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Dual parallelisms

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Abstract. Assume that $\mathcal{P}$ is a parallelism in $PG(3,K)$, for K a field, that admits a collineation group G that fixes one spread Σ and acts transitively on the remaining spreads of $\mathcal{P}$. If G contains suitable central collineations of Σ then it is shown that the dual parallelism is a parallelism that can never be isomorphic to the original. The results show that the Johnson parallelisms of Hall or Knuth type, the Johnson-Pomareda parallelisms of type f and all of the 'derived' parallelisms produce dual parallelisms which are parallelisms but are nonisomorphic to the original parallelism.

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Introduction

Recently, the author and the author with Pomareda have constructed a variety of parallelisms in $PG(2,K)$ where K is a field (see Johnson [4] chapter 18, Johnson [5] and Johnson and Pomareda [6]). In various of these examples of parallelisms $\mathcal{P}$, there is a unique Pappian spread Σ and an affine central collineation group G of Σ with axis ℓ which acts transitively on the spreads of $\mathcal{P}-\{\Sigma\}$. (The other examples are related to such parallelisms by a 'derivation' process.)

In one case, the full central collineation group is employed and the remaining spreads are Hall spreads. In the finite cases, not covered by the above, the remaining spreads are derived Knuth semifield spreads. In the infinite cases, not covered by the above, the remaining spreads are derived conical flock spreads and there can be a variety of such examples.

In general, if $-$ is a parallelism in $PG(3,K)$, for K a field, then by applying a duality δ, there is a corresponding 'dual parallelism' $\Gamma\delta$, that is, a covering of the lines by a set of dual spreads. If all of the spreads of Γ are dual spreads then $\Gamma\delta$ is a parallelism. A major problem is to determine when $\Gamma\delta$ is isomorphic to Γ in the case that $\Gamma\delta$ is a parallelism.

In the only study of this type considered, Pentilla and Williams [8] construct an infinite class of finite parallelisms in $PG(3,q)$ each admitting a group transitive on the spreads of the parallelism where the spreads are all Desarguesian.

Pentilla and Williams are able to show that the dual parallelisms are always non-isomorphic to the original parallelisms.

In this note, we show that all of the parallelisms $\mathcal{P}$ of the author or of Johnson-Pomareda mentioned above have the property that $\mathcal{P}\delta$ is always *non-isomorphic* to $\mathcal{P}$ thus generating a variety of completely new classes of parallelisms. Moreover, there are classes of 'derived' parallelisms and all of their duals are parallelisms which are not isomorphic to the original.

Actually, we provide a result which shows that any of the so-called finite 'deficiency one' transitive parallelisms have dual structures that are parallelisms and are non-isomorphic provided there is a sufficient central collineation subgroup.

Our main result in the finite case is the following theorem.

Theorem 1. *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$. Assume that there exists a Desarguesian spread Σ in $\mathcal{P}$ and an collineation group G of $\mathcal{P}$ which fixes Σ and a component ℓ of Σ and acts transitively on the remaining spreads of $\mathcal{P}$.*

Assume that G contains an elation group E^+ of order q^2.

(1) If G contains a homology of odd order with axis ℓ which does not fix any spread of $\mathcal{P}-\{\Sigma\}$ then the dual parallelism $\mathcal{P}\delta$ is a parallelism that is not isomorphic to $\mathcal{P}$.

(2) If $q+1=2^a$ for some integer a and G contains a homology of order $2^b \geq 8$ then the dual parallelism $\mathcal{P}\delta$ is a parallelism that is not isomorphic to $\mathcal{P}$.

For infinite parallelisms, we prove the following results.

Theorem 2. *Let $\mathcal{P}$ be a parallelism of $PG(3,K)$, for K a skewfield, containing a spread Σ and a central collineation group G^- of Σ with affine axis ℓ.*

If G^- acts two-transitively on the lines of $\Sigma-\{\ell\}$ then K is a field and the dual parallelism $\mathcal{P}\delta$ is a parallelism that is not isomorphic to $\mathcal{P}$.

