Note di Matematica **21,** n. 2, 2002/2003, 1–33.

Some Constructions and Embeddings of the Tilde Geometry

Antonio Pasini
Department of Mathematics, University of Siena
Via del Capitano 15, I-53100 Siena, Italy
pasini@unisi.it

Hendrik Van Maldeghem
Department of Pure Mathematics and Computer Algebra, Gent University
Krijgslaan 281, B-9000 Gent, Belgium
hvm@cage.rug.ac.be

Received: 13 December 2001; accepted: 4 March 2002.

Abstract. Some old and new constructions of the tilde geometry (the flag transitive connected triple cover of the unique generalized quadrangle $\mathsf{W}(2)$ of order $(2,2)$) are discussed. Using them, we prove some properties of that geometry. In particular, we compute its generating rank, we give an explicit description of its universal projective embedding and we determine its homogeneous embeddings.

Keywords: Generalized quadrangle, diagram geometry, tilde geometry, triple cover, universal embedding.

MSC 2000 classification: 51E12; 51E24.

Introduction

The tilde geometry is described by Ronan and Stroth [13] as the coset geometry of the parabolic system of $G = 3^{\cdot}\mathbf{S}(6)$ (non-split extension) formed by two copies P_1, P_2 of $Z_2 \times \mathbf{S}(4)$ with $P_1 \cap P_2 = 2^{3+1}$. Throughout this paper, we denote that geometry $\widetilde{\mathsf{W}(2)}$. Clearly, $\widetilde{\mathsf{W}(2)}$ is a triple cover of the generalized quadrangle $\mathsf{W}(2)$ of order $(2,2)$, with $O_3(G)$ as the deck group. It is in fact the only flag-transitive triple cover of $\mathsf{W}(2)$. It is also the only triple cover of $\mathsf{W}(2)$ with no quadrangle, as the reader can prove by himself.

The collinearity graph of $\widetilde{\mathsf{W}(2)}$ was known long before Ronan and Stroth [13] found the above construction. That graph is the so-called *Foster graph* (Foster [4]; also Smith [15]). A geometric construction of $\widetilde{\mathsf{W}(2)}$ (whence, of the Foster graph) is described by Brouwer, Cohen and Neumaier [1, 13.2.A]. More constructions of $\widetilde{\mathsf{W}(2)}$ have been found later by a number of authors.

In this paper we survey the constructions of $\widetilde{\mathsf{W}(2)}$ we are aware of and we add a few new ones to them. In particular, we present a geometric construction inside $\mathbf{PG}(3,4)$ such that the full automorphism group — including all dualities — of $\widetilde{\mathsf{W}(2)}$ is inherited by the one of $\mathbf{PG}(3,4)$. We discuss the mutual relations between all these constructions and exploit one of them to determine the generating rank $\mathrm{grk}(\widetilde{\mathsf{W}(2)})$ and the (universal) embedding rank $\mathrm{erk}(\widetilde{\mathsf{W}(2)})$ of $\widetilde{\mathsf{W}(2)}$, proving that $\mathrm{grk}(\widetilde{\mathsf{W}(2)}) = \mathrm{erk}(\widetilde{\mathsf{W}(2)}) = 11$. Actually, the equality $\mathrm{erk}(\widetilde{\mathsf{W}(2)}) = 11$ is known since long ago. For instance, that equality is mentioned by Ivanov and Shpectorov [7], who say it follows by computing the 0-homology group of $\widetilde{\mathsf{W}(2)}$ over $\mathbf{GF}(2)$ (which, according to Ronan [12], yields the universal embedding). That computation is not very hard in this case and we don't claim that our proof is much easier than it, but it is more straightforward.

We also determine the homogeneous embeddings of $\widetilde{\mathsf{W}(2)}$ (an embedding being homogeneous if the full type preserving collineation group of the embedded geometry is induced by the automorphism group of the ambient projective space stabilizing the embedded geometry), proving that there exist exactly two of them, apart from the universal one; namely, an embedding in $\mathbf{PG}(5,2)$ (which is known) and a seemingly new one, in $\mathbf{PG}(9,2)$. Along the way, we prove several general facts for embeddings and projections of embeddings.

1 Four known constructions

We have mentioned the construction of $\widetilde{\mathsf{W}(2)}$ as a coset geometry, by Ronan and Stroth [13]. An explicit description of the incidence matrix of $\widetilde{\mathsf{W}(2)}$ has been given by Ito [5], but we are not going to recall it here. We shall only discuss four constructions which look somehow more geometric than those.

1.1 Construction A

We first recall the construction by Brouwer, Cohen and Neumaier [1] (see also Ivanov [6, 2.7.13]). Let $V = V(3,4)$. Given a hyperoval $\mathcal{H}$ of the projective plane $\mathbf{PG}(V)$, let $\mathcal{H}^*$ be the dual hyperoval formed by the six lines of $\mathbf{PG}(V)$ exterior to $\mathcal{H}$. Let $\overline{\mathcal{H}}$ (respectively $\overline{\mathcal{H}}^*$) be the complement of $\mathcal{H}$ (respectively $\mathcal{H}^*$) in the set of points (lines) of $\mathbf{PG}(2,4)$. It is well known that $(\overline{\mathcal{H}}, \overline{\mathcal{H}}^*)$, with the natural incidence relation, is a copy of $\mathsf{W}(2)$.

Let V^+ be the additive group of V. Clearly, V^+ can be regarded as a 6-dimensional vector space over $\mathbf{GF}(2)$. Accordingly, $\mathcal{H}$ can be viewed as a collection of 2-dimensional subspaces of V^+. Let Γ_A be the induced subgeometry of $\mathbf{PG}(V^+)$ defined as follows:

(A1) the points of Γ_A are the nonzero vectors of V^+ that, regarded as vectors of V, span a 1-space belonging to $\overline{\mathcal{H}}$;

(A2) the lines of Γ_A are the 2-dimensional subspaces of V^+ that meet each member of $\mathcal{H}$ trivially and, regarded as sets of vectors of V, span a 2-space belonging to $\overline{\mathcal{H}}^*$.

Then $\Gamma_A \cong \widetilde{\mathsf{W}(2)}$ (see [1]) and the mapping f sending every point and line of Γ_A to the subspace of $\mathbf{PG}(V)$ spanned by it, is a covering from Γ_A to $(\overline{\mathcal{H}}, \overline{\mathcal{H}}^*) \cong \mathsf{W}(2)$, with all fibers of size 3.

1.2 A variation of Construction A

A variation of Construction A is also described by Brouwer, Cohen and Neumaier [1]. They consider the hexacode, namely the subspace U of $V(6,4)$ spanned by the following three vectors, where ω is a generator of the multiplicative group of $\mathbf{GF}(4)$:

$$(0,0,1,1,1,1), \quad (0,1,0,1,\omega,\omega^2), \quad (1,0,0,1,\omega^2,\omega)$$

Every nonzero vector of U has either no or just two zero coordinates. Precisely 18 out of the 63 nonzero vectors of U have no zero coordinates. They span six 1-dimensional subspaces of U, forming a hyperoval $\mathcal{H}$ of $\mathbf{PG}(U)$ ($\cong \mathbf{PG}(2,4)$). The remaining 45 nonzero vectors of U can be regarded as the points of Γ_A. The lines of Γ_A can be described as follows. Regarding U as a subset of $V(6,\mathbb{C})$, with $\omega \neq 1$ a cubic root of 1 in $\mathcal{C}$, consider the usual dot product $\sum_{i=1}^{6} x_i \bar{y}_i$ of $V(6,\mathbb{C})$. Then two points of Γ_A are collinear if and only if they, regarded as vectors of $V(6,\mathbb{C})$, have dot product equal to 2.

1.3 Construction B

We first recall a few well known properties of the Steiner system $\mathcal{S} = S(24,8,5)$ for M_{24} and construct the tilde geometry of rank 3 for M_{24} (Ronan and Stroth [13]), which we call $\Gamma(M_{24})$. As the residues of $\Gamma(M_{24})$ of rank 2 are isomorphic to $\widetilde{\mathsf{W}(2)}$, a construction of $\widetilde{\mathsf{W}(2)}$ will be obtained as a by-product. The construction of $\Gamma(M_{24})$ we shall describe here is contained in Ivanov [6] (also Ivanov, Pasechnik and Shpectorov [8, page 529]).

For Ξ a sextet of $\mathcal{S} = S(24,8,5)$, we denote by $O(\Xi)$ the set of octads that are joins of two tetrads of Ξ and by $T(\Xi)$ the set of trios formed by three octads of $O(\Xi)$. The structure $(O(\Xi), T(\Xi))$, with the natural incidence relation, is isomorphic to $\mathsf{W}(2)$.

Let S be the set of points of $\mathcal{S}$. It is well known that, for every octad O of $\mathcal{S}$, a copy A_O of $\mathbf{AG}(4,2)$ can be defined on the complement $S \setminus O$ of O in S in such a way that the stabilizer of O in M_{24} induces the full automorphism group of A_O on $S \setminus O$. Furthermore, if Ξ is a sextet such that $O \in O(\Xi)$, the four tetrads of Ξ not contained in O form a parallel class of planes of A_O. The affine geometry A_O has 63 classes of parallel lines, but only three of them are formed by lines contained in tetrads of Ξ. We call them the Ξ-classes of O.

Let $\Gamma(M_{24})$ be the geometry of rank 3 defined as follows: $\{0,1,2\}$ is the set of types of $\Gamma(M_{24})$ and:

(1) the 0-elements are the pairs (O,ω) with O an octad of $\mathcal{S}$ and ω a parallel class of lines of A_O;

(2) the 1-elements are the triples $\{(O_i,\omega_i)\}_{i=1}^{3}$ where $\{O_i\}_{i=1}^{3}$ is a trio, ω_i is a parallel class of lines of A_{O_i} for $i=1,2,3$ and ω_i,ω_j induce the same partition on O_k, for $\{i,j,k\}=\{1,2,3\}$;

(3) the 2-elements of $\Gamma(M_{24})$ are the sextets of $\mathcal{S}$;

(4) the incidence relation between 0- and 1-elements is the natural one, namely inclusion;

(5) a 2-element Ξ and a 0-element (O,ω) are declared to be incident precisely when $O\in O(\Xi)$ and ω is a Ξ-class of O;

(6) a 2-element Ξ and a 1-element $\{(O_i,\omega_i)\}_{i=1}^{3}$ are incident if and only if (O_i,ω_i) and Ξ are incident for every $i=1,2,3$.

It is well known that M_{24} is the full automorphism group of $\Gamma(M_{24})$ (see for instance [6]). Also, the residues of the 0-elements of $\Gamma(M_{24})$ are isomorphic to $\mathbf{PG}(3,2)$ and the residues of the 2-elements of $\Gamma(M_{24})$ are isomorphic to $\widetilde{\mathsf{W}(2)}$, as it is clear from the structure of the stabilizers in M_{24} of the elements and the flags of $\Gamma(M_{24})$. In particular, given a 2-element Ξ of $\Gamma(M_{24})$ (namely, a sextet of $\mathcal{S}$), its residue $\mathrm{Res}(\Xi)$ in $\Gamma(M_{24})$ is isomorphic to $\widetilde{\mathsf{W}(2)}$. The function sending $(O,\omega)\in\mathrm{Res}(\Xi)$ to O induces a covering from $\mathrm{Res}(\Xi)$ to $(O(\Xi),T(\Xi))\cong\mathsf{W}(2)$.

Thus, we have got the following model $\Gamma_B := \mathrm{Res}(\Xi)$ of $\widetilde{\mathsf{W}(2)}$ (see also Ivanov [6, Lemma 2.10.2]):

(B1) the points of Γ_B are the pairs (O,ω) with $O\in O(\Xi)$ and ω a Ξ-class of lines of A_O;

(B2) the lines of Γ_B are the triples $\{(O_i,\omega_i)\}_{i=1}^{3}$ where $\{O_i\}_{i=1}^{3}\in T(\Xi)$, ω_i is a Ξ-class of A_{O_i} for $i=1,2,3$ and ω_i,ω_j induce the same partition on O_k, for $\{i,j,k\}=\{1,2,3\}$;

(B3) the incidence relation is the natural one, namely inclusion.

1.4 From Construction B to Construction A

Let Ξ be a sextet of $\mathcal{S} = S(24, 8, 5)$ and $\Gamma_B = \mathrm{Res}(\Xi)$, as in the previous subsection. It is well known ([3]; see also Conway [2] or Ivanov [6, Lemma 2.10.2]) that the stabilizer G of Ξ in M_{24} is a split extensions $G = U : S$ of an elementary Abelian group $U = O_2(G)$ of order 2^6 by the non-split extension $S = 3^{\cdot}\mathbf{S}(6)$. The group U is the elementwise stabilizer of $\mathrm{Res}(\Xi)$ in M_{24}. Also, U is the hexacode, exploited in Subsection 1.2 to describe the variation of Construction A. The subgroup S has two orbits U_0, U_2 on the set of nontrivial elements of U, the elements of U_0 (respectively U_2) being the words of the hexacode with no (exactly two) zero coordinate(s). Given a point (O, ω) of Γ_B, the stabilizer of (O, ω) and Ξ in G is the centralizer in G of an element of U_2. Thus, we have a bijection f from the set of points of Γ_B to U_2, which is the point set of Γ_A. Exploiting the information given in [3, page 94], one can also check that f maps every line of Γ_B onto a line of Γ_A (we omit the details).

1.5 Construction C

The construction we shall describe here has been found by Stroth and Wiedorn [16, section 2.2] (see also Pasini and Wiedorn [11, section 6.3]).

With $\mathcal{A} = \mathbf{AG}(3, 4)$, let $\mathcal{A}^\infty$ be the plane at infinity of $\mathcal{A}$. For a line (a plane) X of $\mathcal{A}$, we denote by X^∞ the point (line) at infinity of X and, given a line L and a point p of $\mathcal{A}$ (two coplanar lines L, M of $\mathcal{A}$), we denote by $[L, p]$ (respectively $[L, M]$) the plane of $\mathcal{A}$ spanned by $L \cup \{p\}$ (respectively $L \cup M$).

Given a hyperoval H of $\mathcal{A}^\infty$ and a point p_0 of $\mathcal{A}$, we define the sets $P(p_0, H)$ and $L(p_0, H)$ as follows:

(1) $P(p_0, H)$ is the set of planes X of $\mathcal{A}$ such that $p_0 \notin X$ and X^∞ is a secant line of H;

(2) $L(p_0, \mathcal{H})$ is the set of lines L of $\mathcal{A}$ such that $p_0 \notin L$, $L^\infty \notin H$ and $[L, p_0]^\infty$ is a secant line of H.

