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Riemannian manifolds structured by a $\mathcal{T}$-parallel exterior recurrent connection

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Abstract. Geometrical and structural properties are proved for Riemannian manifolds which are equipped with a $\mathcal{T}$-parallel exterior recurrent connection.

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Introduction

Riemannian manifolds structured by a $\mathcal{T}$-parallel connection have been defined in [9] and have also been studied in [6]. Let M be a $2m$-dimensional C^∞-manifold and ∇ be the Levi-Civita connection. We recall that if M carries a globally defined vector field $\mathcal{T}(\mathcal{T}^A)$ and the connection forms satisfy

$$\theta^A_B = < \mathcal{T}, e_B \wedge e_A >, \tag{1}$$

where $\wedge$ denotes the wedge product of vector fields, then one says that M is structured by a $\mathcal{T}$-parallel connection. In the present paper we assume in addition that θ^A_B are exterior recurrent forms [2], which means that

$$d\theta^A_B = 2\alpha \wedge \theta^A_B, \qquad \text{where} \qquad \alpha = \mathcal{T}^\flat, \tag{2}$$

having $\mathcal{T}^\flat$ as recurrence form. This implies that the curvature forms Θ^A_B are also exterior recurrent. In consequence of this fact, we adopt the terminology that M is structured by a $\mathcal{T}$-parallel exterior recurrent connection.

For the above mentioned structures, we prove the following properties:

(i) $\mathcal{T}$ is a concurrent vector field and defines an infinitesimal conformal transformation of θ^A_B and Θ^A_B and the differential system ∇_{e_A} corresponding to the vector basis $\mathcal{O} = \{e_A\}$ admits an infinitesimal transformation with generator $\mathcal{T}$;

(ii) $\|\mathcal{T}\|^2$ is an isoparametric function [13], and an eigenfunction of Δ having $4(2m + \|\mathcal{T}\|^2)$ as eigenvalue;

(iii) if V is any parallel vector field, one has by the Weitzenbock formula that

$$(\Delta \mathcal{T}^\flat)V = -4m\|\mathcal{T}\|^2 g(\mathcal{T}, V);$$

(iv) if

$$\Theta^{(p)}_{u^1,\ldots,u^{2p}} = \Theta^{u_2}_{u_1} \wedge \Theta^{u_3}_{u_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1}$$

denotes the Bianchi forms (in the sense of Tachibana [12]), these forms are exterior recurrent with $3(2m-1)\alpha$ as recurrence form;

(v) any vector field X such that

$$\nabla X = X \wedge \mathcal{T}$$

is a skew symmetric Killing vector field [11] and X defines an infinitesimal transformation of the conformal symplectic form Ω, i.e.

$$\mathcal{L}_X \Omega = -2g(X, \mathcal{T})\Omega.$$

In Section 4 we consider some properties of the tangent bundle manifold TM having the manifold M, studied in Section 3, as basis. On TM the canonical vector field $V(V^A)$ $(A = 1, \cdots 2m)$ is called the Liouville vector field [3], and the complete lift [14] Ω^C of the structure 2-form of rank $4m$ on TM is given by

$$\Omega^C = \sum dV^a \wedge \omega^{a^*} + \omega^a \wedge = dV^{a^*}, \qquad a = 1, \cdots m; a^* = a + m. \tag{3}$$

In Section 3, the following relation will be derived (see formula (24)):

$$d\omega^A = \alpha \wedge \omega^A.$$

By exterior differentiation of (3), and taking into account the above formula, one gets

$$d\Omega^C = \alpha \wedge \Omega^C, \tag{4}$$

and

$$\mathcal{L}_V \Omega^C = \Omega^C. \tag{5}$$

The above equations express that the 2-form Ω^C is a homogeneous 2-form of class 1 [4] on TM. Next, the Liouville form μ (i.e. $\mu = V^\flat$) is expressed by

$$\mu = \sum V^A \omega^A \qquad A = 1, \cdots 2m \tag{6}$$

and one finds by exterior differentiation that

$$d\mu = \alpha \wedge \mu + \psi, \tag{7}$$

where we have set

$$\psi = \sum dV^A \wedge \omega^A. \tag{8}$$

One also derives that

$$\mathcal{L}_V \psi = \psi, \tag{9}$$

and this shows that, like Ω^C, the form ψ is a homogeneous 2-form of class 1. Moreover, making use of the vertical operator i_v of Godbillon [3], one calculates that

