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Lie symmetries of differential equations: direct and inverse problems

Francesco Oliveri[i]
Department of Mathematics, University of Messina
Contrada Papardo, Salita Sperone 31, 98166 Messina, Italy
oliveri@mat520.unime.it

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Abstract. This paper reviews some relevant problems arising within the context of Lie group analysis of differential equations either in the direct approach or in the inverse one. For what concerns the direct approach, there are considered two results, the first related to the reduction through an invertible point transformation of a system of PDE's to an equivalent autonomous form, and the second related to the reduction of a nonlinear first order system of PDE's to linear form. Two applications of the results are given. The Navier–Stokes–Fourier model equations for a viscous and heat conducting monatomic gas in a rotating frame are mapped in two different autonomous forms, and some explicit exact solutions are determined. Moreover, the first order system corresponding to the most general second order completely exceptional equation in $(1+1)$ dimensions (which is a Monge–Ampère equation) is reduced to linear form. Finally, within the context of the inverse approach of Lie group analysis, there is introduced the concept of Lie remarkable systems and it is shown that second order Monge–Ampère equations and the third order Monge–Ampère equation in $(1+1)$ dimensions are Lie remarkable.

Keywords: Lie point symmetries, Autonomous differential equations, Linearizable differential equations, Completely exceptional equations, Lie remarkable equations.

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1 Introduction

Lie group theory [1–7] provides extremely general and powerful methods for determining (invariant) solutions to differential equations as well as reducing the order of ordinary differential equations or transforming ordinary and partial differential equations in more convenient forms. In this paper we shall focus our attention on certain aspects related to the Lie point symmetries admitted by partial differential equations (PDE's).

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Roughly speaking, in dealing with Lie group analysis of PDE's, either a direct problem or an inverse one may be considered. In the direct problem, starting with a system of PDE's

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{1}$$

where $\mathbf{x}\in\mathbb{R}^n$ is the set of the independent variables, $\mathbf{u}\in\mathbb{R}^N$ the set of the dependent variables, and $\mathbf{u}^{(r)}$ the set of all partial derivatives of the $\mathbf{u}$'s with respect to the $\mathbf{x}$'s up to the order r, one is interested to find the admitted group of Lie symmetries.

Let us consider a one-parameter (ϵ) Lie group of point transformations

$$\begin{aligned}
x_i^\star &= x_i^\star(x_j,u_B;\epsilon)=x_i+\epsilon\xi_i(x_j,u_B)+O(\epsilon^2),\\
u_A^\star &= u_A^\star(x_j,u_B;\epsilon)=x_i+\epsilon\eta_A(x_j,u_B)+O(\epsilon^2),
\end{aligned}$$

($i,j=1,\dots,n$, $A,B=1,\dots,N$), and the corresponding vector field (infinitesimal operator)

$$\Xi=\sum_{i=1}^{n}\xi_i\frac{\partial}{\partial x_i}+\sum_{A=1}^{N}\eta_A\frac{\partial}{\partial u_A}. \tag{2}$$

The use of the straightforward Lie's algorithm, which requires that the r–order prolongation $\Xi^{(r)}$ of the vector field (2) acting on (1) be zero along the solutions of (1), *i.e.*,

$$\Xi^{(r)}\Delta=0\Big|_{\Delta=0}, \tag{3}$$

provides an overdetermined set of linear differential equations for the infinitesimal generators ξ_i and η_A, whose integration gives the infinitesimal operators Ξ_i, admitted by the system (1); these vector fields span a Lie algebra that can be finite or infinite dimensional.

Many computer algebra packages (Mulie, Dimsym, MathLie, Spde, Symgrp, Relie, ...) exist allowing to easily perform the cumbersome calculations involved in the application of Lie's algorithm and to render almost automatic much of the computation of the Lie symmetries [8].

The Lie symmetries of a given system of PDE's can be used for various tasks:

- for determining invariant solutions, by appending to the given system the invariance surface conditions

$$\sum_{i=1}^{n}\xi_i\frac{\partial u_A}{\partial x_i}=\eta_A,\qquad A=1,\dots,N, \tag{4}$$

  whereupon a *reduced system* involving usually one independent variable less, is obtained;

- for introducing suitable invertible transformations mapping the given system to a more convenient form [6, 9–11]: for instance, we may transform the system to autonomous form (if the given system is nonautonomous), or to a new equivalent autonomous form (if the given system is already autonomous), or to linear form.