It is not difficult to see that every line of $L(p_0, H)$ belongs to exactly two planes of $P(p_0, H)$, every plane of $P(p_0, H)$ contains six lines of $L(p_0, H)$ and the parallelism relation partitions that set of six lines in three pairs.

For two distinct lines $L, M \in L(p_0, H)$, we write $L\pi M$ if L and M are parallel and $[L, M] \in P(p_0, H)$. We denote by $\tilde{\pi}$ the transitive closure of the relation π. By straightforward computations one can check that the classes of $\tilde{\pi}$ have size 3 and, for every such class $\{L_1, L_2, L_3\}$, there exist three distinct planes $X_1, X_2, X_3 \in P(p_0, H)$ such that $L_i, L_j \subset X_k$, for $\{i, j, k\} = \{1, 2, 3\}$. We call $\{X_1, X_2, X_3\}$ the *trihedron* of the class $\{L_1, L_2, L_3\}$. We can now define the geometry Γ_C:

(C1) $P(p_0, H)$ is the set of points of Γ_C;

(C2) the lines of Γ_C are the trihedra of the classes of $\tilde{\pi}$;

(C3) the incidence relation is the natural one, namely inclusion.

Clearly, the stabilizer $G = 3^{\cdot}\mathbf{S}(6)$ of H in $\mathrm{A\Gamma L}(3,4)$ acts faithfully and flag-transitively on Γ_C and it is not difficult to see that stabilizers in G of the points, lines and flags of Γ_C are just as in the tilde geometry $\widetilde{\mathsf{W}(2)}$. Therefore, $\Gamma_C \cong \widetilde{\mathsf{W}(2)}$. The covering from Γ_C to $\mathsf{W}(2)$ is easy to describe: it is the function sending $X \in P(p_0, H)$ to $X^\infty \cap H$.

1 Remark. In [16] and [11], the geometry Γ_C is obtained as the geometry at infinity of a certain circular extension of the dual Petersen graph. In view of that theoretical framework, in [16] and [11] the classes of $\tilde{\pi}$ are taken as lines instead of their trihedra.

1.6 A variation of Construction C

With $\mathcal{A}$, $\mathcal{A}^\infty$, H and p_0 as in the previous subsection, let H^* be the dual hyperoval of $\mathcal{A}^\infty$ formed by the six lines of $\mathcal{A}^\infty$ exterior to H. Thus, $P(p_0, H)$ is the set of planes X of $\mathcal{A}$ such that $p_0 \notin X$ and $X^\infty \notin H^*$. Note also that, if $\Lambda = \{X_1, X_2, X_3\}$ is a line of Γ_C, the lines X_1^∞, X_2^∞, X_3^∞ pass through a common point p of $\mathcal{A}^\infty$, exterior to H. The point p belongs to two lines of H^*, say M_1 and M_2. Let Y_1 and Y_2 be the planes of $\mathcal{A}$ through p_0 with $Y_i^\infty = M_i$ (for $i = 1, 2$). It is not difficult to check that the set

$$H_\Lambda := \{\mathcal{A}^\infty, Y_1, Y_2, X_1, X_2, X_3\}$$

is a dual hyperoval in the star $\mathrm{St}_{\mathcal{P}}(p)$ of p, in the projective geometry $\mathcal{P} = \mathbf{PG}(3,4)$ obtained by adding $\mathcal{A}^\infty$ to $\mathcal{A}$. Furthermore,

$(*)$ H_Λ contains the plane $\mathcal{A}^\infty$ and two planes passing through p_0 and intersecting $\mathcal{A}^\infty$ in lines of H^*.

It is straightforward to check that, for every point p of $\mathcal{A}^\infty$ exterior to H, exactly three out of the 168 hyperovals of $\mathrm{St}_{\mathcal{P}}(p)$ satisfy $(*)$. As $\mathcal{A}^\infty$ contains 15 points exterior to H and Γ_C $(\cong \widetilde{\mathsf{W}(2)})$ has 45 lines, the lines of Γ_C bijectively correspond to dual hyperovals of $\mathrm{St}_{\mathcal{P}}(p)$ satisfying $(*)$ and with $p \in \mathcal{A}^\infty \setminus H$.

We are now ready to rephrase Construction C. Turning to the dual $\mathcal{P}^*$ of $\mathcal{P}$, p_0 and $\mathcal{A}^\infty$ turn into a plane and a point, respectively. We denote them P_0 and a_0. The hyperoval H is now a dual hyperoval of the star $\mathrm{St}_{\mathcal{P}^*}(a_0)$ of a_0, whereas H^* is a hyperoval of $\mathrm{St}_{\mathcal{P}^*}(a_0) \cong \mathbf{PG}(2,4)$. The geometry Γ_C can be described as follows, by means of points and hyperovals of (planes of) $\mathcal{P}^*$:

(C1') the points of Γ_C are the points x of the affine geometry $\mathbf{AG}(3,4) = \mathcal{P}\backslash P_0$ such that $x \neq a_0$ and the line a_0x through a_0 and x does not belong to H^*;

(C2') the lines of Γ_C are the hyperovals of $\mathcal{P}^*$ containing a_0 and meeting P_0 in two points y_1, y_2 such that $a_0y_i \in H^*$ for $i = 1, 2$.

1.7 From Construction C to Construction A

The above can be made more explicit by regarding $\mathcal{P}\backslash P_0$ as the set of vectors of $V = V(3,4)$, with a_0 taken as the zero vector and the 1-spaces of V as the points of P_0. Thus, H^* is a hyperoval of $\mathbf{PG}(V)$ and, if $\{a_0, x_1, x_2, x_3, y_1, y_2\}$ is a line of Γ_C, with $y_1, y_2 \in P_0$, then $\{a_0, x_1, x_2, x_3\} = \{0, x_1, x_2, x_3\}$ is a subgroup of the additive group V^+ of V, whence a 2-subspace of V^+, the latter being regarded as a $\mathbf{GF}(2)$-vector space. Conversely, for every 2-subspace $S = \{0, x_1, x_2, x_3\}$ of V^+, if there are two 1-spaces y_1, y_2 of V contained in the 2-space of V spanned by S and belonging to H^*, then $\{a_0, x_1, x_2, x_3, y_1, y_2\}$ is a line of Γ_C. So,

(C1") the points of Γ_C are the nonzero vectors x of $V = V(3,4)$ such that the 1-space $\langle x \rangle$ of V spanned by x does not belong to H^*;

(C2") the lines of Γ_C are the 2-subspaces $S = \{0, x_1, x_2, x_3\}$ of V^+ such that there are two 1-spaces y_1, y_2 of V that are contained in the 2-space of V spanned by S and that belong to H^*.

The variation (C1"), (C2") of Construction C is clearly the same as Construction A: the hyperoval $\mathcal{H}$ of Construction A is the hyperoval H^* of (C1") and (C2").

1.8 Construction D

In this subsection we recall the construction of $\widetilde{\mathsf{W}(2)}$ by Polster and Van Maldeghem [10]. Let Π_0, Π_1, Π_2 be three copies of the Petersen graph Π. Regarding the vertices of the Petersen graph as pairs of the 5-set $S = \{0, 1, 2, 3, 4\}$, $\{\{i, j\}_k\}_{0 \leq i < j \leq 4}$ is the set of points of Π_k for $k = 0, 1, 2$. The edges of Π_k are pairs $\{\{i, j\}_k, \{i', j'\}_k\}$ with $\{i, j\}$ and $\{i', j'\}$ disjoint pairs of S. We define a geometry Γ_D as follows:

(D1) The points of Γ_D are the points of these graphs, namely the points $p^k_{ij} := \{i, j\}_k \in \Pi_k$, with $0 \leq i, j \leq 4$ $(i \neq j)$ and $k = 0, 1, 2$ (which we call *old points*), plus the following 15 triples of edges, which we call *new points*:

$$p^k_i := \left\{ \{p^k_{i+1,i+2}, p^k_{i+3,i+4}\}, \{p^{k+1}_{i+1,i+3}, p^{k+1}_{i+2,i+4}\}, \{p^{k+2}_{i+1,i+4}, p^{k+2}_{i+2,i+3}\} \right\}$$

for all $i \in \mathbb{Z}$ mod 5 and all $k \in \mathbb{Z}$ mod 3.

(D2) The lines of Γ_D are the edges of Π_0, Π_1 and Π_2.

(D3) The incidence relation between lines and old points is the natural one, inherited from Π_0, Π_1 and Π_2.

(D4) A line e and a new point p are declared to be incident when e is an element of the triple p.

Note that, for every $i = 0, 1, 2, 3, 4$, the set $\{i+1, i+2, i+3, i+4\}$ can be partitioned into two 2-subsets in exactly three ways, namely:

$$\{\{i{+}1, i{+}2\}, \{i{+}3, i{+}4\}\},\ \{\{i{+}1, i{+}3\}, \{i{+}2, i{+}4\}\},\ \{\{i{+}1, i{+}4\}, \{i{+}2, i{+}3\}\}.$$

Let P_i be the triple of these partitions, ordered as above. The definition of p_i^k in (D1) matches P_i with the even permutation π_k of $\{0, 1, 2\}$ sending 0 to k, 1 to $k+1$ and 2 to $k+2$ ($k+1$ and $k+2$ being computed modulo 3). Thus, the new points can be regarded as pairs $p_i^k = (P_i, \pi_k)$, with P_i and π_k defined as above.

As proved in [10], $\Gamma_D \cong \widetilde{\mathsf{W}(2)}$. The covering from Γ_D to $\mathsf{W}(2)$ is easy to describe: with $\overline{S} = \{0, 1, 2, 3, 4, 5\}$, that covering sends an old point p_{ij}^k to the pair $\{i, j\}$ of $\overline{S}$, a new point p_i^k to the pair $\{i, 5\}$ and a line $\{p_{i_1,j_1}^k, p_{i_2,j_2}^k\}$ to the partition of $\overline{S}$ having $\{i_1, j_1\}$ and $\{i_2, j_2\}$ as two of its classes.

1.9 From Construction B to Construction D

The 15 new points form a geometric hyperplane Ω of $\widetilde{\mathsf{W}(2)}$ (in the meaning of Ronan [12]) and no two points of Ω are collinear. So, Ω behaves like an ovoid: every line of $\widetilde{\mathsf{W}(2)}$ meets Ω in exactly one point. In fact, Ω is the preimage of an ovoid of $\mathsf{W}(2)$ via the covering from $\widetilde{\mathsf{W}(2)}$ to $\mathsf{W}(2)$. So, Construction D is a way to recover $\widetilde{\mathsf{W}(2)}$ from the complement $\widetilde{\mathsf{W}(2)} \setminus \Omega$ of Ω in $\widetilde{\mathsf{W}(2)}$.

An easy description of the hyperplane Ω of $\widetilde{\mathsf{W}(2)}$ is offered by Construction B. Let $\Gamma_B = \mathrm{Res}(\Xi)$, as in Subsection 1.3. Picked a tetrad X of Ξ, let Ω be the set of points (O, ω) of Γ_B with $X \subset O$. Then every line of Γ_B meets Ω in exactly one point and Ω is the preimage of an ovoid of $\mathsf{W}(2)$ via the covering from Γ_B to $\mathsf{W}(2)$, as in the previous subsection (final remarks). We can now revisit Construction D in the light of B.

The complement $\Gamma_B(X) := \Gamma_B \setminus \Omega$ of Ω in Γ_B is the disjoint union of three copies Π_0, Π_1, Π_2 of the Petersen graph, which correspond to the three partitions of X in pairs. More explicitly, denoted those partitions ξ_0, ξ_1, ξ_2, numbered in such a way that Π_i corresponds to ξ_i, the vertices of Π_i are the points (O, ω) of $\Gamma_B(X)$ with $\xi_i \subset \omega$ and the edges of Π_i correspond to the lines $\{(O_1, \omega_1), (O_2, \omega_2), (O_3, \omega_3)\}$ of Γ_B where, if $X \subset O_3$, we have $\xi_i \subset \omega_1 \cap \omega_2$.

According to Construction D, we get Γ_B back from Π_1, Π_2, Π_3 by adding the fifteen points of the hyperplane Ω, earlier removed from Γ_B. One can recover them as suitable triples $\{e_0, e_1, e_2\}$ of lines of $\Gamma_B(X)$, with e_i and edge of Π_i for $i = 0, 1, 2$.

1.10 A variation of Construction D

We now mention a "translation" of Construction D into the language of projective planes, viz. $\mathbf{PG}(2,4)$. Since we will not need this construction anymore in the sequel, we mention it without proof.

Given a hyperoval $\mathcal{H}$ of $\mathbf{PG}(2,4)$, let L be a line of $\mathbf{PG}(2,4)$ exterior to $\mathcal{H}$. Let Π be the graph with the ten points of $\mathbf{PG}(2,4)$ not on $\mathcal{H} \cup L$ as vertices, two such points p_1, p_2 being adjacent in Π when the line p_1p_2 through them is a secant of $\mathcal{H}$. It is well known that Π is isomorphic to the Petersen graph.

For $k = 0, 1, 2$, let $\mathbf{P}_k$ be a copy of $\mathbf{PG}(2,4)$ and let $\mathcal{H}_k$ and L_k be corresponding copies of $\mathcal{H}$ and L in $\mathbf{P}_k$. The vertices of the graph Π_k considered in subsection 1.8 are the ten points of $\mathbf{P}_k$ not in $\mathcal{H}_k \cup L_k$ and the edges of Π_k are the 15 secant lines of $\mathcal{H}_k$. Assuming that $(\mathbf{P}_k \setminus L_k) \cap (\mathbf{P}_h \setminus L_h) = \emptyset$ for $0 \leq k < h \leq 2$, we can define Γ_D as follows. Select an arbitrary element $\varepsilon \in \mathbf{GF}(4) \setminus \mathbf{GF}(2)$. For any line M in $\mathbf{PG}(2,4)$, we denote by M_k the copy of M in $\mathbf{P}_k$, $k \in \{0,1,2\}$. Now let M be secant to $\mathcal{H}$. For any $k \in \{0,1,2\}$, there are unique secants M', M'' of $\mathcal{H}$ such that L, M, M' and M'' pass through the same point and the cross-ratio $(L, M; M', M'') = \varepsilon$. For $k \in \{0,1,2\}$ we set $f_{k,1}(M) := M'_{k+1}$ and $f_{k,2}(M) := M''_{k+2}$ (reading subscripts modulo 3). Note that $(L, M; M', M'') = (L, M'; M'', M)$, as we are in $\mathbf{GF}(4)$. Hence $f_{k+1,1}(M') = M''$ and $f_{k+1,2}(M') = M$. Therefore, $\{\{M_k, f_{k,1}(M), f_{k,2}(M)\} \mid M \text{ secant to } \mathcal{H}\}$ is a partition of the set of 45 secants of $\mathcal{H}_i$, $i = 0, 1, 2$, into 15 classes of 3 elements, independent of $k \in \{0,1,2\}$. We call such a class an *ideal set of secants*.