$$i_v \psi = 0, \tag{10}$$

which together with (9) proves that ψ is a Finslerian form. In addition, if $\mathcal{T}^V$ denotes the vertical lift of $\mathcal{T}$, one also finds that

$$\mathcal{L}_{\mathcal{T}^V} \psi = 0,$$

which shows that $\mathcal{T}^V$ defines an infinitesimal automorphism of ψ. Some other properties regarding the principal almost symplectic form $II = \|\mathcal{T}\|^2 \psi$ are also discussed.

1 Preliminaries

Let (M, g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator with respect to the metric tensor g. We assume that M is oriented and ∇ is the Levi-Civita connection of g. Let $\Gamma TM = \Xi(M)$ be the set of sections of the tangent bundle, and

$$\flat \colon TM \xrightarrow{\flat} T^*M \qquad \text{and} \qquad \sharp \colon TM \xleftarrow{\sharp} T^*M \tag{11}$$

the classical isomorphisms defined by g (i.e. $^\flat$ is the index lowering operator, and $^\sharp$ is the index raising operator).

Following [8], we denote by

$$A^q(M, TM) = \Gamma \mathrm{Hom}(\Lambda^q TM, TM), \tag{12}$$

the set of vector valued q-forms ($q < \dim M$), and we write for the covariant derivative operator with respect to ∇

$$d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM). \tag{13}$$

It should be noticed that in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. We denote by $I \in A^1(M, TM)$ the canonical vector valued 1-form of M, which is also called the soldering form of M [2]. Since ∇ is symmetric one has that $d^\nabla(I) = 0$.

A vector field $Z \in \Xi(M)$ which satisfies

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge I \in A^2(M, TM); \qquad \pi \in \Lambda^1 M \tag{14}$$

is defined to be an exterior concurrent vector field [9] (see also [6]). The 1-form π in (14) is called the concurrence form and is defined by

$$\pi = \lambda Z^\flat, \qquad \lambda \in \Lambda^0 M. \tag{15}$$

Let $\mathcal{O} = \{e_A \mid A = 1, \cdots 2m\}$ be a local field of orthonormal frames over M and let $\mathcal{O}^* = \text{covect}\{\omega^A\}$ be its associated coframe. Then E. Cartan's structure equations can be written in indexless manner as

$$\nabla e = \theta \otimes e, \tag{16}$$

$$d\omega = -\theta \wedge \omega, \tag{17}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{18}$$

In the above equations θ (respectively Θ) are the local connection forms in the tangent bundle TM (respectively the curvature 2-forms on M).

2 Manifolds with $\mathcal{T}$-parallel exterior recurrent connection

Let $M(\Omega, \mathcal{T}, g)$ be a $2m$-dimensional manifold with almost symplectic 2-form Ω and with structure vector field $\mathcal{T}(\mathcal{T}^A)$ $(A = 1, \cdots 2m)$. Now, by reference to [9] (see also [6]), we assume that (M, g) is structured by a $\mathcal{T}$-parallel connection, which means that the connection forms satisfy

$$\theta^A_B = < \mathcal{T}, e_B \wedge e_A >, \tag{19}$$

where $\wedge$ stands for the wedge product of vector fields. In addition, we also assume that the connection forms θ^A_B are exterior recurrent [2] with $2\mathcal{T}^\flat$ as recurrence forms, which means that

$$d\theta^A_B = 2\mathcal{T}^\flat \wedge \theta^A_B. \tag{20}$$

Since

$$\theta^A_B = \mathcal{T}^B \omega^A - \mathcal{T}^A \omega^B,$$

it follows that

$$d\mathcal{T}^A = \mathcal{T}^A \alpha, \tag{21}$$

where we have set $\alpha := \mathcal{T}^\flat$. Now, in view of the structure equations (17) and invoking the curvature forms Θ_B^A, one derives

$$\Theta_B^A = \|\mathcal{T}\|^2 \omega^B \wedge \omega^A + \alpha \wedge \theta_B^A. \tag{22}$$