On the contrary, in the inverse problem one chooses a Lie group of symmetries and determine the most general system (having an assigned structure) admitting it [2], or look for the additional constraints to be imposed in order to have the requested invariance.

Various examples of inverse problems relevant in mathematical physics include

- the determination of quasilinear hyperbolic systems which are invariant with respect to a stretching group of transformation in connection to the study of the propagation of weak discontinuity waves into nonconstant states described by self similar solutions [12];
- the determination of the structural form of a quasilinear first order system which results invariant with respect to the Galilean group [13–15].
- the determination of the additional constraints to be imposed to the equations of ideal gas dynamics [16–19] and ideal magneto gas dynamics [20–22] in order their solutions be invariant with respect to generalized stretching transformations and generalized time translations (substitution principles).

2 Reduction to autonomous or to linear form

Reduction to autonomous form provides useful when one is interested to study nonlinear wave propagation in continuous media whose governing equations are nonautonomous, namely when we consider inhomogeneous materials, or axi-symmetric problems, or physical systems in non-inertial reference frames.

To study the propagation of weak discontinuities compatible with nonautonomous quasilinear hyperbolic systems we need to know the state ahead of the front wave, the so called unperturbed state, which is a solution (usually taken constant) of the basic governing equations. But, if the governing system is nonautonomous, there not exist, in general, constant solutions. Nevertheless, if the nonautonomous system can be mapped into autonomous form, we may consider the evolution of weak discontinuities in particular nonconstant states as evolution problems in constant states admitted by the transformed system [23–26].

In these cases the following theorem [10] may reveal useful.

1 Theorem. *The system of partial differential equations*

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{5}$$

can be transformed by an invertible point transformation of the form

$$\widehat{x}_i=\widehat{x}_i(x_j,u_B), \qquad \widehat{u}_A=\widehat{u}_A(x_j,u_B), \tag{6}$$

where $i,j=1,\dots,n$, $A,B=1,\dots,N$, to the autonomous form

$$\widehat{\Delta}\left(\widehat{\mathbf{u}},\widehat{\mathbf{u}}^{(r)}\right)=0 \tag{7}$$

if and only if it is left invariant by n Lie groups of point transformations whose infinitesimal operators Ξ_i $(i=1,\dots,n)$ satisfy the conditions

$$[\Xi_i,\Xi_j]=0, \qquad (i,j=1,\dots,n), \tag{8}$$

$[\cdot,\cdot]$ *denoting the commutator of two operators. The condition (8) means that the operators Ξ_i generate a n–dimensional Abelian Lie algebra.*

The proof consists in determining a set of canonical variables for the admitted infinitesimal operators that in the new variables assume the form

$$\widehat{\Xi}_i=\frac{\partial}{\partial\widehat{x}_i}, \qquad i=1,\dots,n, \tag{9}$$

whereupon the transformed system must necessarily be autonomous.

The Lie symmetries provide useful also for the construction of linearizing transformations. Necessary and sufficient conditions for the existence of invertible mappings linking a nonlinear system (source) of first order PDE's with a linear system (target) of PDE's have been given by Kumei and Bluman [27]. The proof they give does not seem "natural", since it requires that a overdetermined system of PDE's be compatible.

A more natural proof has been given in [9] and it involves the introduction of the canonical variables related to some infinitesimal operators (in general not admitted by the source system of PDE's) but whose linear combination (with coefficients given by arbitrary functions that are solution of a linear system of PDE's) is an admitted group of the source system of PDE's.

2 Theorem. *The nonlinear system of first order PDE's*

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)}\right)=0, \tag{10}$$

can be transformed to the linear form

$$\mathcal{L}(\widehat{\mathbf{x}})[\widehat{\mathbf{u}}] = \mathbf{g}(\widehat{\mathbf{x}}), \tag{11}$$

where $\mathcal{L}(\widehat{\mathbf{x}})$ is a linear first order differential operator, through the invertible point transformation

$$\widehat{x}_i = \widehat{x}_i(x_j, u_B), \qquad \widehat{u}_A = \widehat{u}_A(x_j, u_B) \tag{12}$$