Theorem 3. *(1) Let $\mathcal{P}$ be a parallelism in $PG(3,K)$ where K is a field of odd or zero characteristic. Assume that G^- acts transitively on the spreads not equal to Σ of the parallelism $\mathcal{P}$ and Σ and a given spread ρ of $\mathcal{P}-\{\Sigma\}$ have the following form:*

$$\Sigma : x=0, y=x\begin{bmatrix} u & \gamma t \\ t & u \end{bmatrix} \forall u,t \in K,\ \gamma \text{ a nonsquare in } K$$

and

$$\rho^* \ :\ x=0, y=x\begin{bmatrix} v & f(s) \\ s & v \end{bmatrix} \forall v,s \in K,$$

f some function on K such that

$f(f(t)/\gamma)$ is not identically γt,

where ρ^* denotes the spread derived from ρ by replacing the opposite regulus to

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K.$$

If Σ is Pappian and G contains the full elation group with axis ℓ of Σ and some homology with axis ℓ which does not fix any spread of the parallelism then the parallelism is not isomorphic to the dual parallelism and the dual parallelism is a parallelism.

(2) If $\mathcal{P}$ is one of the parallelisms of Johnson-Pomareda in $PG(3, K)$ where K is the field of real numbers and $\gamma = -1$, then $\mathcal{P}$ is not isomorphic to the dual parallelism and the dual parallelism is a parallelism.

Since the known examples of Johnson and Johnson and Pomareda fit one of these situations, we have the following corollary., part of which was noted in the previous theorem part (2).

Corollary 1. *Let $\mathcal{P}$ be a parallelism in $PG(3, K)$, for K a field.*

If $\mathcal{P}$ is a Johnson parallelism of Hall type, or of Knuth type in [4] or [5] or a Johnson-Pomareda parallelism of type f [6] then the dual parallelism $\mathcal{P}\delta$ is not isomorphic to $\mathcal{P}$.

1 Background

In Johnson [5], the following result was proved.

Theorem 4. *Let $\mathcal{P}$ be a parallelism in $PG(3, K)$, for K a field, that admits a Pappian spread Σ and a collineation group G^- fixing a line ℓ of Σ that acts transitively on the remaining spreads of $\mathcal{P}$.*

(1) If K is finite and if G^- contains the full elation group with axis ℓ then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.

(2) If G^- contains the full elation group with axis ℓ and for ρ a spread of $\mathcal{P} - \{\Sigma\}$, G^-_ρ contains a non-trivial homology (i.e. homology in Σ) then the spreads of $\mathcal{P} - \{\Sigma\}$ are derived conical flock spreads.

We note that in the above case, the dual parallelism is always a parallelism.

Theorem 5. *Let $\mathcal{P}$ be a parallelism of $PG(3, K)$, for K a field, whose spreads are either conical flock spreads or derived conical flock spreads (by reguli). Then the dual parallelism $\mathcal{P}\delta$ is a parallelism.*

PROOF. Now it is shown in Johnson [3] that all conical flock spreads are dual spreads. By Johnson [4], Theorem 24.1, p. 322, any derived conical flock spread is a dual spread since we are deriving by the replacement of a regulus in $PG(3, K)$. Hence, in the situation of the result stated above, then all spreads of the parallelism $\mathcal{P}$ are dual spreads so that the dual parallelism $\mathcal{P}\delta$ is a parallelism. QED

Before we give the proof to our main result, we need to ask what effect a duality will have on the collineation group of a spread.

Theorem 6. *(see e.g. Johnson [2])*

Let Σ be a spread which is a dual spread. If coordinates are chosen so that Σ is

$$x = 0, y = xM \text{ for } M \text{ in a set of matrices} \mathcal{M},$$

then the dual spread $\Sigma\delta$ of Σ has spread set

$$x = 0, y = xM^t \text{ for } M \text{ in a set of matrices} \mathcal{M}, \quad \text{where } M^t \text{ denotes the transpose of } M.$$

Theorem 7. *(see e.g. Biliotti, Jha, Johnson [1])*

Let π be a spread in $PG(3, K)$, for K a field, such that the dual spread $\pi\delta$ is a spread. Let C be an affine central collineation of π with axis ℓ. Let $C = EH$ where E is the normal elation subgroup with axis ℓ and H is a homology group with axis ℓ and coaxis $co(H)$.

Then $\pi\delta$ has a collineation group $C^\delta = \mathcal{E}\mathcal{K}$ where $\mathcal{E}$ is an elation group isomorphic to E and $\mathcal{K}$ is a homology group with axis $co(H)$ and coaxis ℓ.