Since one has

$$d\|\mathcal{T}\|^2 = 2\|\mathcal{T}\|^2 \alpha, \tag{23}$$

then by (21) one gets

$$d\omega^A = \alpha \wedge \omega^A. \tag{24}$$

By exterior differentiation of (22), one derives that

$$d\Theta_B^A = 3\alpha \wedge \Theta_B^A. \tag{25}$$

The above equation expresses the fact that the connection forms being exterior recurrent implies the same property for the curvature forms Θ_B^A also. Taking moreover the Lie derivatives of θ_B^A and Θ_B^A with respect to the structure vector field $\mathcal{T}$, and using (23), one finds

$$\begin{array}{rcl} \mathcal{L}_\mathcal{T} \theta_B^A & = & 2\|\mathcal{T}\|^2 \theta_B^A, \\ \mathcal{L}_\mathcal{T} \Theta_B^A & = & 3\|\mathcal{T}\|^2 \Theta_B^A. \end{array} \tag{26}$$

Hence, $\mathcal{T}$ defines an infinitesimal conformal transformation of both the connection forms and the curvature forms.

On the other hand, by (19) one finds that

$$\nabla e_A = \mathcal{T}^A I - \omega^A \otimes \mathcal{T}, \tag{27}$$

and in this way one gets by (21) also that

$$\nabla \mathcal{T} = \|\mathcal{T}\|^2 I. \tag{28}$$

This shows that $\mathcal{T}$ is a concurrent vector field (it is well known [1] that concurrency is of conformal nature). From (27) and (28) it follows that

$$[\mathcal{T}, e_A] = -\|\mathcal{T}\|^2 e_A, \tag{29}$$

and this proves that the differential system $\{e_A\}$ corresponding to the vector basis admits an infinitesimal transformation with generator $\mathcal{T}$. We also notice that operating on (28) with ∇ (the operator ∇ acts inductively) one gets

$$\nabla(\nabla \mathcal{T}) = \nabla^2 \mathcal{T} = \|\mathcal{T}\|^4 \alpha \wedge I. \tag{30}$$

This shows that $\mathcal{T}$ is an exterior concurrent vector field [10] (see also [7]). In consequence of (30) one may now also write

$$\mathcal{R}(\mathcal{T}, Z) = -(2m-1)\|\mathcal{T}\|^4 g(\mathcal{T}, Z), \qquad Z \in \Xi(M), \tag{31}$$

where $\mathcal{R}$ means the Ricci tensor field of ∇. In the same way one can also calculate that

$$\nabla^3 e_A = \|\mathcal{T}\|^4 (\alpha \wedge \omega^A) \wedge I, \tag{32}$$

and consequently one can conclude that the elements of the vector basis $\{e_A\}$ are exterior concurrent vector fields; in the sequel we will use the terminology of a 2-exterior vector basis for this case.

We recall that a function $f\colon \mathbb{R}^{2m} \to \mathbb{R}$ is called isoparametric [13] if both $\|\mathrm{grad} f\|^2$ and $\mathrm{div}(\mathrm{grad} f)$ are functions of f. In the case under discussion, one has first of all that

$$\mathrm{grad}\|\mathcal{T}\|^2 = \|\mathcal{T}\|^2 \mathcal{T}, \tag{33}$$

from which there follows that

$$\|\ \mathrm{grad}\|\mathcal{T}\|^2\ \|^2 = \|\mathcal{T}\|^4. \tag{34}$$

Next, one also derives that

$$\mathrm{div}\,\mathrm{grad}\|\mathcal{T}\|^2 = 4\left(2m + \|\mathcal{T}\|^2\right)\|\mathcal{T}\|^2, \tag{35}$$

from which one may conclude that $\|\mathcal{T}\|^2$ is an isoparametric function. Next, by the general formula

$$\Delta\mu = -\mathrm{div}\nabla\mu, \qquad \mu \in \Lambda^0 M,$$

where Δ denotes the Laplacian, and in virtue of (33), we see that $\|\mathcal{T}\|^2$ is an eigenfunction of Δ, having $4(2m + \|\mathcal{T}\|^2)$ as eigenvalue of Δ. Recall now that if Z is any vector field, one has

$$\mathrm{tr}\nabla^2 Z = \sum \nabla_{e_A}(\nabla_{e_A} Z).$$

Then, by (30) one derives

$$\mathrm{tr}\nabla^2 \mathcal{T} = 2\|\mathcal{T}\|^2 \mathcal{T}. \tag{36}$$

With $\mathcal{R}$ denoting the Ricci tensor field, one now has

$$\mathcal{R}(\mathcal{T}, V) = -2(2m-1)\|\mathcal{T}\|^2 g(\mathcal{T}, V), \qquad V \in \Xi(M). \tag{37}$$