if and only if it admits the vector field

$$\Xi = \sum_{A=1}^{N} F_A(\widehat{x}_i(x_j, u_B))\Xi_A, \tag{13}$$

where

$$\Xi_A = \sum_{i=1}^{n} \xi_A^i(x_j, u_B)\frac{\partial}{\partial x_i} + \sum_{C=1}^{N} \eta_A^C(x_j, u_B)\frac{\partial}{\partial u_C}, \tag{14}$$

with $\xi_A^i(x_j, u_B)$, $\eta_A^C(x_j, u_B)$ specific functions of their arguments, and the functions $F_A(\widehat{x}_i(x_j, u_B))$ satisfying the linear system

$$\mathcal{L}(\widehat{\mathbf{x}})[\mathbf{F}] = 0, \tag{15}$$

along with the conditions

$$\Xi_A\widehat{x}_i = 0, \qquad [\Xi_A, \Xi_B] = 0, \tag{16}$$

with $i = 1, \ldots, n$, $A, B = 1, \ldots, N$.

Also in this case the proof is constructive and passes through the determination of the canonical variables of the infinitesimal operators Ξ_A (in general not admitted by the system), whereupon the operator Ξ writes as

$$\widehat{\Xi} = \sum_{A=1}^{N} F_A(\widehat{x}_i(x_j, u_B))\frac{\partial}{\partial \widehat{u}_A}, \tag{17}$$

thus implying that the transformed system is linear.

3 The Navier–Stokes–Fourier equations – Reduction to autonomous form

In this section we give an example of application of Theorem 1 to a nonautonomous system involving 4 independent variables.

By considering a gas in rotation about some fixed axis with a constant angular velocity $\boldsymbol{\omega}$, it is convenient to introduce explicitly the inertial frame $\{x_i'\}$ with respect to which the non-inertial system (with coordinates $\{x_i\}$) is rotating about the x_3' axis (where x_3 and the x_3' axes will be assumed coincident).

The ordinary thermodynamics of viscous and heat conducting monatomic gases can be obtained starting from the balance equations for the first 13 moments and using a formal iterative scheme analogous to the Maxwellian iteration [28, 29]. In this case we have only 5 fields (density ρ, velocity v_i, and temperature T) governed by the balance equations for mass, momentum and energy:

$$
\begin{aligned}
&\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_k)}{\partial x_k}=0,\\
&\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial}{\partial x_k}\left\{\rho v_i v_k-\frac{2}{\alpha}\left(\frac{K}{m}\right)T\left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)-\frac{1}{3}\frac{\partial v_r}{\partial x_r}\delta_{ik}\right]\right.\\
&\qquad\left.+\rho\left(\frac{K}{m}\right)T\delta_{ik}\right\}=\rho(f_i+i_i),\\
&\frac{\partial(\rho T)}{\partial t}+\frac{\partial}{\partial x_k}\left\{\rho T v_k-\frac{5}{2\alpha}\left(\frac{K}{m}\right)T\frac{\partial T}{\partial x_k}\right\}\\
&\qquad+\left\{\frac{2}{3}\rho T\delta_{lk}-\frac{4}{3\alpha}T\left[\frac{1}{2}\left(\frac{\partial v_l}{\partial x_k}+\frac{\partial v_k}{\partial x_l}\right)-\frac{1}{3}\frac{\partial v_r}{\partial x_r}\delta_{lk}\right]\right\}\frac{\partial v_l}{\partial x_k}=0,
\end{aligned}
\tag{18}
$$

where i_i are the components of the *specific inertial forces*, f_i are the components of the *specific external forces* acting on the gas, K is the Boltzmann constant, m the mass of a single particle, and the coefficient α is an appropriate constant which describes the interaction between the Maxwellian molecules. If the origins of both frames coincide, since we are considering a time independent rotation, we have only the contributions of the Coriolis and the Centrifugal forces, so that

$$
i_i = 2\epsilon_{ijl}\omega_l v_j + \omega^2 x_i - (\omega_r x_r)\omega_i, \tag{19}
$$

where $\boldsymbol{\omega} = (0,0,\omega)$ and ϵ_{ijl} is the Ricci tensor. Moreover, if $\mathbf{f} = (0,0,-g)$ (with g the gravitational acceleration), what we obtain is

$$
\rho(\mathbf{f}+\mathbf{i}) = (\rho(2\omega v_2+\omega^2 x_1), \rho(-2\omega v_1+\omega^2 x_2), -\rho g). \tag{20}
$$