(D1') The points of Γ_D are the 30 points of $\mathbf{P}_k$ not in $\mathcal{H}_k \setminus L_k$, $k = 0, 1, 2$, together with the 15 ideal sets of secants.

(D2') The lines of Γ are the 45 secants of $\mathcal{H}_k$, $k = 0, 1, 2$.

(D3') Incidence is natural.

In fact, using this variation, one can show directly the equivalence of Constructions C and D. We leave this to the reader.

2 Three new constructions

2.1 Construction E

Embedded $\mathbf{PG}(2,4)$ as a plane P_0 in $\mathbf{PG}(3,4)$, let p_0 be a point of $\mathbf{PG}(3,4)$ not in P_0. Given a hyperoval $\mathcal{H}$ in P_0, let $\mathcal{H}^*$ be the dual hyperoval of P_0 formed by the lines of P_0 exterior to $\mathcal{H}$.

We first note that, given a point $p \in P_0 \setminus \mathcal{H}$ and denoted L_1, L_2, L_3 the three secants of $\mathcal{H}$ through p, the set $(L_1 \cup L_2 \cup L_3) \setminus \mathcal{H}$ is the point set of a Baer subplane B of P_0. We call it the Baer subplane of P_0 *exterior* to $\mathcal{H}$ and *pivoted* on p. Clearly, the lines L_1, L_2, L_3 do not belong to $\mathcal{H}^*$. The remaining four lines of B belong to $\mathcal{H}^*$.

Let now x be a point of $\mathbf{PG}(3,4) \setminus P_0$ different from p_0 and such that $p_0x \cap P_0 \notin \mathcal{H}$. Let B be the Baer subplane of P_0 exterior to $\mathcal{H}$ pivoted on $p := p_0x \cap P_0$. Then $B \cup \{x, p_0\}$ is contained in a unique induced subgeometry of $\mathbf{PG}(3,4)$ isomorphic to $\mathbf{PG}(3,2)$. We call that subgeometry the $\mathbf{GF}(2)$-*closure* of x *relative* to $\mathcal{H}$.

We are now ready to define the geometry Γ_E.

(E1) the points of Γ_E are the points x of $\mathbf{PG}(3,4) \setminus P_0$, different from p_0 and such that the line p_0x does not meet $\mathcal{H}$;

(E2) the lines of Γ_E are the planes X of $\mathbf{PG}(3,4)$ different from P_0 and such that $p_0 \notin X$ and $X \cap P_0$ does not belong to $\mathcal{H}^*$;

(E3) a point x and a plane X of Γ_E are declared to be incident in Γ_E when they are not incident in $\mathbf{PG}(3,4)$, the lines $L := p_0x$ and $L' := X \cap P_0$ are concurrent and X is contained in the $\mathbf{GF}(2)$-closure of x relative to $\mathcal{H}$.

2 Theorem. $\Gamma_E \cong \widetilde{\mathsf{W}(2)}$.

Proof. We shall prove that $\Gamma_E \cong \Gamma_C$, the latter being described as in the variation (C1'), (C2') of Construction C. The point a_0 and the plane P_0 of (C1'), (C2') correspond to p_0 and P_0 of (E1), (E2). The hyperoval $\mathcal{H}$ and the dual hyperoval $\mathcal{H}^*$ of (E1), (E2) can be regarded as the intersections of P_0 with H^* and H respectively, where H^* and H are as in (C1'), (C2').

Given a line $\Xi = \{x_1, x_2, x_3\}$ of Γ_C, let X be the plane of $\mathbf{PG}(3,4)$ containing Ξ. Then $p_0 \in X$ and, as noticed when we have rephrased (C1'), (C2') as (C1"), (C2") (Subsection 1.7) the points p_0, x_1, x_2, x_3 belong to a Baer subplane B_Ξ of X, the remaining three points of which are the intersections $p_i := p_0x_i \cap P_0$, $i = 1,2,3$. For every $i = 1,2,3$, the line $L = \{p_1, p_2, p_2\}$ belongs to the Baer subplane B_i of P_0 exterior to $\mathcal{H}$ and pivoted on p_i. Hence it belongs to the $\mathbf{GF}(2)$-closure $S_{i,\Xi}$ of x_i relative to $\mathcal{H}$. Note that $L = B_\Xi \cap P_0 = X \cap (P_0 \setminus \mathcal{H})$

and $S_{i,\Xi} \cap X = B_\Xi$. Let X_i be the unique plane of $\mathbf{PG}(3,4)$ spanned by the plane of $S_{i,\Xi}$ containing L and different from B_Ξ and B_i. An elementary calculation shows that $X_i = X_j$ for $i,j = 1,2,3$. Hence $X_\Xi := X_i$ is the unique plane of $\mathbf{PG}(3,4)$ incident in Γ_E with x_1, x_2 and x_3.

Let Θ be another line of Γ_C contained in X. (Note that Θ and Ξ are disjoint as lines of Γ_C.) Then $B_\Theta \cap P_0 = L$. So, if $X_\Theta = X_\Xi$, then $S_{i,\Xi} \cap S_{i,\Theta}$ contains B_i, the point p_0 and all points $p_0 p \cap X_\Xi$ for $p \in B_i \setminus L$. This forces $S_{i,\Theta} = S_{i,\Xi}$, whence $B_\Theta = B_\Xi$ and, consequently, $\Theta = \Xi$. Therefore, the function sending every line Ξ of Γ_C to the above defined plane X_Ξ is injective. It is now clear that this function, matched with the identity mapping on the set of points of Γ_C, yields an isomorphism from Γ_C to Γ_E. $\boxed{QED}$

2.2 Construction F

Embedded $\mathbf{PG}(1,4)$ as a line L_0 in $\mathbf{PG}(2,4)$, put $\mathbf{AG}(2,4) := \mathbf{PG}(2,4) \setminus L_0$ and let p_0 be a point of $\mathbf{AG}(2,4)$.

3 Lemma. *Let x be a point of $\mathbf{AG}(2,4)$ different from p_0 and put $p := p_0 x \cap L_0$ and $\{p_1, p_2, p_3, p_4\} := L_0 \setminus \{p\}$. Then six out of the 162 hyperovals of $\mathbf{PG}(2,4)$ contain p_0 and x and admit L_0 as a secant. They bijectively correspond to the six pairs of $L_0 \setminus \{p\}$.*

For $1 \leq i < j \leq 4$, let H_{ij} be the hyperoval on p_0, x, p_i and p_j and let $x_{i,j;1}, x_{i,j;2}$ be the two points of $H_{ij} \cap \mathbf{AG}(2,4)$ different from p_0 and x. Then, for every partition $\{\{p_i, p_j\}, \{p_k, p_h\}\}$ of $L_0 \setminus \{p\}$, the set $\{x_{i,j;1}, x_{i,j;2}, x_{k,h;1}, x_{k,h;2}\}$ is a line of $\mathbf{AG}(2,4)$ with p as the point at infinity. This establishes a bijection between the three partitions of $L_0 \setminus \{p\}$ and the three lines of $\mathbf{AG}(2,4)$ parallel to $p_0 x$ and different from $p_0 x$. In particular, we have

$$H_{ij} \cap H_{kh} \cap \mathbf{AG}(2,4) = \{p_0, x\}$$

for any two distinct (but possibly non-disjoint) pairs $\{i,j\}$, $\{k,h\}$ of $L_0 \setminus \{p\}$.

The proof is straightforward; we leave it for the reader. As a consequence:

4 Corollary. *With x, p as above, let H_1, H_2 be two hyperovals of $\mathbf{PG}(2,4)$ containing p_0 and x and with L_0 as a secant. Let y_1, y_2 (respectively z_1, z_2) be the points of $H_1 \cap \mathbf{AG}(2,4)$ (respectively $H_2 \cap \mathbf{AG}(2,4)$) different from p_0 and x. Then the following are equivalent:*

(a) *the lines $p_0 x, p_0 y_1, p_0 y_2, p_0 z_1, p_0 z_2$ are mutually distinct;*

(b) *the points y_1, y_2, z_1, z_2 are collinear.*

We are now ready to define Γ_F.

(F1) All points of $\mathbf{AG}(2,4)$ different from p_0 are points of Γ_F.

(F2) Γ_F has 30 more points besides the above, namely the triples of points x_1, x_2, x_3 of $\mathbf{AG}(2,4) \setminus \{p_0\}$ contained in some hyperoval of $\mathbf{PG}(2,4)$ passing through p_0 and with two points on L_0.

(F3) The lines of Γ_F are the triples $\{x, \{x, y_1, y_2\}, \{x, z_1, z_2\}\}$ of points of Γ_F such that y_1, y_2, z_1, z_2 satisfy the (equivalent) conditions (a) and (b).

(F4) The incidence relation is the natural one.

5 Theorem. $\Gamma_F \cong \widetilde{\mathsf{W}(2)}$.

Proof. We shall prove that $\Gamma_F \cong \Gamma_C$, but we shall also use something from Subsection 2.1. With p_0, P_0, $\mathcal{H}$ and $\mathcal{H}^*$ as in Subsection 2.1, let P_1 be a plane through p_0 meeting P_0 in a line $L_0 \in \mathcal{H}^*$. The points of $P_1 \setminus L_0$ different from p_0 are the 15 points of Γ_F considered in (F1).

Let x be a point of Γ_C not in P_1. Then the points of P_1 collinear with x in Γ_E belong to the $\mathbf{GF}(2)$-closure S_x of x relative to $\mathcal{H}$ (see Subsection 2.1). The structure S_x meets P_0 in a Baer subplane B_x of P_0 (namely, the Baer subplane of P_0 exterior to $\mathcal{H}$ and pivoted on $p_0x \cap P_0$). Furthermore, if z is a point of $P_1 \cap S_x$, then the point $p_0z \cap L_0$ belongs to B_x. For every point p of B_x different from $p_0x \cap P_0$, the two lines of $\mathcal{H}^*$ through p belong to B_x. Therefore L_0 belongs to B_x and the points of P_1 collinear with x in Γ_C are the three points of $(S_x \cap P_1) \setminus L_0$ different from p_0. They form a conic O_x of the Baer subplane $S_x \cap P_1$ of P_1, with p_0 as the nucleus. Clearly, O_x is a point of Γ_F as defined in (F2).

Let f be the function sending every point x of Γ_C exterior to P_1 to the above defined triple O_x. It is not hard to see that f is injective, and we leave this to the interested reader to check. As the points of Γ_F of type (F2) are as many as the points of Γ_C not in P_1, the function f is surjective, too.

Now let Ξ be a line of Γ_C and let X be the plane of $\mathbf{PG}(3,4)$ spanned by Ξ, the latter being regarded as a triple of points. Then $p_0 \in X$ and the two lines of X through p_0 that do not meet Ξ meet $\mathcal{H}$. Hence none of them belongs to P_1. Consequently, $\Xi \cap P_1$ is a point, say x. Let x_1, x_2 be the remaining two points of Ξ. The points O_1 and O_2 of Γ_F corresponding to x_1 and x_2 via f are triples of points of P_1 containing x. For $i = 1, 2$, let $x_{i,1}$ and $x_{i,2}$ be the two points of O_i different from x.

Put $p := p_0x \cap P_0$, $p_1 := p_0x_1 \cap P_0$ and $p_2 := p_0x_2 \cap P_0$. So, p, p_1, p_2 are the three points of $L := X \cap P_0$ not in $\mathcal{H}$. For $i = 1, 2$, the Baer subplane B_i of P_0 exterior to $\mathcal{H}$ and pivoted on p_i contains p, p_1, p_2 and the points $p_{i,j} := p_0x_{i,j} \cap P_0$ for $j = 1, 2$. Thus, $\{p, p_1, p_2\} \subseteq B_1 \neq B_2$. If $B_1 \cap B_2$ contained a fourth point

$q \notin \{p, p_1, p_2\}$, then it would also contain the points of the lines qp, qp_1 and qp_2 exterior to $\mathcal{H}$. Thus, $B_1 = B_2$, which is not the case. Therefore, p, p_1 and p_2 are the only points of $B_1 \cap B_2$.

Suppose now that one of the lines $p_0x_{1,1}, p_0x_{1,2}$ coincides with one of the lines $p_0x_{2,1}, p_0x_{2,2}$, say $p_0x_{1,1} = p_0x_{2,1}$. Then $B_1 \cap B_2$ contains p, p_1, p_2 and the point $q := p_{1,1} = p_{2,1}$, contrary to what we have remarked above. Therefore, the lines $p_0x_{1,1}$, $p_0x_{1,2}$, $p_0x_{2,1}$ and $p_0x_{2,2}$ are pairwise distinct. So, the triple

$$\{x,\ \{x, x_{1,1}, x_{1,2}\},\ \{x, x_{2,1}, x_{2,2}\}\}$$

satisfies (a) (equivalently, (b)) of Corollary 4. The number of triples as above satisfying (b) of Corollary 4 is easy to compute. It turns out to be equal to 45, which is the number of lines of Γ_C. The isomorphism $\Gamma_F \cong \Gamma_C$ is proved. $\boxed{QED}$

2.3 Construction G

Let $\mathcal{S}$ be any spread of $\mathbf{PG}(3,2)$. We recall that all spreads of $\mathbf{PG}(3,2)$ are regular and projectively equivalent [17, 8.3]. Define Γ_G as follows:

(G1) The points of Γ_G are the 15 points of $\mathbf{PG}(3,2)$ together with the 30 lines of $\mathbf{PG}(3,2)$ that are not contained in $\mathcal{S}$.

(G2) The lines of Γ_G are the 45 planar line pencils containing an element of $\mathcal{S}$.

(G3) A point p of $\mathbf{PG}(3,2)$, regarded as a point of Γ_G, is incident with a line Φ of Γ_G if it belongs to all lines of the pencil Φ.

(G4) A line of $\mathbf{PG}(3,2)$ not belonging to $\mathcal{S}$, regarded as a point of Γ_G, is incident with a line Φ of Γ_G if it belongs to the pencil Φ.