Then, by reference to [8], if V is a parallel vector field, one has the Weitzenbock formula:

$$(\Delta \mathcal{T}^\flat)V = \mathcal{R}(V, \mathcal{T}) - < \mathrm{tr}\nabla^2 \mathcal{T}, V >= -4m\|\mathcal{T}\|^2 g(\mathcal{T}, V). \tag{38}$$

On the other hand, regarding the almost symplectic form Ω, one writes with standard notation

$$\Omega = \sum \omega^a \wedge \omega^{a^*}, \qquad a = 1, \cdots m, a^* = a + m. \tag{39}$$

Taking the exterior derivative of Ω, and in view of (24), one finds that

$$d\Omega = 2\alpha \wedge \Omega, \qquad \alpha = \mathcal{T}^\flat. \tag{40}$$

This affirms the fact that Ω defines a locally conformal symplectic structure on M having α as covector of Lee. Then, as is known from [5], calling the mapping $Z \to -i_Z\Omega = {}^\flat Z$ the symplectic isomorphism, one has

$$-{}^\flat\mathcal{T} = i_{\mathcal{T}}\Omega = \sum(\mathcal{T}^a \omega^{a^*} - \mathcal{T}^{a^*}\omega^a), \tag{41}$$

and by (21) and (24) one finds that

$$\mathcal{L}_{\mathcal{T}}\Omega = 2\|\mathcal{T}\|^2\Omega. \tag{42}$$

Hence, following a known definition [5], the above equation means that $\mathcal{T}$ defines a infinitesimal conformal transformation of Ω. On the other hand, regarding the curvature forms, we recall that the Bianchi forms in the sense of Tachibana [12] are defined by

$$\Theta^{(p)}_{u^1,\ldots,u^{2p}} = \Theta^{u_2}_{u_1} \wedge \Theta^{u_3}_{u_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1}. \tag{43}$$

Then, by exterior differentiation one gets from (43)

$$d\left(\Theta^{(p)}_{u^1,\ldots,u^{2p}}\right) = 3(2m-1)\alpha \wedge \Theta^{(p)}_{u^1,\ldots,u^{2p}}, \tag{44}$$

and we may consequently observe that the Bianchi forms $\Theta^{(p)}_{u^1,\ldots,u^{2p}}$ are exterior recurrent, with $3(2m-1)\alpha$ as recurrence form.

In another perspective, let X be any vector field on M; if the covariant differential of X is the wedge product of X with the structure vector field $\mathcal{T}$, this means that X is a skew symmetric Killing vector field (in the sense of [11]), i.e.

$$\nabla X = X \wedge \mathcal{T} = \alpha \otimes X - X^\flat \otimes \mathcal{T}. \tag{45}$$

One may also remark that the above relation is indeed in correspondence with Rosca's lemma [11] concerning skew-symmetric Killing and conformal skew-symmetric Killing vector fields.

$$dX^\flat = 2X \wedge X^\flat.$$

In this case, the differentials of the components of X, i.e. dX^A satisfy

$$dX^A = -g(X, \mathcal{T})\omega^A + X^A\alpha. \tag{46}$$

In view of the mentioned facts, and taking the Lie derivative of Ω with respect to X, one calculates that

$$\mathcal{L}_X\Omega = -2g(X, \mathcal{T})\Omega. \tag{47}$$

This proves the property that any skew symmetric Killing vector field X, having the structure vector field $\mathcal{T}$ as generative, defines an infinitesimal conformal transformation of the conformal symplectic form Ω.