By reference to the construction of Johnson in the following theorem, we mean the construction given in [4], chapter 18 (also see list below).

Theorem 8. *(Johnson and Pomareda [7]) If $\mathcal{P}$ admits as a collineation group the full central collineation group G of Σ with a given axis ℓ that acts two-transitive on the remaining spread lines then*

(1) Σ is Pappian,

(2) $\mathcal{P}$ is a parallelism,

(3) the spreads of $\mathcal{P} - \{\pm\}$ are Hall, and

(4) G acts transitively on the spreads of $\mathcal{P} - \{\pm\}$.

(5) Moreover, $\mathcal{P}$ is one of the parallelisms of the construction of Johnson.

1. 1 The Known Examples

The first two theorems are from Johnson [4].

Theorem 9. *Let Σ be a Pappian spread in $PG(3, K)$ for K a field. Assume that there exists a regulus R which is contained in at least two distinct Pappian spreads Σ and Σ'. Let L be a fixed component of Σ and let G denote the full group of central collineations of the affine translation plane $\mathcal{A}$ associated with Σ with axis L.*

Consider the set of spreads $\{\Sigma' g; g \in G\}$ *and form the Hall spreads* $\overline{\Sigma' g}$ *by derivation of each* Rg. *Let* $\overline{\Sigma'} g$ *denote the associated derived spreads that are images under elements* g *of* G.

(1) $\overline{\Sigma' g} = \overline{\Sigma'} g$; *there is a group of* Σ *acting transitively on the set of Hall spreads.*

(2) $\Sigma \cup \{\ \overline{\Sigma' g}\ ; g \in G\}$ *is a parallelism consisting of one Pappian spread and the remaining spreads are Hall spreads.*

Using the previous construction, we may obtain another parallelism by the derivation of Σ and $\overline{\Sigma'}$.

Theorem 10. *Under the assumptions of the previous theorem, let* $\overline{\Sigma}$ *denote the Hall spread obtained by the derivation of* R *and let* $\mathcal{P}$ *denote the previously constructed parallelism.*

Then $\overline{\Sigma} \cup \Sigma' \cup \{\mathcal{P} - \{\Sigma, \overline{\Sigma'}\}\}$ *is a parallelism of* $PG(3, K)$.

Definition 1. For purposes of description, we shall refer to any parallelism constructed via the central collineation group of a Pappian spread within $PG(3, K)$, a 'Johnson parallelism of Hall type'. The parallelism constructed as directly above shall be termed a 'derived Johnson parallelism of Hall type'.

The following results in the finite case are from Johnson [5].

Theorem 11. *Let* q *be odd equal to* $p^{2^b z}$ *where* z *is an odd integer* > 1. *Assume that* $2^a \parallel (q-1)$ *then there exists a nonidentity automorphism* σ *of* $GF(q)$ *such that* $2^a \mid (\sigma - 1)$.

Let γ_2 *and* γ_1 *be nonsquares of* $GF(q)$ *such that the equation* $\gamma_2 t^\sigma = \gamma_1 t$ *implies that* $t = 0$.

(1) Then, there exists a parallelism $\mathcal{P}_{\gamma_2,\sigma}$ *of derived Knuth type with* $q^2 + q$ *derived Knuth planes and one Desarguesian plane.*

(2) The collineation group of this parallelism contains the central collineation group of the Desarguesian plane with fixed axis ℓ *of order* $q^2 2^a (q+1)$.

Theorem 12. *For each parallelism of type* $\mathcal{P}_{\gamma,\sigma}$, *there is a parallelism consisting of one Hall spread, one Knuth semifield spread, and* $q^2 + q - 1$ *derived Knuth spreads. We shall call such parallelisms the 'derived' parallelisms of the* $\mathcal{P}_{\gamma,\sigma}$*-parallelisms.*

Definition 2. The parallelisms above shall be called the 'Johnson parallelism of Knuth type' and the 'derived Johnson parallelisms of Knuth type'.

The following results are from Johnson and Pomareda [6].

