PG(3,q)$ obtained by the group and derivation constructions.

25 Conclusion. If q is even, there are at least $2[(q/2-1)/2r]$ mutually non-isomophic parallelisms obtained by the group and derivation constructions.

26 Theorem. *Let* K *be an infinite field which admits a non-square. Let the automorphism group have cardinality* $\mathcal{A}_o$.

(1) Then there are at least card $K/\mathcal{A}_o$ *mutually non-isomorphic parallelisms.*

(2) Let K *be any subfield of the reals. Then there are at least* card K *mutually non-isomorphic parallelisms.*

In particular, if K *is the field of real numbers then there are* $2^{\varkappa_o}$ *mutually non-isomorphic parallelisms.*

(3) If K *is the field of real numbers then there are* 2^{χ_o} *mutually non-isomorphic group Johnson parallelisms and* 2^{χ_o} *mutually non-isomorphic derived Johnson parallelisms in* $PG(3,K)$.

27 Theorem. *Let* K *be a skewfield and* Σ *a spread in* $PG(3,K)$*. Assume that there exists a partial parallelism* $\mathcal{P}$ *containing* Σ *that admits as a* *collineation group a central collineation group G of Σ with axis ℓ that acts two-transitively on the remaining lines of Σ.*

Then

(1) Σ is Pappian,

(2) the spreads of $\mathcal{P}-\{\Sigma\}$ are Hall and

(3) $\mathcal{P}$ is a Johnson parallelism.

4 Applications of Beutelspacher's Construction

We have noted that when K_1 is infinite, line parallelisms are essentially ubiquitous by Beutelspacher. On the other hand, there is little control on the spreads in the sense that there is essentially no knowledge or classification regarding the spreads or the parallelisms. The previous section gives a construction of a large variety of line parallelisms in $PG(3, K_1)$, where K_1 is an arbitrary field that admits a quadratic field extension K_2. Furthermore, there are classification results using collineation groups that may potentially be used in the extension of parallelisms in higher dimensional projective spaces.

Using the extension theorem, we can construct line parallelisms in projective spaces as follows:

Assume that there is a set of fields $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_k \subseteq K_{k+1}$ such that K_{i+1} is a quadratic extension of K_i for $i = 1, 2, \ldots, k$. To begin, assume that $k = 5$.

To construct a line parallelism of $PG(2^3 - 1, K_1)$, we consider K_4 as a K_2-vector space and take the Pappian spread of all 1-dimensional K_2-subspaces of K_4. This becomes a Pappian spread S_4 of $PG(2^3 - 1, K_1)$. Any two components g and h of S_4 generate a 2-dimensional K_2-subspace which becomes a $PG(3, K_1)$ and S_4 induces a Pappian spread in $PG(3, K_1)$ coordinatizable by K_2. By the method of the previous section, we may first choose any line parallelism of such a $PG(3, K_1)$ that contains a Pappian spread coordinatizable by K_2 and consider that spread induced by S_4. We then may choose any line parallelism $\mathcal{P}_2$ of $PG(2^2 - 1, K_2)$. Since, K_3 exists, our previous method allows the existence of a line parallelism here. Hence, we have obtained a variety of parallelisms in $PG(2^3 - 1, K_1)$ by the extension of Beutelspacher's construction.

Since $k = 5$ in our initial situation, we may also construct a variety of parallelisms in $PG(2^3 - 1, K_2)$. To continue, take K_5 as a K_2-vector space of dimension 2^3 and form the set of 1-dimensional K_2-subspaces of K_5 which then becomes a Pappian line spread S_5 of $PG(2^4 - 1, K_1)$. This Pappian line spread also induces in any associated $PG(3, K_1)$ determined by two components g and h the Pappian line spread coordinatizable by K_2. Another application of the extension theorem produces a line parallelism in $PG(2^4 - 1, K_1)$.

So, in the general case $z = 1, 2, \ldots, k$, we may similarly construct line parallelisms in $PG(2^a - 1, K_1)$ for $a = 2, 3, \ldots, k-1$.

28 Theorem. *(1) Let K_1 be a field and $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_k \subseteq K_{k+1}$ such that K_{i+1} is a quadratic field extension of K_i for $i = 1, 2, \ldots, k$. Assume that V_z is a 2^z-dimensional K_z-vector space. Then, there exist line parallelisms in $PG(2^z - 1, K_1)$ for all $z = 2, \ldots, k$.*

(2) If there is an infinite sequence of fields each a quadratic extension field of the previous then then exist line parallelisms in $PG(2^z - 1, K_1)$ for all positive integers z.

(3) If there is another sequence $K_1 = K_1' \subseteq K_2' \subseteq K_3' \subseteq \cdots \subseteq K_k' \subseteq K_{k+1}'$, such that K_{i+1}' is a quadratic field extension of K_i', then, there exist another set of line parallelisms in $PG(2^z - 1, K_1)$, for all $z = 2, 3, \ldots, k$.

If K_2 is not isomorphic to K_2' and the parallelisms are chosen so that there is a unique Pappian spread $\mathcal{R}$ and $\mathcal{R}'$ in a parallelism using the corresponding sequence then none of these line parallelisms can be isomorphic to their analogues of (1).

PROOF. Assume that there is an infinite sequence of fields. Whenever there is a construction of a line parallelism in $PG(2^a - 1, K_1)$, there is an analogous construction in $PG(2^a - 1, K_2)$. Then using the line parallelism in $PG(2^a - 1, K_2)$ together with a choice of parallelism for each associated $PG(3, K_1)$ containing a spread induced from a Pappian line spread of $PG(2^{a+1} - 1, K_1)$ taken as constructed from the 1-dimensional K_2-subspaces of K_{a+2}, there is a constructed line parallelism in $PG(2^{a+1} - 1, K_1)$.

If we have a parallelism of $PG(2^a - 1, K_1)$ containing $\mathcal{R}$ and a parallelism of $PG(2^a - 1, K_1)$ containing $\mathcal{R}'$ but all other spreads are non-Pappian then an isomorphism from one to the other must map $\mathcal{R}$ onto $\mathcal{R}'$, implying that $PG(V - 1, K_2)$ is isomorphic to $PG(V - 1, K_2')$, for an appropriate vector space V, which, in turn, implies that K_2 is isomorphic to K_2'. $\boxed{QED}$

29 Example. Let K_1 be the field of rational numbers. Then there are $\varkappa_o$-distinct infinite sequences of fields each a quadratic extension of the previous field. Each such sequence provides a set of line parallelisms as above. Furthermore, there are choices of infinitely mutually non-isomorphic subfields K_2 quadratic over K_1. Any such sequence with the choice of exactly one Pappian spread within the parallelism produces an infinite number of mutually non-isomorphic line parallelisms of each order.

5 Final Remarks

We have shown that given any field K that admits a quadratic extension F, there are a variety of line parallelisms in $PG(3,K)$ each containing a Pappian spread. However, there are many other examples of such line parallelisms containing a Pappian spread in the author's work and in the construction methods of the author and Pomareda (see Johnson [6], and Johnson-Pomareda [10], [7], [9]). Hence, we have not attempted to give any sort of classification of the tremendous variety of line parallelisms that may be constructed using the extension of Beutelspacer's theorem.

We end with a problem:

30 Problem. Using isomorphism conditions of $PG(3,L)$, for various fields L, determine conditions that ensure that the parallelisms constructed in higher dimensional projective spaces are mutually non-isomorphic.

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