The system (18) is invariant with respect to a 12–parameter Lie group whose

Lie algebra is spanned by the following vector fields:

$$
\begin{aligned}
\Xi_1 &= x_2\partial_{x_1} - x_1\partial_{x_2} + v_2\partial_{v_1} - v_1\partial_{v_2},\\
\Xi_2 &= 2t\partial_t + (2x_1 + 2\omega x_2 t)\partial_{x_1} + (2x_2 - 2\omega x_1 t)\partial_{x_2} + (2x_3 - gt^2)\partial_{x_3}\\
&+ (2\omega v_2 t + 2\omega x_2)\partial_{v_1} + (2\omega v_1 t + 2\omega x_1)\partial_{v_2} + 2gt\partial_{v_3} - 2\rho\partial_\rho,\\
\Xi_3 &= -t\partial_t - \omega x_2 t\partial_{x_1} + \omega x_1 t\partial_{x_2} + gt^2\partial_{x_3} + (v_1 - \omega v_2 t - \omega x_2)\partial_{v_1}\\
&+ (v_2 + \omega v_1 t + \omega x_1)\partial_{v_2} + (v_3 + 2gt)\partial_{v_3} + \rho\partial_\rho + 2T\partial_T,\\
\Xi_4 &= \partial_{x_3}, \qquad \Xi_5 = t\partial_{x_3} + \partial_{v_3}, \qquad \Xi_6 = \partial_t,\\
\Xi_7 &= \sin(\omega t)\partial_{x_1} + \cos(\omega t)\partial_{x_2} + \omega\cos(\omega t)\partial_{v_1} - \omega\sin(\omega t)\partial_{v_2},\\
\Xi_8 &= t\sin(\omega t)\partial_{x_1} + t\cos(\omega t)\partial_{x_2}\\
&+ (\omega t\cos(\omega t) + \sin(\omega t))\partial_{v_1} + (-\omega t\sin(\omega t) + \cos(\omega t)\partial_{v_2},\\
\Xi_9 &= -\cos(\omega t)\partial_{x_1} + \sin(\omega t)\partial_{x_2} + \omega\sin(\omega t)\partial_{v_1} + \omega\cos(\omega t)\partial_{v_2},\\
\Xi_{10} &= -t\cos(\omega t)\partial_{x_1} + t\sin(\omega t)\partial_{x_2}\\
&+ (\omega t\sin(\omega t) - \cos(\omega t))\partial_{v_1} + (\omega t\cos(\omega t) + \sin(\omega t))\partial_{v_2},\\
\Xi_{11} &= \left[(gt - g\omega^2 t^2 - 2\omega^2 x_3)\sin(\omega t) - 2g\omega t\cos(\omega t)\right]\partial_{x_1}\\
&+ \left[(gt - g\omega^2 t^2 - 2\omega^2 x_3)\cos(\omega t) + 2g\omega t\sin(\omega t)\right]\partial_{x_2}\\
&+ 2\omega^2(x_1\sin(\omega t) + x_2\cos(\omega t))\partial_{x_3}\\
&+ \left[(g - 2\omega^2 v_3)\sin(\omega t) - (g\omega^3 t^2 - g\omega t + 2\omega^3 x_3 + 2g\omega)\cos(\omega t)\right]\partial_{v_1}\\
&+ \left[(g - 2\omega^2 v_3)\cos(\omega t) + (g\omega^3 t^2 - g\omega t + 2\omega^3 x_3 + 2g\omega)\sin(\omega t)\right]\partial_{v_2}\\
&+ 2\omega^2\left[(v_1\sin(\omega t) + v_2\cos(\omega t)) + \omega(x_1\cos(\omega t) - x_2\sin(\omega t))\right]\partial_{v_3},\\
\Xi_{12} &= \left[(-g\omega^2 t^2 - 2\omega^2 x_3 + 2g)\cos(\omega t) + 2g\omega t\sin(\omega t)\right]\partial_{x_1}\\
&+ \left[(g\omega^2 t^2 + 2\omega^2 x_3 - 2g)\sin(\omega t) + 2g\omega t\cos(\omega t)\right]\partial_{x_2}\\
&+ 2\omega^2(x_1\cos(\omega t) - x_2\sin(\omega t))\partial_{x_3}\\
&+ \left[(g\omega^3 t^2 + 2\omega^3 x_3)\sin(\omega t) - 2\omega^2 v_3\cos(\omega t)\right]\partial_{v_1}\\
&+ \left[(g\omega^3 t^2 + 2\omega^3 x_3)\cos(\omega t) + 2\omega^2 v_3\sin(\omega t)\right]\partial_{v_2}\\
&+ 2\omega^2\left[\cos(\omega t)(v_1 - \omega x_2) - \sin(\omega t)(v_2 + \omega x_1)\right]\partial_{v_3}.
\end{aligned}
$$