We let Σ_2 be a spread in $PG(3, \mathcal{R})$, defined by a function f:

$$x = 0, y = x \begin{bmatrix} u & -f(t) \\ t & u \end{bmatrix} \forall u, t \in \mathcal{R}$$

where f is a function such that $f(t) = t$ implies that $t = 0$ and $f(0) = 0$.

Thus, if a spread exists then the two spreads Σ_1 and Σ_2 share exactly the regulus $\mathcal{D}$ with partial spread:

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in \mathcal{R}.$$

Lemma 1. *Let f be any continuous strictly increasing function on the field of real numbers such that $\lim_{x \longmapsto \pm\infty} f(t) = \pm\infty$.*

(1) Then Σ_2 is a spread.

(2) Let $G^- = EH^-$ where H^- denotes the homology group of Σ_1 (or rather the associated affine plane) whose elements are given by

$$\left\langle \begin{bmatrix} u & -t & 0 & 0 \\ t & u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; u^2 + t^2 = 1 \right\rangle.$$

and where E denotes the full elation group with axis $x = 0$.

(2) Then G^- is transitive on the set of reguli of Σ_1 that share $x = 0$.

Theorem 13. *Under the above assumptions, assume also that f is symmetric with respect to the origin in the real Euclidean 2-space and $f(t_o + r) = f(t_o) + r$ for some t_o and r in the reals implies that $r = 0$.*

Then $\Sigma_1 \cup \Sigma_2^* g$ for all $g \in G^-$ and where Σ_2^* denotes the derived spread of Σ_2 by derivation of $\mathcal{D}$, is a partial parallelism $\mathcal{P}_f$ in $PG(3, \mathcal{R})$.

Theorem 14. *The above construction produces a parallelism if and only if $f(t) - t$ is surjective.*

We note in the following section on derived parallelisms that there is always a set of derived parallelisms in this setting.

Definition 3. The parallelisms constructed above are called the 'Johnson-Pomareda parallelisms of type f' and the 'derived Johnson-Pomareda parallelisms of type f'.

2 The Proofs of the Main Results

It now remains to show that $\mathcal{P}$ is not isomorphic to $\mathcal{P}\delta$ when $\mathcal{P}$ is a parallelism of the type stated in the main theorems listed in the introduction.

Assume that $\mathcal{P}\delta$ is isomorphic to $\mathcal{P}$ by an element σ of $\Gamma L(4, K)$. The natural duality of the projective space will map a spread Σ of the form $x = 0, y = xM$

into a spread of the form $x = 0, y = xM^{-t}$. Since Σ is Pappian this spread retains the form $x = 0, y = xM^t$. Coordinates for Σ can easily chosen so that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is a collineation of the projective space that reconfigures the spread into its original form. Hence, we may assume that σ leaves Σ invariant.

We are assuming that $\mathcal{P}$ admits the central collineation subgroup EH of G that also acts on Σ. The above results show that $\mathcal{P}\delta$ admits a collineation group G^δ containing the collineation group $C^\delta = \mathcal{E}\mathcal{K}$ where $\mathcal{E}$ is an elation group isomorphic to E and $\mathcal{K}$ is a homology group with axis $co(H)$ and coaxis ℓ with $\Sigma\delta$.

Assume that the parallelism is finite.

First assume that σ leaves ℓ invariant. In this case, we have the following group $EH\mathcal{K}$ acting as an automorphism group of Σ and $\mathcal{P}$. The elation group E^+ of order q^2 is partitioned into a set of $q+1$ subgroups of order $q-1$ each of which fixes exactly q derived conical flock spreads. Any homology group H permutes these elation groups. Let H have an affine homology h of prime order u which does not fix a second spread of the parallelism. If the order of h divides $q-1$ but does not divide $q+1$, then h must fix one of the above elation groups and permute the set of q derived conical flock spreads fixed by the associated elation group. Hence, h then would fix one of these spreads. So, if h does not fix a spread other than Σ, it follows that the order of h must divide $q+1$ but not $q-1$. We point out that if $q+1 = 2^a$ then the above argument requires some refinement since then there is not such prime order u which divides $q+1$ but not $q-1$.