Summing up, we state the following

Theorem 1. *Let $M(\Omega, \mathcal{T}, \alpha)$ be a $2m$-dimensional Riemannian manifold structured by a $\mathcal{T}$-parallel exterior recurrent connection. In this case, the structure vector field $\mathcal{T}$ is concurrent and defines an infinitesimal conformal transformation of the connection forms θ^A_B, of the curvature forms Θ^A_B and of the conformal symplectic form Ω. In addition, one has the following properties:*

(i) $\|\mathcal{T}\|^2$ *is an isoparametric function;*

(ii) *the differential system $\{e_A\}$ admits an infinitesimal transformation with generator $\mathcal{T}$, i.e.*

$$[\mathcal{T}, e_A] = \|\mathcal{T}\|^2 e_A;$$

(iii) *all the basis vector fields e_A are 2-exterior concurrent vector fields, i.e.*

$$\nabla^3 e_A = 2\|\mathcal{T}\|^2(\alpha \wedge \omega^A) \wedge I, \qquad \alpha = \mathcal{T}^\flat.$$

(iv) $\|\mathcal{T}\|^2$ *is an eigenfunction of Δ having $4(2m + \|\mathcal{T}\|^2)$ as eigenvalue of Δ;*

(v) *if V denotes any parallel vector field, then one has the Weitzenbock formula*

$$\Delta\alpha(V) = \mathcal{R}(V, \mathcal{T}) - < tr\nabla^2\mathcal{T}, V >= -4m\|\mathcal{T}\|^2 g(\mathcal{T}, V);$$

(vi) *if $\Theta^{(p)}_{u_1,\ldots,u_{2p}} = \Theta^{u_2}_{u_1} \wedge \Theta^{u_3}_{u_2} \wedge \cdots \wedge \Theta^{2p}_{2p-1}$ means the Bianchi form of type $(2p, 2p)$, in the sense of Tachibana, then $\Theta^{(p)}_{u_1,\ldots,u_{2p}}$ is exterior recurrent with $3(2m-1)\alpha$ as recurrence form;*

(vii) *any skew symmetric Killing vector field X, having $\mathcal{T}$ as generative, defines an infinitesimal conformal transformation of Ω, i.e.*

$$\mathcal{L}_X\Omega = -2g(X,\mathcal{T})\Omega.$$

3 Geometry of the tangent bundle

In this section we will discuss some properties of the tangent bundle manifold TM having as basis manifold M studied in Section 3. Denote by $V(V^A)$ $(A = 1, \cdots 2m)$ the Liouville vector field (or the canonical vector field on TM [4]). Accordingly, one may consider the set

$$B^* = \{\omega^A, dV^A \mid A = 1, \cdots 2m\}$$

as an adapted cobasis in TM (see also [6]). Let T^r_s be the set of all tensor fields of type (r,s) on M. It is well known [14] that the vertical and complete lifts are linear mappings of T^r_sM into $T^r_s(TM)$, and one has

$$(\mathcal{T}_1 \otimes \mathcal{T}_2)^C = \mathcal{T}_1{}^V \otimes \mathcal{T}_2{}^C + \mathcal{T}_1{}^C \otimes \mathcal{T}_2{}^V. \tag{48}$$

Hence, in the case under discussion we may define the complete lift Ω^C of the structure conformal 2-form Ω of M to be the 2-form of rank $4m$ on TM given by

$$\Omega^C = \sum(dV^a \wedge \omega^{a^*} + \omega^a \wedge dV^{a^*}), \qquad a = 1, \cdots m; a^* = a + m. \tag{49}$$

On the other hand, the Liouville vector field V is expressed by

$$V = \sum V^A \frac{\partial}{\partial V^A}; \tag{50}$$

it is also known that the associated basic 1-form

$$\mu = \sum V^A \omega^A \tag{51}$$

is called the Liouville form. (Alternatively, one can also write that $\mu = V^\flat$.)

Next, taking the Lie differential of Ω^C with respect to the Liouville vector field V and taking into account (24), one finds that

$$\mathcal{L}_V\Omega^C = \Omega^C. \tag{52}$$

Hence, with reference to [4], the above equation proves that Ω^C is a homogeneous 2-form of class 1 on TM.

Taking moreover the Lie differential of Ω^C with respect to the structure vector field $\mathcal{T}$, one also derives that

$$\mathcal{L}_{\mathcal{T}}\Omega^C = \|\mathcal{T}\|^2\Omega^C. \tag{53}$$

The above equation shows that $\mathcal{T}$ defines also for Ω^C an infinitesimal conformal transformation.