Since the governing system involves 4 independent variables (the time t and the 3 spatial coordinates x_i) its reduction to autonomous form through the application of Theorem 1 requires the existence of a 4–dimensional Abelian Lie subalgebra. By direct inspection it is easily seen that either the Lie subalgebra $\mathcal{L}_1$ generated by $\{\Xi_2, \Xi_5, \Xi_8, \Xi_{10}\}$ or the Lie subalgebra $\mathcal{L}_2$ generated by $\{\Xi_3, \Xi_4, \Xi_7, \Xi_9\}$ are Abelian.

By considering the Lie subalgebra $\mathcal{L}_1$, we may construct the following invertible point transformation

$$\begin{aligned}
\widehat{x}_1 &= \frac{x_1}{t}\cos(\omega t) - \frac{x_2}{t}\sin(\omega t), \qquad \widehat{x}_2 = \frac{x_1}{t}\sin(\omega t) + \frac{x_2}{t}\cos(\omega t),\\
\widehat{x}_3 &= \frac{x_3}{t} + \frac{gt}{2}, \qquad \widehat{t} = \ln(t),\\
\rho &= \frac{1}{t}\widehat{\rho}, \qquad T = \widehat{T},\\
v_1 &= \widehat{v}_1\cos(\omega t) + \widehat{v}_2\sin(\omega t) + \frac{x_1}{t} + \omega x_2,\\
v_2 &= -\widehat{v}_1\sin(\omega t) + \widehat{v}_2\cos(\omega t) + \frac{x_2}{t} - \omega x_1,\\
v_3 &= \widehat{v}_3 + \frac{x_3}{t} - \frac{gt}{2},
\end{aligned}$$

where the new dependent variables $\widehat{\rho}$, $\widehat{T}$, $\widehat{v}_1$, $\widehat{v}_2$ and $\widehat{v}_3$ depend on the new independent variables $\widehat{t}$, $\widehat{x}_1$, $\widehat{x}_2$ and $\widehat{x}_3$.

In the new variables the Navier–Stokes–Fourier system (18) writes in the autonomous form

$$\begin{aligned}
&\frac{\partial\widehat{\rho}}{\partial\widehat{t}} + \frac{\partial(\widehat{\rho}\widehat{v}_k)}{\partial\widehat{x}_k} + 2\widehat{\rho} = 0,\\
&\frac{\partial(\widehat{\rho}\widehat{v}_i)}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left\{\widehat{\rho}\widehat{v}_i\widehat{v}_k - \frac{2}{\alpha}\left(\frac{K}{m}\right)\widehat{T}\left[\frac{1}{2}\left(\frac{\partial\widehat{v}_i}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_i}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{ik}\right]\right.\\
&\qquad \left. +\widehat{\rho}\left(\frac{K}{m}\right)\widehat{T}\delta_{ik}\right\} + 3\widehat{\rho}\widehat{v}_i = 0, \qquad (21)\\
&\frac{\partial(\widehat{\rho}\widehat{T})}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left\{\widehat{\rho}\widehat{T}\widehat{v}_k - \frac{5}{2\alpha}\left(\frac{K}{m}\right)\widehat{T}\frac{\partial\widehat{T}}{\partial\widehat{x}_k}\right\}\\
&\qquad + \left\{\frac{2}{3}\widehat{\rho}\widehat{T}\delta_{lk} - \frac{4}{3\alpha}\widehat{T}\left[\frac{1}{2}\left(\frac{\partial\widehat{v}_l}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_l}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{lk}\right]\right\}\frac{\partial\widehat{v}_l}{\partial\widehat{x}_k} + 4\widehat{\rho}\widehat{T} = 0.
\end{aligned}$$

It is worth of noticing that this system has the same form as the original system with the exception of the inhomogeneity terms.