First assume that $q+1 \neq 2^a$, so that there is a homology of order of prime order u dividing $q+1$ but not $q-1$. We know that the order of G is divisible by $q^2(q+1)$. Represent the Desarguesian affine plane in the standard manner and if the axis of the homology group is $x = 0$ and the co-axis is $y = 0$, then we obtain an element of the matrix form: $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for some $a \in GF(q^2)$, where the order of a is u and $GF(q^2)$ is the field coordinatizing Σ. From the transpose theorem given above, it follows that we not have an element of the same order with axis $y = 0$ and co-axis $x = 0$ and of the matrix form: $\left[\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right]$ where b also has order u. Since we are in a field $GF(q^2)$, it follows that $\langle a \rangle = \langle b \rangle$. Hence, within the group there is a collineation of the following form: $\left[\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right]$ where c has order u. This is a kernel homology of the associated Desarguesian affine plane with spread Σ. Now this group fixes every regulus of Σ containing ℓ and hence fixes each derived regulus and thus fixes each derived conical flock spread; i.e. fixes each spread of the parallelism (see Johnson [5] to see that the structure is as claimed).

So, we have a kernel homology group of the associated Desarguesian affine plane π_Σ that acts as a collineation group of each derived flock spread. Moreover, this group does not leave any Baer subplane of π_Σ invariant because if it did the group would induce a kernel homology group on a subplane of order q which would force the group order to divide $q-1$.

Hence, u^a divides q^2+1. But, u^a divides $q+1$ which divides q^2-1 and $(q^2+1, q^2-1) = (2, q^2-1)$, a contradiction since u is odd.

Now assume that $q+1 = 2^a$. In this setting, it follows that $q = p$ is an odd prime. Let h be a homology of order 2^a which does not fix a spread not equal to Σ of the parallelism. Then, it follows that $2^a \geq 4$ since every involutory homology fixes the standard regulus. Assume that, in fact, $2^a \geq 8$. The above argument shows that there exists homologies of order 8 with axis $x = 0$ and coaxis $y = 0$ and homologies of order 8 with axis $y = 0$ and coaxis $x = 0$ and these generated a kernel homology of order 8 of the Desarguesian affine plane π_Σ. As above, this group must fix each spread of the parallelism but can fix no line of any spread. However, the involution in the cyclic kernel homology group fixes all spread lines not in Σ but since 4 does not divide $q-1$, we are forced to have 4 dividing q^2+1, a contradiction.

Now assume that the isomorphism σ does not leave ℓ invariant. Then, clearly there is a group isomorphic to $SL(2,q^2)$ generated by the elation groups. This subgroup is normal and does not contain affine homologies so we have a group of order divisible by $u^a q^2(q^4-1)$. Again, the above analysis shows that there is a kernel homology group of order u^a fixing a regulus and this implies a contradiction as above. This completes the proof in the finite case.

We now assume that the parallelism is infinite.

Lemma 2. *(1) A parallelism of infinite Johnson type cannot admit the kernel homology group of the fixed Pappian spread.*

(2) A parallelism of Johnson-Pomareda type cannot admit the kernel homology group of the fixed Pappian spread corresponding to the determinant 1 *group.*

PROOF. Suppose so. Then a kernel element g must fix each regulus net on the special line and then fix each parallelism. This means that the group g must act semi-regularly on each derived conical flock spread. But, in this case, this means that in the conical flock spread, there is a collineation g which fixes each component of a regulus net and fixes no other component. Moreover, the collineation acting in $\Gamma L(4,K)$ on the conical flock spread is linear since it comes from a kernel homology of Σ.

Let the conical flock spread be given in the following form:

$$x = 0, y = x \begin{bmatrix} u + g(t) & f(t) \\ t & u \end{bmatrix} \forall u, t \in K,$$

where f and g are the defining functions on K. We are assuming that the regulus fixed componentwise by g is the standard regulus. we may assume this without loss of generality. Since g fixes the standard regulus componentwise, it follows that $g = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$ where

$$A = \begin{bmatrix} w + a_o s & b_o s \\ s & w \end{bmatrix}.$$

In the Johnson type, we have that the associated derived conical flock spreads are Hall spreads so that the conical flock spreads are Pappian. This means that

$$(g(t), f(t)) = (a_1 t, b_1 t) \ \forall t \in K.$$

In the Johnson-Pomareda type, we have that

$$g(t) = 0 \ \forall t \in K.$$

Hence, for the Johnson type, we have, for all $u, t \in K$:

$$\begin{bmatrix} w + a_o s & b_o s \\ s & w \end{bmatrix}^{-1} \begin{bmatrix} u + a_1 t & b_1 t \\ t & u \end{bmatrix} \begin{bmatrix} w + a_o s & b_o s \\ s & w \end{bmatrix}$$
$$= \begin{bmatrix} u^* + a_o t^* & b_o t^* \\ t^* & u^* \end{bmatrix} \text{ for some } u^*,\ t^* \text{ depending on } u \text{ and } t.$$