6 Theorem. $\Gamma_G \cong \widetilde{\mathsf{W}(2)}$.

Proof. We shall prove that $\Gamma_G \cong \Gamma_F$. The points of Γ_F are points and suitable triples of points of $P_1 \setminus (L_0 \cap \{p_0\})$. The latter can be regarded as the set of nonzero vectors of $V = V(2,4)$. The additive group V^+ of V can be viewed as a copy of $V(4,2)$ and the five 1-subspaces of V are lines of $\mathbf{PG}(V^+) \cong \mathbf{PG}(3,2)$, forming a spread $\mathcal{S}$ of $\mathbf{PG}(3,2)$. So, the points of Γ_F are the nonzero vectors of V^+ (namely, the points of $\mathbf{PG}(3,2)$) and the 2-subspaces of V^+ different from the members of $\mathcal{S}$ (compare variation (C1"), (C2") of Construction C, Subsection 1.7). The lines of Γ_F are characterized by condition (b) of Corollary 4. The isomorphism $\Gamma_G \cong \Gamma_F$ is now straightforward. $\boxed{QED}$

2.4 A few variations of Construction G

The Klein correspondence from $\mathbf{PG}(3,2)$ to the hyperbolic quadric $Q^+(5,2)$ sends the spread $\mathcal{S}$ of the previous subsection to an ovoid $\mathcal{O}$ of $Q^+(5,2)$, namely a set of (five) mutually non-collinear points meeting each plane of $Q^+(5,2)$ in one point. It also sends Γ_G to the following substructure of $Q^+(5,2)$:

(G1') the points are the 30 points of $Q^+(5,2) \setminus \mathcal{O}$ and the planes of $Q^+(5,2)$ belonging to one of the two families of generators of planes of $Q^+(5,2)$;

(G2') the lines are the 45 lines of $Q^+(5,2)$ incident to some element of $\mathcal{O}$;

(G3') incidence is natural.

We can also formulate another construction by taking images of the elements of Γ_G under a symplectic polarity of $\mathbf{PG}(3,2)$ for which all elements of $\mathcal{S}$ are singular lines. We leave this to the interested reader.

3 Dualities and polarities

We first recall a few definitions. A *duality* (also, *correlation*) of a geometry Γ of rank 2 is a non-type-preserving automorphism of Γ. If Γ admits a duality then it is said to be *self-dual*. The involutory dualities of Γ are called *polarities*. An element x of Γ is *absolute* for a given polarity δ of Γ if x and x^δ are incident.

Let Γ be a projective plane or the point-plane system of a 3-dimensional projective geometry and let δ be a polarity of Γ. Following [18], if the set of absolute points of Γ is a proper (but possibly empty) subspace of Γ, then we call δ a *pseudo polarity* (but we warn that, in spite of this name, pseudo polarities are indeed polarities).

We now turn to $\widetilde{\mathsf{W}(2)}$. Considering its definition as a coset geometry, it is not so difficult to see that $\widetilde{\mathsf{W}(2)}$ is self-dual. (Indeed, $\mathrm{Out}(3^{\cdot}\mathbf{S}(6)) = Z_2$ and every representative in $\mathrm{Aut}(3^{\cdot}\mathbf{S}(6))$ of the nontrivial element of $\mathrm{Out}(3^{\cdot}\mathbf{S}(6))$ induces a duality on $\widetilde{\mathsf{W}(2)}$.)

However, the above is also clear from Construction E. With the notation of Subsection 2.1, let δ be a duality of the projective space $\mathbf{PG}(3,4)$ permuting p_0 with P_0 and the points of $\mathcal{H}$ with the planes through p_0 intersecting P_0 in lines of $\mathcal{H}^*$. Then δ induces a duality on Γ_E ($\cong \widetilde{\mathsf{W}(2)}$).

Note that we can even choose a pseudo polarity of $\mathbf{PG}(3,4)$ as δ. More explicitly, let δ be a pseudo polarity of $\mathbf{PG}(3,4)$ permuting p_0 with P_0, $\mathcal{H}$ with the set of planes on p_0 meeting P_0 in lines of $\mathcal{H}^*$ and such that the absolute points (planes) of δ are the points of a plane P through p_0 with $P \cap P_0 \in \mathcal{H}^*$ (the planes containing a given point $p \in \mathcal{H}$). Then δ induces a polarity on Γ_E.

It is an elementary exercise to calculate that, for any line $L \in \mathcal{H}^*$ and any point $p \in \mathcal{H}$, there exists exactly one pseudo polarity of P_0 interchanging $\mathcal{H}$ with $\mathcal{H}^*$ and such that all points on L are absolute points and all lines through p are absolute lines. Hence there are $36 = 6 \times 6$ such pseudo polarities. Such a pseudo polarity can be extended in exactly three different ways to a pseudo polarity of $\mathbf{PG}(3,4)$ permuting p_0 and P_0.

On the other hand, every polarity of Γ_E induces a polarity of $\mathsf{W}(2)$, the latter being regarded as the geometry formed by the points and lines of P_0 exterior to $\mathcal{H}$ and $\mathcal{H}^*$, respectively. Furthermore, every polarity of $\mathsf{W}(2)$ arises from a pseudo polarity of P_0. Hence $\widetilde{\mathsf{W}(2)}$ ($\cong \Gamma_E$) admits exactly $36 \times 3 = 108$ polarities (whereas $6! \times 3$ is the total number of dualities of $\widetilde{\mathsf{W}(2)}$).

Now let us see what the set of absolute points of a polarity of $\widetilde{\mathsf{W}(2)}$ looks like. Let δ be a pseudo polarity of $\mathbf{PG}(3,4)$ inducing a polarity of $\Gamma_E \cong \widetilde{\mathsf{W}(2)}$, as above. Let L be the unique line of P_0 containing all absolute points of δ in P_0 and let x be any point of Γ_E. If x^δ is incident in Γ_E with x, then p_0x meets the line $P_0 \cap x^\delta$ and, consequently, the point $p_0x \cap P_0$ is absolute for the pseudo polarity induced by δ in $P_0 = \mathbf{PG}(2,4)$. So, all points of Γ_E that are absolute for δ belong to the plane P on p_0 and L. Also, every point x of Γ_E in P is mapped by δ onto a plane which is incident with exactly one point of Γ_E on the line p_0x. Since there are three such points, either one or three of them are absolute. But it is an easy exercise to see that, if a point x of Γ_E is absolute for δ (in Γ_E), then none of the four points at distance 3 (in the incidence graph of Γ_E) from the line x^δ of Γ_E and lying in the plane P can be absolute. These four points are collinear in $\mathbf{PG}(3,4)$. Thus, we obtain exactly one absolute point on each line of P through p_0.

Let $\mathcal{O}$ be the set of absolute points together with the point p_0. We have just shown that, for each point $x \in \mathcal{O} \setminus \{p_0\}$, there is a line $L_x \neq L$ with L, L_x, p_0x concurrent and such that L_x is disjoint from $\mathcal{O}$. It is easy to deduce from this property that $\mathcal{O}$ is a hyperoval. All such hyperovals are projectively equivalent with respect to p_0 and L. Furthermore, as the 15 points of Γ_E in P form an ovoid of Γ_E (compare Subsection 2.2, final Remark), $\mathcal{O} \setminus \{p_0\}$ is contained in an ovoid of Γ_E.

The above can be summarized as follows:

7 Proposition. *The geometry* $\widetilde{\mathsf{W}(2)}$ *admits* 108 *polarities. They form one conjugacy class in the group of all* (*possibly non-type-preserving*) *automorphisms of* $\widetilde{\mathsf{W}(2)}$. *Every polarity of* $\widetilde{\mathsf{W}(2)}$ *has five absolute points and these five points belong to an ovoid of* $\widetilde{\mathsf{W}(2)}$. *Also, the full automorphism group of* Γ_E *(including* *dualities) is induced by the full automorphism group of* $\mathbf{PG}(3,4)$.

4 Embeddings and generating rank

4.1 Preliminaries

In this subsection we shall recall some definitions and a few general statements on embeddings. We will turn back to $\widetilde{\mathsf{W}(2)}$ in the remaining subsections. We will state the propositions of this introductory subsection in a more general form than strictly needed in this paper, but our exposition will not become much longer because of that.

Let $\Gamma = (P, \mathcal{L})$ be a point-line geometry, with P (respectively $\mathcal{L}$) as the set of points (lines). We assume that Γ is connected, that every point (line) of Γ is incident to at least two lines (points) and that no two distinct lines have the same set of points, so that the lines of Γ can be regarded as subsets of P.

A *subspace* of Γ is a subset $S \subseteq P$ such that, for every line $L \in \mathcal{L}$, either $L \subseteq S$ or L has at most one point in S. Given a subset $X \subseteq P$, the *span* $\langle X \rangle_\Gamma$ of X in Γ is the minimal subspace of Γ containing X, namely the intersection of all subspaces of Γ containing X. If $\langle X \rangle_\Gamma = P$ then X is said to *span* (also, to *generate*) Γ. The *generating rank* $\mathrm{grk}(\Gamma)$ of Γ is the minimal size of a spanning set of Γ.

Embeddings. Henceforth we also assume that no two distinct lines of Γ meet in more than one point. Then we can consider projective embeddings of Γ. According to [18], given a division ring $\mathbb{K}$, a *lax embedding* of Γ *defined over* $\mathbb{K}$ is an injective mapping ε from the set of elements (points and lines) of Γ to the projective geometry $\mathbf{PG}(V)$ of 1- and 2-dimensional subspaces of a $\mathbb{K}$-vector space V, sending points to points and lines to lines and such that:

(I) for every point p and every line L of Γ, we have $p \in L$ if and only $p^\varepsilon \in L^\varepsilon$;

(II) P^ε spans $\mathbf{PG}(V)$.

If furthermore

(III) for every line L of Γ, ε maps the set of points of L onto the set of points of L^ε,

then we say that ε is *full*. On the other hand, we say that ε is *flat* if it satisfies the following:

(IV) for every point p of Γ, denoted by $p^\perp$ the set of points of Γ collinear with p or equal to p, the set $(p^\perp)^\varepsilon$ spans a plane of $\mathbf{PG}(V)$.

Clearly, if Γ admits a full embedding defined over $\mathbb{K}$, then all lines of Γ have $1 + |\mathbb{K}|$ points. In the finite case, this condition uniquely determines $\mathbb{K}$ and we may omit to mention $\mathbb{K}$ explicitly.

Note that the full embeddings as defined above are the projective embeddings in the meaning of Ronan [12] (also Shult [14]). Accordingly, we will use the shortened expression *projective embedding* only for full embeddings. When the embeddings we consider are possibly non-full, we shall explicitly call them lax embeddings.

Morphisms of embeddings. Following Ronan [12] and Shult [14], we define morphisms of embeddings as follows: Given two lax embeddings $\varepsilon : \Gamma \to \mathbf{PG}(V)$ and $\eta : \Gamma \to \mathbf{PG}(W)$ defined over the same division ring $\mathbb{K}$, a *morphism* $\varphi : \varepsilon \to \eta$ is a semilinear mapping $\varphi : V \to W$ such that $\eta = \varphi\varepsilon$. If the semilinear mapping φ is invertible, then we say that φ is an *isomorphism* from ε to η. (Needless to say, we take the identity mappings on V and W as the identity morphisms of ε and η.) If there exists a morphism from ε to η, then we say that η is a *quotient* of ε. If an isomorphism from ε to η exists, then we say that ε and η are *isomorphic* and we write $\varepsilon \cong \eta$, as usual.

The following is obvious:

8 Lemma. *Let φ_1, φ_2 be semilinear mappings from V to W and, given a collection $\mathcal{X}$ of subspaces of V of dimension at least 2, let $(\mathcal{X}, \sim)$ be the graph with vertex set $\mathcal{X}$, where $X \sim Y$ if $X \cap Y \neq 0$, for any two distinct members X, Y of $\mathcal{X}$. Assume the following:*

(*i*) *the graph $(\mathcal{X}, \sim)$ is connected;*

(*ii*) *the set $\mathcal{X}$ generates V;*

(*iii*) *for every $X \in \mathcal{X}$, the semilinear mappings induced by φ_1 and φ_2 on X are proportional,* i.e.*, they only differ by a scalar factor.*

Then φ_1 and φ_2 are proportional.

The previous lemma implies the next proposition which, in spite of its importance and straightforwardness, does not seem to have been ever mentioned in the literature:

9 Proposition. *Suppose the embedding $\varepsilon : \Gamma \to \mathbf{PG}(V)$ is full and let $\eta : \Gamma \to \mathbf{PG}(W)$ be a quotient of ε. Then the morphism from ε to η is unique, modulo a scalar factor.*

Proof. Given morphisms φ_1 and φ_2 from ε to η, let $\mathcal{X} = \mathcal{L}^\varepsilon = \{L^\varepsilon\}_{L \in \mathcal{L}}$, where $\mathcal{L}$ is the set of lines of Γ. For every member X of $\mathcal{X}$, the equality $\varphi_1\varepsilon = \varphi_2\varepsilon = \eta$, the injectivity of η and the fact that ε is full, force φ_1 and φ_2 to induce proportional semilinear mappings on X. Clearly, (i) and (ii) of Lemma 8 hold on $\mathcal{X}$, as Γ is connected and P^ε spans $\mathbf{PG}(V)$. The conclusion follows from Lemma 8. $\boxed{QED}$

So, the category $\mathrm{Emb}(\Gamma)$ of the full projective embeddings of Γ defined over $\mathbb{K}$, with morphisms defined as above, is a preorder. We may also regard it as a poset, by taking its objects modulo isomorphisms. We will do so in the sequel, writing $\varepsilon \geq \eta$ when η is a quotient of ε.

The universal embedding. Suppose that $\mathrm{Emb}(\Gamma)$ admits a unique maximal element, namely it contains an embedding $\tilde{\varepsilon} : \Gamma \to \mathbf{PG}(\widetilde{V})$ such that $\tilde{\varepsilon} \geq \varepsilon$ for every $\varepsilon \in \mathrm{Emb}(\Gamma)$. Then, following Shult [14], we call $\tilde{\varepsilon}$ the (*absolutely*) *universal* embedding of Γ and $\dim(\widetilde{V})$ the *embedding rank* of Γ. The embedding rank of Γ will be denoted $\mathrm{erk}(\Gamma)$ in the sequel. Clearly, $\mathrm{erk}(\Gamma) \leq \mathrm{grk}(\Gamma)$.