A direct inspection of the transformed system (21) allows us for the determination of some simple classes of solutions to the system (18). For instance, by searching for the solutions of the transformed system (21) depending only on $\widehat{t}$, one gets the following solution to the original system (18):

$$\begin{aligned}
\rho &= \frac{\rho_0}{t^3}, \qquad T = \frac{T_0}{t^2},\\
v_1 &= \frac{v_{1_0}\cos(\omega t) + v_{2_0}\sin(\omega t)}{t} + \frac{x_1}{t} + \omega x_2,
\end{aligned}$$

$$
\begin{aligned}
v_2 &= \frac{-v_{1_0}\sin(\omega t) + v_{2_0}\cos(\omega t)}{t} + \frac{x_2}{t} - \omega x_1,\\
v_3 &= \frac{v_{3_0}}{t} + \frac{x_3}{t} - \frac{gt}{2},
\end{aligned}
$$

where ρ_0, v_{1_0}, v_{2_0}, v_{3_0} and T_0 are arbitrary constants.

Also, by assuming in (21) $\widehat{v}_1 = \widehat{v}_2 = \widehat{v}_3 = 0$, we get the following solution to the original system:

$$
\begin{aligned}
\rho &= \frac{1}{t^3\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)}}, \qquad T = \frac{T_0}{t^2\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)}},\\
v_1 &= \frac{x_1}{t} + \omega x_2, \qquad v_2 = \frac{x_2}{t} - \omega x_1, \qquad v_3 = \frac{x_3}{t} - \frac{gt}{2},
\end{aligned}
$$

where T_0 is a positive constant, and ψ is a positive solution of Laplace equation

$$
\frac{\partial^2\psi}{\partial\widehat{x}_1^2} + \frac{\partial^2\psi}{\partial\widehat{x}_2^2} + \frac{\partial^2\psi}{\partial\widehat{x}_3^2} = 0. \tag{22}
$$

In the case of the subalgebra $\mathcal{L}_2$ we are able to introduce the invertible point transformation

$$
\begin{aligned}
\widehat{x}_1 &= x_1\cos(\omega t) - x_2\sin(\omega t), \qquad \widehat{x}_2 = x_1\sin(\omega t) + x_2\cos(\omega t),\\
\widehat{x}_3 &= x_3 + \frac{gt^2}{2}, \qquad \widehat{t} = \ln(t),\\
\rho &= \frac{1}{t}\widehat{\rho}, \qquad T = \frac{1}{t^2}\widehat{T},\\
v_1 &= \frac{\widehat{v}_1}{t}\cos(\omega t) + \frac{\widehat{v}_2}{t}\sin(\omega t) + \omega x_2,\\
v_2 &= -\frac{\widehat{v}_1}{t}\sin(\omega t) + \frac{\widehat{v}_2}{t}\cos(\omega t) - \omega x_1,\\
v_3 &= \frac{\widehat{v}_3}{t} - gt,
\end{aligned}
$$

where the new dependent variables $\widehat{\rho}$, $\widehat{T}$, $\widehat{v}_1$, $\widehat{v}_2$ and $\widehat{v}_3$ depend on the new independent variables $\widehat{t}$, $\widehat{x}_1$, $\widehat{x}_2$ and $\widehat{x}_3$.

In these new variables the Navier–Stokes–Fourier system (18) writes in the autonomous form