First assume that the isomorphism between the parallelism and dual parallelism fixes the axis of the central collineation group. Moreover, since we have the 'full homology group' with axis $x = 0$ and coaxis $y = 0$ and the full homology group with axis $y = 0$ and coaxis $x = 0$, we would obtain the full kernel homology group of Σ acting on the parallelism and hence fixing all spreads of the parallelism.

If the isomorphism does not leave ℓ invariant, since we have a full homology group with axis ℓ, the group generated is clearly $GL(2, K^2)$. Hence, here we also obtain the full kernel homology group of Σ acting on the parallelism.

If we vary w, s, u, t and insist that the above images all belong to a second Pappian spread Σ', it is straightforward (but a little long) to check out that the only way this could occur is if $a_o = a_1$ and $b_0 = b_1$; that is that $\Sigma = \Sigma'$, which does not occur.

If the parallelisms is a Johnson-Pomareda type, we must have:

$$\begin{bmatrix} w & -s \\ s & w \end{bmatrix}^{-1} \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \begin{bmatrix} w & -s \\ s & w \end{bmatrix}$$

$$= \begin{bmatrix} u^* & f(t^*) \\ t^* & u^* \end{bmatrix} \text{ for some } u^*,\ t^* \text{ depending on } u \text{ and } t.$$

Now the group G is transitive on the components of $\Sigma - \{\ell\}$, so if the isomorphism moves ℓ then the group generated is doubly transitive on the components of Σ. Then, there must be a homology group with determinant 1 (of the type mentioned in the list of parallelism) with coaxis ℓ since the group is transitive on the components. This means that we may assume that we have both types of homology groups. So, we have a generated kernel homology group of determinant 1 acting on the parallelism.

So, in any case, we always have a generated kernel homology group of determinant 1 type acting on the parallelism.

In the above equation, we may assume that $s \neq 0$ for infinitely many kernel type homologies.

Hence, we obtain:

$$\begin{bmatrix} w & -s \\ s & w \end{bmatrix}^{-1} \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \begin{bmatrix} w & -s \\ s & w \end{bmatrix}$$

$$= \begin{bmatrix} (uw + st)w + (wf(t) + su)s & \\ & -s(-su + wt) + (-sf(t) + wu)w \end{bmatrix}$$

so that the $(1,1)$- and $(2,2)$-entries are equal. This implies that

$$swf(t) = -swt$$

for all s, w such that $w^2 + s^2 = 1$. Hence, this implies that the two spreads are equal, contrary to assumption. This shows that the Johnson type and the Johnson-Pomareda type parallelisms are not isomorphic to their duals. The result given which characterizes the Johnson type parallelisms applies to complete our first stated theorem for the infinite case. However, in the statement of the second theorem for the infinite case, we did not assume the full extent of the homology group, nor did we assume that the field K is the field of real numbers, merely that the fixed spread is Pappian of the given form and the full elation group exists and there is some homology with the same axis which does not fix any spread. In particular, we may assume in the above equation that we have:

$$\begin{bmatrix} w & \gamma s \\ s & w \end{bmatrix}^{-1} \begin{bmatrix} u & f(t) \\ t & u \end{bmatrix} \begin{bmatrix} w & \gamma s \\ s & w \end{bmatrix}$$

$$= \begin{bmatrix} (uw - \gamma st)w + (wf(t) - \gamma su)s & \\ & \gamma s(-su + wt) + (-sf(t) + wu)w \end{bmatrix}$$

which implies:

$$2wsf(t) = 2ws\gamma t.$$

Now since the indicated kernel homology of the Pappian spread Σ does not fix another spread of the parallelism, it follows that we may assume that $s \neq 0$. However, it may be possible that $w = 0$ in this more general setting, otherwise we have the same contradiction as above or the characteristic is 2 which has been excluded by assumption.