The reader is referred to Kasikova and Shult [9] for conditions sufficient for the existence of the universal embeddings. We only remark here that, when all lines of Γ have size 2 (as in the geometry $\widetilde{\mathsf{W}(2)}$), the universal embedding exists (provided that $\mathrm{Emb}(\Gamma) \neq \emptyset$, of course). It arises from the *universal representation module* of Γ (Ivanov [6]), namely the group $M(\Gamma)$ presented by the following relations on the set of generators $\{r_x\}_{x \in P}$:

$$\begin{array}{lrl}
(1) & r_x^2 = 1, & \text{for all } x \in P; \\
(2) & r_x r_y r_z = 1, & \text{for all } \{x, y, x\} \in \mathcal{L}; \\
(3) & r_x r_y = r_y r_x, & \text{for all } x, y \in P.
\end{array}$$

Kernels. Turning back to the general case, given a lax embedding $\varepsilon : \Gamma \to \mathbf{PG}(V)$ and a quotient $\eta : \Gamma \to \mathbf{PG}(W)$ of ε, let φ be a morphism from ε to η. By replacing η with $\alpha\eta$ for a suitable automorphism α of W if necessary, we may assume that φ is linear. As the restriction of φ to P^ε is injective, the kernel $U := \mathrm{Ker}(\varphi)$, regarded as a (possibly empty) subspace of $\mathbf{PG}(V)$, satisfies the following:

(V) $\langle U \cup \{x\} \rangle \cap P^\varepsilon = \{x\}$ for all points $x \in P^\varepsilon$.

Also, modulo replacing η and φ with $\beta\eta$ and $\beta\varphi$ for a suitable isomorphism $\beta : W \to V/U$, we may also assume that $W = V/U$ and φ is the quotient map from V to V/U.

Conversely, if a subspace U of V satisfies (V) then, denoted by φ_U the quotient map from V to V/U, the mapping $\varepsilon_U := \varphi_U \varepsilon$ is an embedding and φ_U is a morphism from ε to ε_U. We call ε_U the *quotient* of ε by U. Accordingly, the subspaces of V satisfying (V) will be said to *define quotients* of ε.

Note that, if ε is non-full, different subspaces of V satisfying (V) might define isomorphic quotients of ε. However,

10 Proposition. *Suppose that ε is full and let U_1, U_2 be distinct subspaces of V satisfying* (V)*. Then $\varepsilon_{U_1} \ncong \varepsilon_{U_2}$.*

Proof. For $i = 1, 2$, let φ_i the quotient map from V to V/U_i. Suppose there exists an isomorphism α from ε_{U_1} to ε_{U_2}. Then both $\alpha\varphi_1$ and φ_2 are morphisms from ε to ε_{U_2}. Hence φ_2 and $\alpha\varphi_1$ are proportional as semilinear mappings from V to V/U_2, by Proposition 9. Therefore (α is linear and) $U_1 = U_2$. $\boxed{QED}$

Homogeneous embeddings. Given a lax embedding $\varepsilon : \Gamma \to \mathbf{PG}(V)$ of Γ, we denote by $\mathrm{Aut}_\varepsilon(\Gamma)$ the stabilizer of $\Gamma^\varepsilon := (P^\varepsilon, \mathcal{L}^\varepsilon)$ in $\mathbf{P\Gamma L}(V)$, regarded as group of automorphisms of Γ. The embedding ε is said to be *homogeneous* when $\mathrm{Aut}_\varepsilon(\Gamma)$ is the full (type-preserving) automorphism group of Γ. Clearly, the universal embedding (when it exists) is homogeneous.

We shall now state a few properties of homogeneous embeddings. In view of that, we need the following definition. A subset A of the point set of a projective space $\mathbf{PG}(V)$ is called *rigid* if the pointwise stabilizer of A in the linear projective group $\mathbf{PGL}(V)$ is trivial.

11 Lemma. *Let V be an n-dimensional vector space over some skew field $\mathbb{K}$. Let P be a set of points of $\mathbf{PG}(V)$ and let U be a subspace of V such that $\langle U, x\rangle \cap P = x$, for all $x \in P$ (compare Condition* (V)*). Let $\varphi : (\mathbf{PG}(V) \setminus \mathbf{PG}(U)) \to \mathbf{PG}(V/U)$ be the projection map naturally associated to the quotient map $V \to V/U$. Suppose that there is a collection $\mathcal{X}$ of subspaces of V of dimension at least* 2 *satisfying conditions* (i) *and* (ii) *of Lemma 8 and such that:*

(i) *$\mathbf{PG}(X) \cap P$ is rigid in $\mathbf{PG}(X)$ for all $X \in \mathcal{X}$;*

(ii) *the restriction of φ to $\mathbf{PG}(X)$ is injective, for all $X \in \mathcal{X}$.*

Let α be a permutation of P and suppose that there exist collineations α_1 and α_2 of $\mathbf{PG}(V)$ and $\mathbf{PG}(V/U)$, respectively, stabilizing P and $\varphi(P)$, respectively, and such that $\alpha_1(x) = \alpha(x)$ and $\alpha_2(\varphi(x)) = \varphi(\alpha(x))$, for all $x \in P$. If the associated field automorphisms of α_1 and α_2 coincide, then α_1 stabilizes U.

Proof. The semilinear projective function $\alpha_2^{-1}\varphi\alpha_1 : \mathbf{PG}(V) \to \mathbf{PG}(V/U)$ has by assumption trivial associated field automorphism. Hence it is linear. Furthermore, it coincides with φ on the set P. Hence it coincides with φ on $\mathbf{PG}(X)$ for every $X \in \mathcal{X}$, by (i) and (ii). The conclusion follows from Lemma 8. $\boxed{QED}$

12 Lemma. *Suppose that each line of Γ has at least three points and, given two lax embeddings ε_1 and ε_2 of Γ in $\mathbf{PG}(V_1)$ and $\mathbf{PG}(V_2)$ respectively (with*

V_1 and V_2 defined over the same division ring $\mathbb{K}$), let $\varphi : V_1 \to V_2$ be a linear mapping such that $\varepsilon_2 = \varphi\varepsilon_1$. Let α be a type-preserving automorphism of Γ such that $\alpha_i\varepsilon_i = \varepsilon_i\alpha$ for suitable $\alpha_i \in \mathbf{P\Gamma L}(V_i)$ and $i = 1, 2$. Suppose that one of the following conditions is satisfied:

(1) *ε_1 is full, or*

(2) *$\alpha_i \in \mathbf{PGL}(V_i)$ for $i = 1, 2$.*

Then Ker(φ) is stabilized by α_1.

PROOF. Let $\mathcal{X}$ be the set of lines of $\varepsilon_1(\Gamma)$. Then $\mathcal{X}$ satisfies conditions (i) and (ii) of Lemma 8. Condition (i) of Lemma 11 follows from the injectivity of ε_1 and the hypothesis that all lines of Γ have at least three points. Condition (ii) of Lemma 11 follows from the equality $\varepsilon_2 = \varphi\varepsilon_1$ and the injectivity of ε_2. Finally, Conditions (1) and (2) both independently imply that the field automorphisms associated to α_1 and to α_2 coincide. It is also clear that we can view V_2 as a quotient space of V_1 with φ as associated quotient mapping. The conclusion now follows from Lemma 11. $\boxed{QED}$

Note that Condition (2) of Proposition 12 is automatically satisfied if, for instance, the skew field $\mathbb{K}$ has no nontrivial field automorphisms. Also, the fact that we assume that φ is linear is not really a restriction; we can always apply a "field automorphism" to ε_2 and obtain an isomorphic embedding.

In particular, if we take for ε_1 the universal full embedding, then Lemma 12 just says the following.

13 Proposition. *If Γ admits a universal embedding $\tilde{\varepsilon} : \Gamma \to \mathbf{PG}(\tilde{V})$, then all full homogeneous embeddings of Γ are quotients of $\tilde{\varepsilon}$ by a subspace U of $\tilde{V}$ satisfying Condition* (V) *and stabilized by* $\mathrm{Aut}_{\tilde{\varepsilon}}(\Gamma)$.

The Grassmannian of a flat embedding. Let ε be a flat lax embedding of $\Gamma = (P, \mathcal{L})$ in $\mathbf{PG}(V)$ with $\dim(V) > 3$ and let Δ be the Grassmannian of lines of $\mathbf{PG}(V)$, namely the geometry with the lines of $\mathbf{PG}(V)$ as points and the point-plane flags of $\mathbf{PG}(V)$ as lines, with the natural incidence relation. Suppose furthermore that $\mathbb{K}$ is a field. Then Δ admits a (full) embedding δ in $\mathbf{PG}(V \wedge V)$. As ε is assumed to be flat, for every point $p \in P$ of Γ the pair $\{p^\varepsilon, (p^\perp)^\varepsilon\}$ is a point-plane flag of $\mathbf{PG}(V)$, *i.e.*, a line of Δ. Thus, the embedding $\delta : \Delta \to \mathbf{PG}(V \wedge V)$ induces a lax embedding ε_δ of the dual of Γ into the span of $\mathcal{L}^{\varepsilon\delta}$ in $\mathbf{PG}(V \wedge V)$. We call ε_δ the *Grassmannian* of ε.

14 Proposition. *If ε is homogeneous, then ε_δ is also homogeneous.*

(Clear, as δ is homogeneous.) Furthermore,

15 Proposition. *The embedding ε_δ is non-flat.*

PROOF. We recall that, given two distinct point-plane flags $F_1 = \{p_1, \pi_1\}$ and $F_2 = \{p_2, \pi_2\}$ of $\mathbf{PG}(V)$, their images F_1^δ and F_2^δ are coplanar as lines of $\mathbf{PG}(V \wedge V)$ if and only if either $\pi_1 = \pi_2$ or $p_1 = p_2$ and the planes π_1, π_2 meet in a line. It easily follows from this remark that, if ε_δ were flat, then, for any line $l \in \mathcal{L}$ of Γ, the set $l^\perp$ of lines of Γ that meet l non-trivially is sent by ε to a set of mutually coplanar lines of $\mathbf{PG}(V)$. However, as all lines of Γ have at least two points and every point of Γ belongs to at least two lines, the above forces $(l^\perp)^\varepsilon$ to be contained in a plane of $\mathbf{PG}(V)$. An easy inductive argument now shows that all of Γ^ε is contained in a given plane of $\mathbf{PG}(V)$. Therefore and since $\mathbf{PG}(V) = \langle P^\varepsilon \rangle$ (by (II)), $\mathbf{PG}(V)$ is a plane, contrary to the assumption $\dim(V) > 3$. So, ε_δ cannot be flat. $\boxed{QED}$

Note. In the above construction we have assumed that $\dim(V) > 3$ and that $\mathbb{K}$ is a field, but we can repeat that construction when $\dim(V) = 3$ as well. In that case, $\mathbf{PG}(V)$ is a plane, Δ is just the dual of the plane $\mathbf{PG}(V)$ and ε_δ is the dual of ε.

Note also that, when $\dim(V) > 3$, we need $\mathbb{K}$ to be a field in order to embed Δ in $\mathbf{PG}(V \wedge V)$, but there is no need for this assumption when $\dim(V) = 3$.

Example. The generalized quadrangle $\mathsf{W}(2)$ has a flat embedding in the 3-dimensional projective space $\mathbf{PG}(3, 2)$; this embedding is homogeneous. The Grassmannian corresponds to the representation of $\mathsf{W}(2)$ as a nonsingular quadric in $\mathbf{PG}(4, 2)$; it is also homogeneous, but not flat. Furthermore, the embedding of $\mathsf{W}(2)$ in $PG(2, 4)$ given by $(\overline{\mathcal{H}}, \overline{\mathcal{H}}^*)$ (see Construction A) is a plane embedding, and the Grassmannian is just the dual, which, in this particular case, can be chosen to be identical to the original.

Lax morphisms and isomorphisms. Given two lax embeddings $\varepsilon : \Gamma \to \mathbf{PG}(V)$ and $\eta : \Gamma \to \mathbf{PG}(W)$, let $\Gamma^\varepsilon = (P^\varepsilon, \mathcal{L}^\varepsilon)$ and $\Gamma^\eta = (P^\eta, \mathcal{L}^\eta)$ be the images of Γ via ε and η, respectively. According to our definition of morphisms, a semilinear mapping $\varphi : V \to W$ inducing an isomorphism from Γ^ε to Γ^η is a morphism from ε to $\eta\alpha_\varphi$ for an automorphism α_φ of Γ, but it is not a morphism from ε to η, except when α_φ is the identity. We call φ a *lax morphism* from ε to η (a *lax isomorphism* if it is invertible). If a lax isomorphism exists between ε and η, then we write $\varepsilon \sim \eta$.

According to these definitions, a lax automorphism of ε is an isomorphism from ε to $\varepsilon\alpha$ for some $\alpha \in \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}_\varepsilon(\Gamma)$ is the full group of lax automorphisms of ε, taken modulo scalar factors, while the automorphisms of ε form the pointwise stabilizer of P^ε in $\mathrm{Aut}_\varepsilon(\Gamma)$ (which is trivial when ε is full, by Proposition 9). Also:

16 Proposition. *Suppose that ε is full and homogeneous and let U_1, U_2 be subspaces of V satisfying* (V). *Then $\varepsilon_{U_1} \sim \varepsilon_{U_2}$ if and only if U_1 and U_2 belong to the same orbit of* $\mathrm{Aut}_\varepsilon(\Gamma)$.

(Compare Proposition 10.) We omit the easy proof.

Note. In spite of the distinction we have drawn between isomorphisms and lax isomorphisms, when one says that an embedding satisfying certain properties is unique up to isomorphisms (as the universal embedding, for instance), it makes no difference if one thinks of isomorphisms or lax isomorphisms.

4.2 A generating set of $\widetilde{\mathsf{W}(2)}$

Referring to Construction D, let X be the following set of points:

$$X := \{p^0_{ij}\}_{\{i,j\}\neq\{0,1\}} \cup \{p^1_{0,1}, p^2_{0,1}\}$$

17 Lemma. *The set X spans Γ_D.*

Proof. The span of X contains all lines of Γ_D containing the edges $\{p^0_{ij}, p^0_{rs}\}$, with $\{i,j,r,s\}$ a 4-subset of $\{0,1,2,3,4\}$ and $\{i,j\} \neq \{0,1\} \neq \{r,s\}$. Thus, we get all 'new' points of Γ_D but p^0_2, p^0_4 and p^1_3. The point $p^1_{0,1}$ is collinear in Γ_D with each of p^1_2, p^2_3 and p^1_4 whereas $p^2_{0,1}$ is collinear with each of p^2_2, p^0_3 and p^2_4. Hence we also get the points p^k_{ij} with $\{i,j\} \subset \{2,3,4\}$ and $k = 1,2$. We can now reach all points of Π_1. For instance, we obtain $p^1_{0,2}$ from the line through p^2_1 and $p^1_{3,4}$. Analogously, we get $p^1_{0,3}$, $p^1_{0,4}$, $p^1_{1,2}$, $p^1_{1,3}$ and $p^1_{1,4}$. Similarly, we reach all points of Π_2. The points we have got so far are enough to reach p^0_2, p^0_4 and p^1_3. The point $p^0_{0,1}$ still remains, but we can get it from any of the three lines of Γ_D on it. $\boxed{QED}$

18 Corollary. *The generating rank of $\widetilde{\mathsf{W}(2)}$ is at most* 11.

Proof. Clear by the above lemma and since $|X| = 11$. $\boxed{QED}$

4.3 The universal embedding

It is known (Ivanov and Shpectorov [7]) that the embedding rank of $\widetilde{\mathsf{W}(2)}$ is 11. In this section we will give our own proof of that result, but we shall prove more than that, namely the following:

19 Theorem. *Given a division ring $\mathbb{K}$, there exists a lax embedding $\varepsilon : \widetilde{\mathsf{W}(2)} \to \mathbf{PG}(10,\mathbb{K})$ if and only if $\mathrm{char}(\mathbb{K}) = 2$. If that is the case, then ε is uniquely determined* (*up to isomorphisms*) *and it is full in a suitable subgeometry of $\mathbf{PG}(10,\mathbb{K})$ defined over $\mathbf{GF}(2)$.*

In particular, $\widetilde{\mathsf{W}(2)}$ admits a unique embedding $\tilde{\varepsilon}$ in $\mathbf{PG}(10,2)$. By Corollary 18, the embedding rank of $\widetilde{\mathsf{W}(2)}$ is at most 11. Therefore,

20 Corollary. *The embedding* $\tilde{\varepsilon} : \widetilde{\mathsf{W}(2)} \to \mathbf{PG}(10,2)$ *is absolutely universal and* $\mathrm{grk}(\widetilde{\mathsf{W}(2)}) = \mathrm{erk}(\widetilde{\mathsf{W}(2)}) = 11$.