By exterior derivation of the Liouville form μ defined by (51), and taking into account (24), one gets that

$$d\mu = \alpha \wedge \mu + dV^A \wedge \omega^A. \tag{54}$$

Introducing the notation

$$\psi = \sum dV^a \wedge \omega^a, \tag{55}$$

and by reference to (24), it follows that

$$d\psi = \alpha \wedge \psi, \tag{56}$$

which shows that ψ is an exterior recurrent form with α as recurrence form. Then, since one first calculates that

$$i_V\psi = \mu, \qquad \alpha(V) = 0, \tag{57}$$

one finally gets that

$$\mathcal{L}_V\psi = \psi, \tag{58}$$

which shows that, as Ω^c, the form ψ is also a homogeneous 2-form of class 1.

We remind that the vertical operator i_v in the sense of [3] possesses by definition the following properties:

$$i_v\lambda = 0, \qquad i_v\omega^A = 0, \qquad i_v dV^A = \omega^A, \tag{59}$$

from which one calculates that

$$i_v\psi = 0. \tag{60}$$

On behalf of (58) and (60) we conclude from this that ψ is a Finslerian form [3].

In another order of ideas, we recall that the vertical lift Z^V [14] of any vector field Z on M with components Z^A is expressed by

$$Z^V = \begin{pmatrix} 0 \\ Z^A \end{pmatrix} = Z^A \frac{\partial}{\partial v^A}, \qquad (A = 1, \cdots 2m).$$

Therefore, in the case under consideration, the vertical lift $\mathcal{T}^V$ of $\mathcal{T}$ is given by

$$\mathcal{T}^V = \sum \mathcal{T}^A \frac{\partial}{\partial V^A}, \qquad A \in \{1, \cdots 2m\}, \tag{61}$$

and by (55) one finds respectively that

$$i_{\mathcal{T}^V} \psi = \alpha, \qquad \text{and} \qquad \mathcal{L}_{\mathcal{T}^V} \psi = 0. \tag{62}$$

On behalf of the above, one may conclude that $\mathcal{T}^V$ defines an infinitesimal automorphism of the 2-form ψ.

Finally, consider the 2-form

$$II = f\psi; \tag{63}$$

following [4], f is called the energy scalar. Now, in view of (23), one has

$$dII = f\left(\frac{df}{f} + \frac{d\|\mathcal{T}\|^2}{2\|\mathcal{T}\|^2}\right) \wedge II. \tag{64}$$

By reference to [4] and in case that

$$\frac{df}{f} + \frac{d\|\mathcal{T}\|^2}{2\|\mathcal{T}\|^2} = 0,$$

this shows that II can then be seen as the canonical symplectic form of the $4m$-dimensional manifold TM. Finally, we set

$$r = f\mathsf{v},$$

where $\mathsf{v} = \frac{1}{2}\sum (V^A)^2$ denotes the Liouville function; then, by reference to [4], the pair (r, II) defines a regular mechanical system (in the sense of Klein) having r as kinetic energy.

Theorem 2. *Let TM be the tangent bundle manifold having as basis the conformal symplectic manifold $M(\Omega, \mathcal{T}, \alpha)$ structured by a $\mathcal{T}$-parallel connection and having $\alpha = \mathcal{T}^\flat$ as covector of Lee. Let V, μ, and v, be the Liouville vector field, the Liouville form, and the Liouville function of TM respectively. One has the following properties:*

(i) *the complete lift Ω^C on TM of the conformally symplectic form Ω of M, is a homogeneous 2-form of class 1, i.e.*

$$\mathcal{L}_V \Omega^C = \Omega^C;$$

(ii) *the vertical lift $\mathcal{T}^V$ of $\mathcal{T}$ defines an infinitesimal automorphism of the 2-form $\psi = \sum dV^A \wedge \omega^A, (A = 1, \cdots 2m)$;*

(iii) *if f stands for the energy function of M, then the 2-form $II = f\psi$ is the canonical symplectic form on TM $\left(\frac{df}{f} + \frac{d\|\mathcal{T}\|^2}{2\|\mathcal{T}\|^2} = 0\right)$, and the pair (r, II), consisting of the scalar $r = f\mathsf{v}$ and the 2-form $f\psi$, defines a regular mechanical system (in the sense of Klein) on TM.*

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