So, assume that $w = 0$. Then, we obtain:

$$\begin{bmatrix} 0 & \gamma s \\ s & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & f(t) \\ t & 0 \end{bmatrix} \begin{bmatrix} 0 & \gamma s \\ s & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma t \\ f(t)/\gamma & 0 \end{bmatrix}$$

which implies that

$$f(f(t)/\gamma) = \gamma t \; \forall t \in K.$$

We have excluded this possibility so we have the proof to the theorem. We note that it might be possible for a function f to have this property and still satisfy the assumptions of Johnson-Pomareda. This completes the proofs to the three main theorems listed in the introduction. QED

3 The Derived Parallelisms

For every parallelism of Johnson type or of Johnson-Pomareda type, there is an associated parallelism called the 'derived parallelism'.

The following in Johnson [5] illustrates the technique:

Let Σ be any Pappian spread in $PG(3, K)$ and let Σ' any spread which shares a regulus R with Σ such that Σ' is derivable with respect to R. Assume that there exists a subgroup G^- of the central collineation group G with fixed axis L with the following properties:

(0) : Σ and Σ' share exactly R,

(i) : Every line skew to L and not in Σ is in $\Sigma' G^-$,

(ii) : G^- is transitive on the reguli that share L and

(iii) : a collineation g of G^- such that for $L' \in \Sigma'$ then $L'g \in \Sigma'$ implies that g is a collineation of Σ'.

Let $(Rg)^*$ denote the opposite regulus to Rg.

We shall call this construction the 'regulus construction technique'

Theorem 15. *Under the above assumptions,* $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ *for all* $g \in G^-$ *is a parallelism of* $PG(3, K)$ *consisting of one Pappian spread* Σ *and the remaining spreads derived* Σ'*-spreads.*

Theorem 16. *Assume that* $\Sigma \cup \{(\Sigma' g - Rg) \cup (Rg)^*$*for all* $g \in G^-$*is a parallelism. Then* $\{\Sigma - R\} \cup R^* \cup \Sigma' \cup \{(\Sigma' g - Rg) \cup (Rg)^*$ *for all* $g \in G^- - \{1\}\}$ *is a parallelism. In this case, the spreads are Hall,* Σ' *(undetermined) and derived* Σ' *type spreads.*

We call the second construction the 'derived parallelism construction'. We then ask the question of whether the dual parallelism of a derived parallelism is a parallelism and, if so, whether this parallelism could be isomorphic to the original derived parallelism. However, we don't yet know that the original parallelism (the pre-derived parallelism) has a dual parallelism which is also a parallelism. So, anything that we say in general about the derived parallelisms will have some hypothesis regarding the pre-derived parallelism.

We note that all spreads of the derived parallelism are either spreads of the original parallelism with two exceptions: We have derived two of these spreads. However, since the derivation is accomplished via a regulus, it follows that the derived spreads are also dual spreads if and only if the original spreads are dual spreads. Hence, if the original parallelism has a dual parallelism which is also a parallelism then the derived parallelism has a dual parallelism which is a parallelism. The question is now whether such a dual parallelism is isomorphic to the derived parallelism.

Theorem 17. *Let* $\mathcal{P}$ *be a parallelism in* $PG(2, K)$ *which has been constructed by the regulus construction technique, for* K *is a field of order* > 3. *Assume that there is a unique Pappian spread and that the remaining spreads are all derived conical flock spreads. Let* $\mathcal{P}^*$ *denote any derived parallelism. Then there is a unique Hall spread, a unique conical flock spread and the remaining spreads are derived conical flock spreads. We assume that for each regulus* R *of* Σ *containing a fixed line* ℓ *, there are exactly one spread* ρ_R *of* $\mathcal{P}$ *containing the opposite regulus* R^*.

Then the dual $\mathcal{P}\delta$ *is isomorphic to* $\mathcal{P}$ *if and only if the dual* $\mathcal{P}^*\delta$ *is isomorphic to* $\mathcal{P}^*$.