Proof of Theorem 19. Let $V = V(11,\mathbb{K})$ be the 11-dimensional left vector space over $\mathbb{K}$ and let $X = \{p^0_{ij}\}_{\{i,j\}\neq\{0,1\}} \cup \{p^1_{0,1}, p^2_{0,1}\}$ be the spanning set of $\widetilde{\mathsf{W}(2)}$ considered in Subsection 4.2. If there exists a lax embedding $\varepsilon : \widetilde{\mathsf{W}(2)} \to \mathbf{PG}(V)$, then the restriction ε_X of ε to X is a bijection from X to a spanning set $\varepsilon(X)$ of $\mathbf{PG}(V)$, corresponding to a basis B of V. We may always give the vectors of B indices like those of the corresponding points of X, thus denoting by e^k_{ij} the vector $e \in B$ such that $\langle e\rangle = \varepsilon_X(p^k_{ij})$ (where $k = 0$ and $\{i,j\} \neq \{0,1\}$ or $k = 1,2$ and $\{i,j\} = \{0,1\}$).

The proof of Lemma 17 shows that, modulo multiplying the vectors of B by suitable scalar, there exist nonzero scalars $k_{3;2}$, $k_{4;1}$ and $k_{4;2}$ such that, with

$$\begin{array}{lll@{\qquad}lll@{\qquad}lll}
v^0_0 & := & e^0_{1,2}+e^0_{3,4} & v^1_0 & := & e^0_{1,4}+e^0_{2,3} & v^2_0 & := & e^0_{1,3}+e^0_{2,4}\\
v^0_1 & := & e^0_{0,4}+e^0_{2,3} & v^1_1 & := & e^0_{0,2}+e^0_{3,4} & v^2_1 & := & e^0_{0,3}+e^0_{2,4}\\
v^1_2 & := & e^0_{0,4}+e^0_{1,3} & v^2_2 & := & e^0_{0,3}+e^0_{1,4} & v^0_3 & := & e^0_{0,4}+e^0_{1,2}\\
v^2_3 & := & e^0_{0,2}+k_{3;2}e^0_{1,4} & v^1_4 & := & e^0_{0,3}+k_{4;1}e^0_{1,2} & v^2_4 & := & e^0_{0,2}+k_{4;2}e^0_{1,3}
\end{array}$$

ε sends p^k_i to $\langle v^k_i\rangle$ for $(i,k) \neq (2,0),(4,0),(3,1)$. Considering now $p^1_{0,1}$ and $p^2_{0,1}$, there exist nonzero scalars $k_{2,3;i}$ and $k_{2,4;i}$ $(i = 1,2)$ such that, with

$$\begin{array}{lllll}
v^1_{2,3} & := & k_{2,3;1}e^1_{0,1}+v^1_4 & = & k_{2,3;1}e^1_{0,1}+e^0_{0,3}+k_{4;1}e^0_{1,2}\\
v^1_{2,4} & := & k_{2,4;1}e^1_{0,1}+v^2_3 & = & k_{2,4;1}e^1_{0,1}+e^0_{0,2}+k_{3;2}e^0_{1,4}\\
v^1_{3,4} & := & e^1_{0,1}+v^1_2 & = & e^1_{0,1}+e^0_{0,4}+e^0_{1,3}\\
v^2_{2,3} & := & k_{2,3;2}e^2_{0,1}+v^2_4 & = & k_{2,3;2}e^2_{0,1}+e^0_{0,2}+k_{4;2}e^0_{1,3}\\
v^2_{2,4} & := & k_{2,4;2}e^2_{0,1}+v^0_3 & = & k_{2,4;2}e^2_{0,1}+e^0_{0,4}+e^0_{1,2}\\
v^2_{3,4} & := & e^2_{0,1}+v^2_2 & = & e^2_{0,1}+e^0_{0,3}+e^0_{1,4}
\end{array}$$

ε sends $p^h_{i,j}$ to $\langle v^h_{i,j}\rangle$, for $\{i,j\} \subset \{2,3,4\}$ and $h = 1,2$. Next, for $i = 0,1$, $j = 2,3,4$ and $h = 1,2$ and $v^h_{i,j}$ defined as below for suitable nonzero scalars $k_{i,j;h}$, ε sends $p^h_{i,j}$ to $\langle v^h_{i,j}\rangle$.

$$\begin{array}{lll}
v^1_{0,2} & := & k_{0,2;1}v^2_1+v^1_{3,4}\\
 & = & k_{0,2;1}(e^0_{0,3}+e^0_{2,4})+e^1_{0,1}+e^0_{0,4}+e^0_{1,3}\\
v^1_{0,3} & := & k_{0,3;1}v^0_1+v^1_{2,4}\\
 & = & k_{0,3;1}(e^0_{0,4}+e^0_{2,3})+k_{2,4;1}e^1_{0,1}+e^0_{0,2}+k_{3;2}e^0_{1,4}\\
v^1_{0,4} & := & k_{0,4;1}v^1_1+v^1_{2,3}\\
 & = & k_{0,4;1}(e^0_{0,2}+e^0_{3,4})+k_{2,3;1}e^1_{0,1}+e^0_{0,3}+k_{4;1}e^0_{1,2}
\end{array}$$

$$
\begin{array}{rcl}
v^1_{1,2} & := & k_{1,2;1}v^1_0 + v^1_{3,4} \\
 & = & k_{1,2;1}(e^0_{1,4} + e^0_{2,3}) + e^1_{0,1} + e^0_{0,4} + e^0_{1,3} \\
v^1_{1,3} & := & k_{1,3;1}v^0_0 + v^1_{2,4} \\
 & = & k_{1,3;1}(e^0_{1,2} + e^0_{3,4}) + k_{2,4;1}e^1_{0,1} + e^0_{0,2} + k_{3;2}e^0_{1,4} \\
v^1_{1,4} & := & k_{1,4;1}v^2_0 + v^1_{2,3} \\
 & = & k_{1,4;1}(e^0_{1,3} + e^0_{2,4}) + k_{2,3;1}e^1_{0,1} + e^0_{0,3} + k_{4;1}e^0_{1,2} \\
v^2_{0,2} & := & k_{0,2;2}v^0_1 + v^2_{3,4} \\
 & = & k_{0,2;2}(e^0_{0,4} + e^0_{2,3}) + e^2_{0,1} + e^0_{0,3} + e^0_{1,4} \\
v^2_{0,3} & := & k_{0,3;2}v^1_1 + v^2_{2,4} \\
 & = & k_{0,3;2}(e^0_{0,2} + e^0_{3,4}) + k_{2,4;2}e^2_{0,1} + e^0_{0,4} + e^0_{1,2} \\
v^2_{0,4} & := & k_{0,4;2}v^2_1 + v^2_{2,3} \\
 & = & k_{0,4;2}(e^0_{0,3} + e^0_{2,4}) + k_{2,3;2}e^2_{0,1} + e^0_{0,2} + k_{4;2}e^0_{1,3} \\
v^2_{1,2} & := & k_{1,2;2}v^2_0 + v^2_{3,4} \\
 & = & k_{1,2;2}(e^0_{1,3} + e^0_{2,4}) + e^2_{0,1} + e^0_{0,3} + e^0_{1,4} \\
v^2_{1,3} & := & k_{1,3;2}v^1_0 + v^2_{2,4} \\
 & = & k_{1,3;2}(e^0_{1,4} + e^0_{2,3}) + k_{2,4;2}e^2_{0,1} + e^0_{0,4} + e^0_{1,2} \\
v^2_{1,4} & := & k_{1,4;2}v^0_0 + v^2_{2,3} \\
 & := & k_{1,4;2}(e^0_{1,2} + e^0_{3,4}) + k_{2,3;2}e^2_{0,1} + e^0_{0,2} + k_{4;2}e^0_{1,3}
\end{array}
$$

The point p^0_2 is collinear with $p^1_{0,3}, p^1_{1,4}$ and with $p^2_{1,3}, p^2_{0,4}$. Therefore, the quadruple $\{v^1_{0,3}, v^1_{1,4}, v^2_{1,3}, v^2_{0,4}\}$ spans a 3-space. Similarly for the quadruples

$$\{v^1_{0,2}, v^1_{1,3}, v^2_{0,3}, v^2_{1,2}\} \quad \text{and} \quad \{v^1_{0,4}, v^1_{1,2}, v^2_{0,2}, v^2_{1,4}\}.$$

So, each of the 3-by-11 matrices formed by the above three quadruples of vectors has rank 3. By straightforward computations one can see that the above forces

$$(*) \begin{cases} k_{3;2} = k_{2,3;2} = k_{0,i;h} = k_{1,j;h} \\ k_{2,4;h} = k_{4;h} = k_{2,3;1} = -1 \end{cases}$$

for $i = 2, 3$, $j = 2, 3, 4$ and $h = 1, 2$. However, we have more collinearities to exploit: for instance, p^2_2 is collinear with $p^1_{0,4}, p^1_{1,3}$ and $p^2_{3,4}, p^2_{0,1}$. This condition forces $k_{4;1} = k_{0,4;1} = k_{1,3;1} = -k_{3;2}$, which is compatible with $(*)$ only if $1 = -1$, hence $\mathrm{char}(\mathbb{K}) = 2$.

It remains to prove that the image of $\widetilde{\mathsf{W}(2)}$ is contained in the $\mathbf{GF}(2)$-span of the basis B. Notice first that, as $\mathrm{char}(\mathbb{K}) = 2$, $(*)$ forces

$$k_{3;2} = k_{0,i;h} = k_{1,j;h} = k_{2,s;h} = k_{4;h} = k_{0,4;1} = 1$$

for $i = 2, 3$, $j = 2, 3, 4$, $s = 3, 4$ and $h = 1, 2$. Thus, 18 out of the 21 scalars introduced so far are equal to 1. Considering that each of the points p^1_2, p^2_2,

p_3^0, p_3^2, p_4^1 and p_4^2 is collinear with a pair of points in each of Π_1 and Π_2 and arguing as above, one can see that the remaining three scalars are also equal to 1. So, the following are the linear "dependences" that follow from the collinearity conditions considered so far:

$$(1)\left\{\begin{array}{lcll} v_{0,3}^1+v_{1,4}^1 &=& v_{0,4}^2+v_{1,3}^2 & \text{(corresponding to } p_2^0)\\ v_{0,4}^1+v_{1,2}^1 &=& v_{0,2}^2+v_{1,4}^2 & \text{(corresponding to } p_3^1)\\ v_{0,2}^1+v_{1,3}^1 &=& v_{0,3}^2+v_{1,2}^2 & \text{(corresponding to } p_4^0)\end{array}\right.$$

$$(2)\left\{\begin{array}{lclcl} v_{0,1}^1+v_{3,4}^1 &=& v_{0,3}^2+v_{1,4}^2 &=& v_2^1\\ v_{0,4}^1+v_{1,3}^1 &=& v_{0,1}^2+v_{3,4}^2 &=& v_2^2\\ v_{0,2}^1+v_{1,4}^1 &=& v_{0,1}^2+v_{2,4}^2 &=& v_3^0\\ v_{0,1}^1+v_{2,4}^1 &=& v_{0,4}^2+v_{1,2}^2 &=& v_3^2\\ v_{0,1}^1+v_{2,3}^1 &=& v_{0,2}^2+v_{1,3}^2 &=& v_4^1\\ v_{0,3}^1+v_{1,2}^1 &=& v_{0,1}^2+v_{2,3}^2 &=& v_4^2\end{array}\right.$$

In view of (1), ε maps p_2^0, p_4^0 and p_3^1 onto $\langle v_4^0\rangle$, $\langle v_4^0\rangle$ and $\langle p_3^1\rangle$ respectively, where:

$$\begin{array}{lcl} v_2^0 &:=& v_{0,3}^1+v_{1,4}^1=v_{1,3}^2+v_{0,4}^2\\ v_4^0 &:=& v_{0,2}^1+v_{1,3}^1=v_{0,3}^2+v_{1,2}^2\\ v_3^1 &:=& v_{0,4}^1+v_{1,2}^1=v_{0,2}^2+v_{1,4}^2\end{array}$$

The point $p_{0,1}^1$ remains to be considered. We have $\varepsilon(p_{0,1}^1)=\langle v_{0,1}^0\rangle$ where $v_{0,1}^0$ is such that

$$\langle v_{0,1}^0\rangle=\langle rv_{3,4}^0+v_2^0\rangle=\langle sv_{2,4}^0+v_3^1\rangle=\langle tv_{2,3}^0+v_4^0\rangle$$

for suitable nonzero scalars r,s,t. The vectors $v_{2,3}^0$, v_4^0, $v_{2,4}^0$, v_3^1, $v_{3,4}^0$, v_2^0 have been defined above and it follows from their definition that:

$$\begin{array}{lcl} rv_{3,4}^0+v_2^0 &=& e_{0,2}^0+\cdots+e_{1,4}^0+e_{2,3}^0+e_{2,4}^0+re_{3,4}^0\\ sv_{2,4}^0+v_3^1 &=& e_{0,2}^0+\cdots+e_{1,4}^0+e_{2,3}^0+se_{2,4}^0+e_{3,4}^0\\ tv_{2,3}^0+v_4^0 &=& e_{0,2}^0+\cdots+e_{1,4}^0+te_{2,3}^0+e_{2,4}^0+e_{3,4}^0\end{array}$$

These vectors are proportional to the same vector $v_{0,1}^0$ if and only if $r=s=t=1$. So, we may set

$$v_{0,1}^0:=\ e_{0,2}^0+e_{0,3}^0+\cdots+e_{1,4}^0+e_{2,3}^0+e_{2,4}^0+e_{3,4}^0$$

and we get

$$(3)\quad v_{0,1}^0=v_{3,4}^0+v_2^0=v_{2,4}^0+v_3^1=v_{2,3}^0+v_4^0$$

The 'only if' part and the second claim of the theorem are proved. On the other hand, it is easy to check that the relations (1), (2) and (3) are consistent with the definitions of the vectors involved in them. This is sufficient to obtain the 'if' part, too. $\boxed{QED}$

4.4 Another description of the universal embedding

In this subsection we shall give a completely geometric construction of the universal embedding $\tilde{\varepsilon}$ of $\widetilde{\mathsf{W}(2)}$, but we first describe a full embedding ε_1 of $\widetilde{\mathsf{W}(2)}$ in $\mathbf{PG}(5,2)$, implicit in Construction A. Note that the dimension of this embedding is minimal. Indeed, as $\widetilde{\mathsf{W}(2)}$ has 45 points whereas $2^{d+1}-1 < 45$ when $d < 5$, no embedding of $\widetilde{\mathsf{W}(2)}$ in $\mathbf{PG}(d,2)$ exists if $d < 5$.

Let $\mathcal{S}$ be a regular line spread of $\mathbf{PG}(5,2)$, *i.e.*, a set of 21 pairwise disjoint lines of $\mathbf{PG}(5,2)$ such that the 3-dimensional space generated by two arbitrary members of $\mathcal{S}$ contains five elements of $\mathcal{S}$. Note that $\mathcal{S}$ endowed with these 5-sets is a copy of $\mathbf{PG}(2,4)$. Let $\mathcal{H} \subseteq \mathcal{S}$ be a set of six lines with the property that each three of them generate $\mathbf{PG}(5,2)$. (This 6-set corresponds to a hyperoval in $\mathbf{PG}(2,4)$). Then the 45 points of $\mathbf{PG}(5,2)$ not incident with any member of $\mathcal{H}$, together with the lines of $\mathbf{PG}(5,2)$ which are contained in 3-spaces generated by two elements of $\mathcal{H}$, that do not belong to $\mathcal{S}$ and do not meet any member of $\mathcal{H}$, define the embedding ε_1.

Note that the triples of opposite points of $\widetilde{\mathsf{W}(2)}$ are mapped by ε_1 onto the lines of $\mathcal{S}\setminus\mathcal{H}$. (We recall that two points of $\widetilde{\mathsf{W}(2)}$ are said to be *opposite* when they have distance 4 in the collinearity graph $\widetilde{\mathsf{W}(2)}$. This happens if and only if those points are mapped onto the same point of $\mathsf{W}(2)$ by any covering of $\widetilde{\mathsf{W}(2)}$ onto $\mathsf{W}(2)$.)

We are now ready to give a geometric description of the universal embedding $\tilde{\varepsilon}$ of $\widetilde{\mathsf{W}(2)}$. Given a 6-dimensional subspace V_1 of $\tilde{V} := V(11,2)$, we embed $\mathbf{PG}(5,2)$ as $PG(V_1)$ in $\mathbf{PG}(\tilde{V}) \cong \mathbf{PG}(10,2)$ and, chosen a complement V_0 of V_1 in $\tilde{V}$, inside $\mathbf{PG}(V_0) \cong \mathbf{PG}(4,2)$ we consider a nonsingular quadric $Q(4,2)$ (the points and lines of which form a copy of $\mathsf{W}(2)$). Further, we consider an arbitrary covering f from $\widetilde{\mathsf{W}(2)}$ (as embedded in $\mathbf{PG}(V_1)$) to $\mathsf{W}(2)$ (as represented by $Q(4,2)$ in $\mathbf{PG}(V_0)$).

Let $\mathcal{P}$ be the set of 45 points of $\mathbf{PG}(\tilde{V})$ obtained in the following way: If x is a point of $\widetilde{\mathsf{W}(2)}$, let x' be the point different from x and x^f on the line xx^f. Then $x' \in \mathcal{P}$, and all points of $\mathcal{P}$ are obtained in this way. The mapping sending x to x' is a bijection from the point-set of $\widetilde{\mathsf{W}(2)}$ to $\mathcal{P}$. Moreover, if three point x, y, z of $\widetilde{\mathsf{W}(2)}$ are collinear, then the points x^f, y^f and z^f are collinear and the corresponding points x', y', z' of $\mathcal{P}$ are also collinear: indeed, they form a line on the hyperbolic quadric defined by the three lines xx^f, yy^f and zz^f. Hence we obtain a full embedding $\tilde{\varepsilon}$ of $\widetilde{\mathsf{W}(2)}$ in $\mathbf{PG}(\tilde{V})$.

21 Theorem. *The embedding $\tilde{\varepsilon}$ constructed above is the universal one.*

Proof. As we have already proved that the universal embedding of $\widetilde{\mathsf{W}(2)}$

is 10-dimensional, we only need to show that $\mathcal{P}$ spans $\mathbf{PG}(\tilde{V})$. To that end, we consider three pairwise opposite points x_1, x_2, x_3 of $\widetilde{\mathsf{W}(2)}$, embedded in $\mathbf{PG}(V_1)$ via ε_1. As noted before, these points form a line L of $\mathbf{PG}(5,2)$, hence of $\mathbf{PG}(V_1)$. The corresponding points x_1', x_2', x_3' of $\mathcal{P}$ lie in a plane π spanned by L and x_1^f $(= x_2^f = x_3^f)$ and it is easy to see that π is also spanned by x_1', x_2', x_3'. Hence $\mathcal{P}$ spans both $\mathbf{PG}(V_1)$ and $\mathbf{PG}(V_0)$, and consequently it spans $\mathbf{PG}(\tilde{V})$. $\boxed{QED}$

22 Corollary. *The embedding ε_1 is homogeneous.*

Proof. Let $G_{\mathcal{P}}$ be the stabilizer of $\mathcal{P}$ in $G = \mathbf{PGL}(\tilde{V})$. As $\tilde{\varepsilon}$, being universal, is homogeneous, every automorphism of $\widetilde{\mathsf{W}(2)}$ is induced by an element of $G_{\mathcal{P}}$. Also, we remark that ε_1 is precisely the projection of $\tilde{\varepsilon}$ from $\mathbf{PG}(V_0)$ onto $\mathbf{PG}(V_1)$ (that is immediate from the construction of $\tilde{\varepsilon}$). Thus, in order to prove the corollary we only need to prove that $G_{\mathcal{P}}$ stabilizes $\mathbf{PG}(V_0)$. In fact, our proof will also show that $\mathbf{PG}(V_1)$ is stabilized by $G_{\mathcal{P}}$.

Let $X = \{x_1, x_2, x_3\}$ be a triple of points of $\mathcal{P}$ mutually opposite in $\widetilde{\mathsf{W}(2)}$. As noticed above, X spans a plane π. So, X is a conic of π. The nucleus of X belongs to $Q(4,2)$ embedded in $\mathbf{PG}(V_0)$ whereas the unique line L of π exterior to X is contained in $\mathbf{PG}(V_1)$ and belongs to $\mathcal{S} \setminus \mathcal{H}$. However, $G_{\mathcal{P}}$ stabilizes the set of triples as above and acts transitively on it. Hence $G_{\mathcal{P}}$ stabilizes both the set of points of $Q(4,2)$ and the set of lines $\mathcal{S} \setminus \mathcal{H}$, acting transitively on each of those two sets. Consequently, $G_{\mathcal{P}}$ stabilizes both $\mathbf{PG}(V_0)$ and $\mathbf{PG}(V_1)$. $\boxed{QED}$

Remark. The "opposite" projection, namely from $\mathbf{PG}(V_1)$ onto $\mathbf{PG}(V_0)$, yields a projection of the universal embedding of $\widetilde{\mathsf{W}(2)}$ onto the universal embedding of $\mathsf{W}(2)$.

4.5 Homogeneous full embeddings

Since we know the universal embedding $\tilde{\varepsilon}$ of $\widetilde{\mathsf{W}(2)}$, we know in principle all full embeddings of $\widetilde{\mathsf{W}(2)}$. Indeed, each of them is a quotient of $\tilde{\varepsilon}$ by a subspace of $\tilde{V}$ satisfying condition (V) of Subsection 4.1. In particular, every subspace of $\tilde{V}$ contained in V_0 defines a quotient of $\tilde{\varepsilon}$.

However, non-trivial subspaces of $\tilde{V}$ also exist that define quotients of $\tilde{\varepsilon}$ but are not contained in V_0. For instance, let U be a 2-dimensional subspace of V_1 corresponding to a line of $\mathcal{H}$. It is not difficult to see that L, as a line of $\mathbf{PG}(\tilde{V})$ is skew with any line of $\mathbf{PG}(\tilde{V})$ through two distinct points of $\mathcal{P}$. Hence L satisfies condition (V) and therefore it defines a quotient of $\tilde{\varepsilon}$.

So, considering that $\mathcal{H}$ contains six lines, that each of them has three points and V_0 has 374 subspaces (including 0 and V_0 among them), we get at least 398 subspaces of $\tilde{V}$ satisfying (V). Therefore, by Proposition 10, there are at least

398 non-isomorphic full embeddings of $\widetilde{\mathsf{W}(2)}$ (but many of them are mutually laxly isomorphic, by Proposition 16). The classification of all subspaces of $\tilde{V}$ satisfying (V) is a tiring job, which we have not accomplished. Instead, we will determine the homogeneous full embeddings of $\widetilde{\mathsf{W}(2)}$.

The universal embedding $\tilde{\varepsilon}$ is homogeneous. The embedding ε_1 of $\widetilde{\mathsf{W}(2)}$ in $\mathbf{PG}(5,2)$ described in the previous subsection is also homogeneous (Corollary 22). We shall prove that, besides $\tilde{\varepsilon}$ and ε_1, only one homogeneous embedding exists, which seems to be new. It is 9-dimensional and gives a 'sporadic' inclusion $3 \cdot \mathbf{S}(6) \leq \mathbf{SL}_{10}(2)$). We call this new embedding $\hat{\varepsilon}$.

With the notation of Subsection 4.4, we can describe $\hat{\varepsilon}$ as the quotient of $\tilde{\varepsilon}$ by the 1-dimensional subspace N of V_0 corresponding to the nucleus of $Q(4,2) \subset \mathbf{PG}(V_0)$. Thus, with $\hat{V} = V_1 \oplus W$, $W := V_0/N$, we can also obtain $\hat{\varepsilon}$ by the same construction used for $\tilde{\varepsilon}$ in Subsection 4.4, but replacing $\tilde{V}$ with $\hat{V}$ and exploiting the standard embedding of $\mathsf{W}(2)$ in $\mathbf{PG}(W) \cong \mathbf{PG}(3,2)$ instead of $Q(4,2)$ in $\mathbf{PG}(V_0) \cong \mathbf{PG}(4,2)$.

As $Aut_{\tilde{\varepsilon}}(\widetilde{\mathsf{W}(2)})$ $(= \ \mathrm{Aut}(\widetilde{\mathsf{W}(2)}))$ stabilizes V_0 and $Q(4,2)$, it also stabilizes N. Therefore, by Proposition 13,

23 Lemma. *The embedding $\hat{\varepsilon}$ is homogeneous.*

In fact,

24 Theorem. *The embeddings $\tilde{\varepsilon}$, $\hat{\varepsilon}$ and ε_1 are the only homogeneous embeddings of $\widetilde{\mathsf{W}(2)}$.*

Proof. In view of Proposition 13, we only need to prove that N and V_0 are the only non-trivial proper subspaces of $\tilde{V}$ stabilized by $G = \ \mathrm{Aut}_{\tilde{\varepsilon}}(\widetilde{\mathsf{W}(2)})$ and satisfying condition (V) of Subsection 4.1. Let U be such a subspace. The conclusion is clear if $U \subseteq V_0$, since G induces in $\mathbf{PG}(V_0)$ the full automorphism group of $Q(4,2)$. So suppose U is not contained in V_0. We search for a contradiction.

Note first that, as noticed at the beginning of Subsection 4.4, the (vector) dimension of U is at most 5. Since $U \not\subseteq V_1$ by assumption, there is a one-space $x \in U \setminus V_1$. Let y be the projection of x from V_0 onto V_1. Then y^G is either the union of the 2-subspaces of V_1 corresponding to lines of $\mathcal{H}$ or the union of the 2-subspaces of V_1 corresponding to lines of $\mathcal{S} \setminus \mathcal{H}$. In either case y^G generates V_1, implying that the projection of U from V_0 onto V_1 is onto (using the fact that G stabilizes both V_0 and V_1, and hence each element of G commutes with the projection map). However, this forces $\dim(U) \geq 6$, which is impossible. $\boxed{QED}$

4.6 Another description of $\hat{\varepsilon}$

The following is straightforward:

25 Lemma. *The 5-dimensional embedding ε_1 is flat.*

Hence we may consider the Grassmannian of ε_1, which we shall denote $\mathrm{gr}(\varepsilon_1)$. As $\widetilde{\mathsf{W}(2)}$ is self-dual, $\mathrm{gr}(\varepsilon_1)$ is also a full embedding of $\widetilde{\mathsf{W}(2)}$.

26 Theorem. $\mathrm{gr}(\varepsilon_1) \cong \hat{\varepsilon}$.

PROOF. The embedding $\mathrm{gr}(\varepsilon_1)$ is homogeneous and non-flat, by Propositions 14 and 15. Hence, by Theorem 24 and Lemma 25, $\mathrm{gr}(\varepsilon_1)$ is isomorphic to either $\hat{\varepsilon}$ or $\tilde{\varepsilon}$. So, it only remains to show that $\mathrm{gr}(\varepsilon_1) \ncong \tilde{\varepsilon}$. Of course, this could be proved with a computer, or with tedious computations. We now present an entirely geometric argument. We start with a few preliminary observations. Let L and M be skew lines of $\mathbf{PG}(3,2)$. Then there are precisely six lines skew to both and they are divided in two sets of three forming two complementary reguli. Furthermore, denoted by Σ the Grassmannian of lines of $\mathbf{PG}(3,2)$ and by σ the embedding of Σ into $\mathbf{PG}(5,2)$, those two reguli are mapped by σ onto two conics $C_{L,M}$ and $C'_{L,M}$ of the image Σ^σ of Σ via σ (which is the Klein quadric).