PROOF. Let $\mathcal{P} = \Sigma \cup \rho \cup (\mathcal{P} - \{\rho, \Sigma\})$ and $\mathcal{P}^* = \Sigma^* \cup \rho^* \cup (\mathcal{P} - \{\rho, \Sigma\})$, where Σ is the unique Pappian spread of $\mathcal{P}$ and ρ is a derived conical flock spread. We assume that Σ and ρ^* share a unique regulus R. Hence, in $\mathcal{P}^*$, there is a unique conical flock spread ρ^* and a Hall spread Σ^*, although there may be other Hall spreads. Then, in $\mathcal{P}\delta$, there is a unique Pappian spread $\Sigma\delta$ and there is a unique Pappian spread $\rho^*\delta$ in $\mathcal{P}^*\delta$.

First assume that there is a unique Hall spread in $\mathcal{P}^*$. Then, if σ is an isomorphism mapping $\mathcal{P}^*\delta$ onto $\mathcal{P}^*$ then σ must map $\rho^*\delta$ onto ρ^* and map $\Sigma^*\delta$ onto Σ^* . Assuming that R is the regulus of Σ used in constructing Σ^*, this implies that $R^*\delta$ maps to R^* under σ using Johnson and Pomareda [7]. Hence, $R\delta\sigma = R$ which implies that $\Sigma\delta\sigma = \Sigma$. But, since ρ^* is the unique conical flock spread in $\mathcal{P}$, it follows that $\rho^*\delta\sigma = \rho^*$ and since $R\sigma\delta = R$, we have that $\rho\delta\sigma = \rho$. Since the remaining spreads of $\mathcal{P}^*$ are also spreads of $\mathcal{P}$, it follows that σ induces an isomorphism from $\mathcal{P}\delta$ onto $\mathcal{P}$.

Assume that σ maps $\Sigma^*\delta$ onto a Hall spread γ of $\mathcal{P}^*$. Now let R denote the regulus of Σ obtained in the construction of Σ^* and so R is in the conical flock spread ρ^*, and ρ^* and Σ share exactly this regulus and no other components. Let D^* denote the unique regulus shared by the Pappian spread γ^* (obtained by the derivation of γ) and the Pappian spread Σ. By Johnson and Pomareda [7], $R^*\delta\sigma = D$ implying that $R\delta\sigma = D^*$. Hence, we have $\rho^*\delta\sigma = \rho^*$ implying that ρ^* shares the regulus R, D^* of Σ so that we can only have $R = D^*$ and so $R^* = D$, which, in turn, implies that $\gamma = \Sigma^*$. Thus, $\Sigma^*\delta\sigma = \Sigma^*$. But, this implies that $\rho\delta\sigma = \rho$ and $\Sigma\delta\sigma = \Sigma$ and, as above, this means that σ induces an isomorphism from $\mathcal{P}^*\delta$ onto $\mathcal{P}^*$. QED

Corollary 2. *Let $\mathcal{P}$ be either a Johnson parallelism of Hall type, derived Johnson parallelism of Hall type, Johnson parallelism of Knuth type, derived Johnson parallelism of Knuth type, Johnson-Pomareda parallelism of type f or derived Johnson-Pomareda parallelism of type f.*

Then the dual parallelism $\mathcal{P}\delta$ is a parallelism which is not isomorphic to $\mathcal{P}$.

4 Maximal Partial Parallelisms

Let $\mathcal{P}$ be a partial parallelism which cannot be properly extended to a partial parallelism. Then we say that $\mathcal{P}$ is a 'maximal' partial parallelism. In Johnson and Pomareda [7], it is pointed out that one obtains a parallelism as above if and only if the function has the property that $f(t)-t$ is surjective when K is the field of real numbers. Since all of the spreads of the partial parallelism are derived conical flock spreads or Pappian, it follows that the dual partial parallelism $\mathcal{P}\delta$ is also a partial parallelism which is maximal. The above arguments also show that such a maximal partial parallelism cannot be isomorphic to its dual. Moreover, the derived partial parallelism is also maximal and cannot be isomorphic to its dual.

Theorem 18. *In any Johnson-Pomareda maximal partial parallelism $\mathcal{P}$, any derived partial parallelism $\mathcal{P}^*$ is a maximal partial parallelism and the duals $\mathcal{P}\delta$ and $\mathcal{P}^*\delta$ are maximal partial parallelisms which are not isomorphic to $\mathcal{P}$ or $\mathcal{P}^*$ respectively.*

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