27 Lemma. *The conics $C_{L,M}$ and $C'_{L,M}$ have the same nucleus $n_{L,M}$ and*

$$\{L^\sigma, M^\sigma, n_{L,M}\}$$

is a line of $\mathbf{PG}(5,2)$.

PROOF. The two reguli are contained in a unique copy Q of $W(2)$: the lines of Q are the lines of the two reguli plus all lines meeting both L and M. So we have all lines of Q and, consequently, all points of Q, too. The image of the dual of Q by σ is a quadric $Q^* = Q(4,2)$ with nucleus $n_{L,M}$. The point $n := n_{L,M}$ is also the nucleus of any conic of the quadric Q^* corresponding to a hyperbolic line of the generalized quadrangle Q^*. However our two reguli are dual hyperbolic lines of Q whence, regarded as conics of Q^*, they have n as their nucleus. We shall now prove that L^σ, M^σ and n are collinear as points of $\mathbf{PG}(5,2)$.

We recall that the symplectic polarity π associated with the Klein quadric Σ^σ maps each tangent hyperplane to its tangent point and any other hyperplane to the nucleus of its intersection with Σ^σ. All lines of $\mathbf{PG}(3,2)$ meeting both L and M are lines of Q. So, denoted by H_L and H_M the hyperplanes of $\mathbf{PG}(5,2)$ tangent to Σ^σ at L^σ and M^σ, we have $H_L \cap H_M \cap \Sigma^\sigma \subseteq Q^*$. Hence $W := H_L \cap H_M$ is a 3-dimensional subspace of $\mathbf{PG}(5,2)$ that meets Σ^σ in a hyperbolic quadric $W \cap \Sigma^\sigma \cong Q^+(3,2)$ and the three hyperplanes of $\mathbf{PG}(5,2)$ containing W are H_L, H_M and the hyperplane spanned by Q^*. The polarity π sends them to three collinear points, namely L^σ, M^σ and n. $\boxed{QED}$

We can now finish the proof of Theorem 26. Let $\mathcal{S}$ be a regular line spread of $\mathbf{PG}(V_1) \cong \mathbf{PG}(5,2)$ and $\mathcal{H}$ a set of six lines of $\mathcal{S}$ corresponding to a hyperoval of $\mathbf{PG}(2,4)$, as in Subsection 4.4. Take three elements L_1, L_2, L_3 of $\mathcal{H}$. For

$1 \leq i < j \leq 3$, let L_iL_j be the 3-space of $\mathbf{PG}(V_1)$ spanned by L_i and L_j. As noticed above, L_iL_j contains two reguli formed by lines skew with both L_i and L_j. One of them, say $R^-_{i,j}$, is formed by lines of $\mathcal{S}$ and the other one, say $R^+_{i,j}$, is formed by lines of the image of $\widetilde{\mathsf{W}(2)}$ in $\mathbf{PG}(V_1)$ via ε_1. As, given a covering $\gamma : \widetilde{\mathsf{W}(2)} \to W(2)$, the lines of $\mathcal{S} \setminus \mathcal{H}$ are the fibers of γ, every line of $R^-_{i,j}$ is mapped by γ onto one point of $W(2)$ and, consequently, all lines of $R^+_{i,j}$ are mapped by γ onto the same line of $W(2)$.

Let Δ the Grassmannian of lines of $\mathbf{PG}(V_1)$ and δ the embedding of Δ into $\mathbf{PG}(15, 2)$. Then the Grassmannian of lines of L_iL_j is a subgeometry of Δ and δ induces on it its natural embedding in $\mathbf{PG}(5, 2)$. Accordingly, $R_{i,j}$ is mapped by δ onto a conic of $\mathbf{PG}(15, 2)$ and, if $n_{i,j}$ is the nucleus of that conic, the points L^δ_i, L^δ_j and $n_{i,j}$ form a line of $\mathbf{PG}(15, 2)$. So, the three points $n_{1,2}, n_{1,3}$ and $n_{2,3}$ form a line of the plane π of $\mathbf{PG}(15, 2)$ spanned by L^δ_1, L^δ_2 and L^δ_3. Precisely, $\{L_1, L_2, L_3\}$ is a conic of π and $\{n_{1,2}, n_{1,3}, n_{2,3}\}$ is the line of π exterior to that conic.

However, as noticed above, for $1 \leq i < j \leq 3$ any covering $\gamma : \widetilde{\mathsf{W}(2)} \to W(2)$ maps the three lines of $R^+_{i,j}$ onto the same line of $W(2)$. So, denoted the dual of $\widetilde{\mathsf{W}(2)}$ by $(\widetilde{\mathsf{W}(2)})^*$, the conic $(R^+_{i,j})^\delta$ corresponds to a triple $X_{i,j}$ of mutually opposite points of $(\widetilde{\mathsf{W}(2)})^*$. Furthermore, as $L_iL_j \cap L_hL_k$ contains no point of $\widetilde{\mathsf{W}(2)}$ for $\{i, j\} \neq \{h, k\}$, the set $X_{1,2} \cup X_{1,3} \cup X_{2,3}$ is an anti clique of the collinearity graph of $(\widetilde{\mathsf{W}(2)})^*$. However, $\widetilde{\mathsf{W}(2)}$ is self-dual. Thus, the embedding $\varepsilon :=$ gr(ε_1) (which is induced by δ) can also be regarded as an embedding of $\widetilde{\mathsf{W}(2)}$. It follows from the above that it has the following property: there are three triples of points $X_{1,2}$, $X_{1,3}$ and $X_{2,3}$ of $\widetilde{\mathsf{W}(2)}$ such that:

(1) each of the triples $X_{1,2}, X_{1,3}$ and $X_{2,3}$ consists of mutually opposite points and the join $X_{1,2} \cup X_{1,3} \cup X_{2,3}$ is an anti clique of the collinearity graph of $\widetilde{\mathsf{W}(2)}$;

(2) $X^\varepsilon_{i,j}$ is a conic, for $1 \leq i < j \leq 3$;

(3) the nuclei $n_{1,2}, n_{1,3}$ and $n_{2,3}$ of the conics $X^\varepsilon_{1,2}$, $X^\varepsilon_{1,3}$ and $X^\varepsilon_{2,3}$ form a line of the codomain $\mathbf{PG}(V)$ of ε.

Suppose now $\varepsilon = \tilde{\varepsilon}$. Then, given $X_{1,2}, X_{1,3}$ and $X_{2,3}$ satisfying (1), condition (2) is satisfied and the nuclei $n_{1,2}, n_{1,3}$ and $n_{2,3}$ (defined as in (3)) are points of the quadric $Q(4, 2) \subset \mathbf{PG}(V_0)$. In view of (1), they do not form a line of $Q(4, 2)$. Consequently, they do not form a line of $\mathbf{PG}(V_0)$ at all, contrary to what is claimed in (3). Hence $\varepsilon \neq \tilde{\varepsilon}$. Therefore, $\varepsilon = \hat{\varepsilon}$. $\boxed{QED}$

4.7 Another characterization of ε_1

As stated in Lemma 25, the embedding ε_1 is flat.

28 Theorem. *The embedding* ε_1 *is the only flat embedding of* Γ.

Proof. We first recall that neither $\tilde{\varepsilon}$ nor $\hat{\varepsilon}$ are flat. This follows from Theorem 26 and Proposition 15 for $\hat{\varepsilon}$ and, consequently, it holds for $\tilde{\varepsilon}$, too.

Suppose ε is a flat embedding of $\Gamma := \widetilde{\mathsf{W}(2)}$, obtained by projecting the point-set $\mathcal{P}$ of $\Gamma^{\tilde{\varepsilon}}$ from a subspace U of $\tilde{V}$ onto $\tilde{V}/U$. We have $\dim(U) \leq 5$. Indeed, if otherwise, $\mathbf{PG}(\tilde{V}/U)$ has not enough points to house Γ. We shall show that $U = V_0$, thus proving the Corollary.

Let p be a point of Γ^{ε_1}. Regarded p as a nonzero-vector of $\tilde{V}$, p belongs to the subspace V_1 of $\tilde{V}$. Let p' be the corresponding point of $Q(4,2)$ in V_0. Then $p+p'$ is in $\mathcal{P}$. Let $x_{i,j}$, $i=1,2,3$ and $j=1,2$, be the points of $\widetilde{\mathsf{W}(2)}$ collinear with p in Γ^{ε_1}, with indices taken in such a way that the lines of Γ^{ε_1} through p are $L_i = \{p, x_{i,1}, x_{i,2}\}$ for $i = 1,2,3$, and let $x'_{i,j}$ be the point of $Q(4,2)$ corresponding to $x_{i,j}$. Then $x_{i,j}+x'_{i,j}$ is a point of $\mathcal{P}$. Since the universal embedding is non-flat, the lines L_1, L_2, L_3 form a conic in the star of $p+p'$. Let L be the nucleus of that conic. Then L is a line of $\mathbf{PG}(\tilde{V})$ through $p+p'$, it is contained in the 3-space P_p of $\mathbf{PG}(\tilde{V})$ spanned by the points $x_{i,j}+x'_{i,j}$ and contains all points of P_p with the property that the projection from such point is injective on the points collinear with $p+p'$. So, there are precisely two such points and they are obtained as sum of three points picked on each of the lines L_1, L_2, L_3. Explicitly, we can write them as follows:

$$\begin{array}{rl} s_1 := & x_{1,1}+x'_{1,1}+x_{2,1}+x'_{2,1}+x_{3,1}+x'_{3,1} \quad \text{and} \\ s_2 := & x_{1,1}+x'_{1,1}+x_{2,1}+x'_{2,1}+x_{3,2}+x'_{3,2} \end{array}$$

As ε_1 is flat, we may assume without loss of generality that $x_{1,1}, x_{2,1}, x_{3,1}$ form a line of $\mathbf{PG}(V_1)$. Then $s_1 = x'_{1,1}+x'_{2,1}+x'_{3,1}$. So, $s_1 \in L'$, where L' is the nucleus of the conic formed by the lines $\{p', x'_{i,1}, x'_{i,1}+p'\}$ in the star of p'. Clearly, $L' = \{p', N, p'+N\}$, where N is the nucleus of $Q(4,2)$. If $s_1 = N$ then, by transitivity of $\mathrm{Aut}_{\varepsilon_1}(\Gamma)$ on the point-set P^{ε_1} of Γ^{ε_1} and by the fact that this group stabilizes N, the point N would belong to every space P_p, for all $p \in P^{\varepsilon_1}$, which implies that projecting from N would yield a flat embedding, contrary to the fact that $\hat{\varepsilon}$ is non-flat. Hence $s_1 = N+p'$. Thus, since $x_{1,1}+x_{2,1}+x_{3,2} = x_{3,1}+x_{3,2} = p$, we obtain $s_2 = p+N$. Hence our space U contains either $N+p$ or $N+p'$, for each point $p \in P^{\varepsilon_1}$. Notice that all points $N+p$ belong to $\langle N, V_1\rangle$, which has (vector) dimension 7.

Suppose first that U contains all points $N+p'$. Then $U \supseteq V_0$, as these points generate V_0. Hence $U = V_0$ since, as noticed above, U has vector dimension at most 5. In this case we are done.

Suppose now that $U \neq V_0$. Then U does not contain V_0 and, by the above argument, it contains at least one point of the form $N + p$. The intersection of U with V_0 has at most 10 points of the form $N + p'$. (It has precisely that number of points when $U \cap V_0$ is a hyperplane of V_0 meeting $Q(4, 2)$ in an elliptic quadric). As each point $N + p'$ comes from three points of P^{ε_1}, U contains at least 15 points of the form $N+p$, and they are all distinct. So U meets $\langle N, V_1\rangle$ in a space of vector dimension at least 4, which implies that U meets V_0 in a space of vector dimension at most 2 (if it contains N) or 1 (if it does not contain N). In any case, U contains at most one point $N+p'$, hence there must be at least 42 points in U of the form $N+p$. This forces $\dim(U) \geq 6$; a contradiction. $\boxed{QED}$

References

[1] A. E. Brouwer, A. M. Cohen, A. Neumaier: Distance Regular Graphs, Springer 1989.

[2] J. H. Conway: *Three lectures on exceptional groups*, in 'Finite Simple Groups' (M.B. Powell and G. Higman eds.), Academic Press (1971), 215–247.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson: Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[4] R. M. Foster: A census of trivalent symmetric graphs, unpublished, University of Waterloo, 1966.

[5] T. Ito: *Bipartite distance-regular graphs of valency three*, Lin. Alg. Appl. **46** (1982), 195–213.

[6] A. A. Ivanov: Geometry of Sporadic Groups I. Petersen and tilde geometries, Cambridge Univ. Press, 1999.

[7] A. A. Ivanov, S. V. Shpectorov: *The flag-transitive tilde and Petersen-type geometries are all known*, Bull. Amer. Math. Soc. **31** (1994), 173–184.

[8] A. A. Ivanov, D. V. Pasechnik, S. V. Shpectorov: *Non-Abelian representations of some sporadic geometries*, J. Algebra **181** (1996), 523–557.

[9] A. Kasikova, E. E. Shult: *Absolute embeddings of point-line geometries*, J. Algebra **238** (2001), 265–291.

[10] B. Polster, H. Van Maldeghem: *Some constructions of small generalized polygons*, J. Combin. Theory Ser. A **96** (2001), 162–179.

[11] A. Pasini, C. Wiedorn: *Local Parallelisms, shrinkings and geometries at infinity*, to appear in Quaderni di Matematica, II Univ. di Napoli.

[12] M. A. Ronan: *Embeddings and hyperplanes of discrete geometries*, European J. Combin. **8** (1987), 179–185.

[13] M. A. Ronan, G. Stroth: *Minimal parabolic geometries for the sporadic groups*, European J. Combin. **5** (1984), 59–91.

[14] E. E. Shult: *Embeddings and hyperplanes of Lie incidence geometries*, in 'Groups of Lie Type and Their Geometries' (W.M. Kantor and L. Di Martino eds.), Cambridge Univ. Press (1995), 215–232.

[15] D. H. Smith: *Primitive and imprimitive graphs*, Quart. J. Math. Oxford, **22** (1971), 551–557.

[16] G. Stroth, C. Wiedorn: *c-extensions of P- and T-geometries*, J. Combin. Theory Ser. A **93** (2001), 261–280.

[17] J. A. Thas: *Projective geometry over a finite field*, Chapter 7 of 'Handbook of Incidence Geometry' (F. Buekenhout ed.), North-Holland 1995.

[18] H. Van Maldeghem: Generalized Polygons, Birkhäuser, Basel 1998.