$$
\begin{aligned}
&\frac{\partial\widehat{\rho}}{\partial\widehat{t}} + \frac{\partial(\widehat{\rho}\widehat{v}_k)}{\partial\widehat{x}_k} - \widehat{\rho} = 0,\\
&\frac{\partial(\widehat{\rho}\widehat{v}_i)}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left\{\widehat{\rho}\widehat{v}_i\widehat{v}_k - \frac{2}{\alpha}\left(\frac{K}{m}\right)\widehat{T}\left[\frac{1}{2}\left(\frac{\partial\widehat{v}_i}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_i}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{ik}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad +\widehat{\rho}\left(\frac{K}{m}\right)\widehat{T}\delta_{ik}\Big\} - 2\widehat{\rho}\widehat{v}_i = 0, \qquad (23)\\
&\frac{\partial(\widehat{\rho}\widehat{T})}{\partial\widehat{t}} + \frac{\partial}{\partial\widehat{x}_k}\left\{\widehat{\rho}\widehat{T}\widehat{v}_k - \frac{5}{2\alpha}\left(\frac{K}{m}\right)\widehat{T}\frac{\partial\widehat{T}}{\partial\widehat{x}_k}\right\}\\
&\quad +\left\{\frac{2}{3}\widehat{\rho}\widehat{T}\delta_{lk} - \frac{4}{3\alpha}\widehat{T}\left[\frac{1}{2}\left(\frac{\partial\widehat{v}_l}{\partial\widehat{x}_k} + \frac{\partial\widehat{v}_k}{\partial\widehat{x}_l}\right) - \frac{1}{3}\frac{\partial\widehat{v}_r}{\partial\widehat{x}_r}\delta_{lk}\right]\right\}\frac{\partial\widehat{v}_l}{\partial\widehat{x}_k} - 3\widehat{\rho}\widehat{T} = 0.
\end{aligned}
$$

and also in this case the transformed system has the same form as the original system with the exception of the inhomogeneity terms.

The solution of the transformed system (23) depending only on $\widehat{t}$ gives the following solution to the original system:

$$
\begin{aligned}
\rho &= \rho_0,\\
v_1 &= v_{1_0}\cos(\omega t) + v_{2_0}\sin(\omega t) + \omega x_2,\\
v_2 &= -v_{1_0}\sin(\omega t) + v_{2_0}\cos(\omega t) - \omega x_1,\\
v_3 &= v_{3_0} - gt,\\
T &= T_0,
\end{aligned}
$$

where ρ_0, v_{1_0}, v_{2_0}, v_{3_0} and T_0 are arbitrary constants.

Also, by assuming in (23) $\widehat{v}_1 = \widehat{v}_2 = \widehat{v}_3 = 0$, we get the solution:

$$
\begin{aligned}
\rho &= \frac{t}{\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)}},\\
T &= T_0 t\sqrt{\psi(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3)},\\
v_1 &= \omega x_2, \qquad v_2 = -\omega x_1, \qquad v_3 = -gt,
\end{aligned}
$$

where T_0 is a positive constant, and ψ is a positive solution of Poisson equation

$$
\frac{\partial^2\psi}{\partial\widehat{x}_1^2} + \frac{\partial^2\psi}{\partial\widehat{x}_2^2} + \frac{\partial^2\psi}{\partial\widehat{x}_3^2} + \frac{4\alpha m}{5KT_0} = 0. \qquad (24)
$$

4 Monge–Ampère equation — Reduction to linear form

The equation

$$
\kappa_1(u_{tt}u_{xx} - u_{tx}^2) + \kappa_2 u_{tt} + \kappa_3 u_{tx} + \kappa_4 u_{xx} + \kappa_5 = 0, \qquad (25)
$$

where the coefficients κ_i ($i = 1, \dots 5$) are constant, is the celebrated Monge-Ampère equation [30].

By means of the positions

$$t = x_1, \qquad x = x_2, \qquad u_t = u_1, \qquad u_x = u_2, \tag{26}$$

the equation (25) can be written under the form of a first order system:

$$\begin{aligned}
&\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0,\\
&\kappa_1 \left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right) + \kappa_2 \frac{\partial u_1}{\partial x_1} + \kappa_3 \frac{\partial u_1}{\partial x_2} + \kappa_4 \frac{\partial u_2}{\partial x_2} + \kappa_5 = 0.
\end{aligned} \tag{27}$$

By straightforward analysis [31], it is easily recognized that the system (27) is invariant with respect to the one-parameter Lie group of point transformations whose infinitesimal operator is

$$\Xi = \sum_{k=1}^{2} F_k(\widehat{x}_1, \widehat{x}_2)\Xi_k, \qquad \Xi_k = \alpha_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial u_k}, \tag{28}$$

where α_1 and α_2 are (not both vanishing) constants such that

$$\kappa_1 + \kappa_2\alpha_2 + \kappa_4\alpha_1 + \kappa_5\alpha_1\alpha_2 = 0, \tag{29}$$

whereas

$$\widehat{x}_1 = x_1 - \alpha_1 u_1, \qquad \widehat{x}_2 = x_2 - \alpha_2 u_2, \tag{30}$$

and the functions $F_1(\widehat{x}_1, \widehat{x}_2)$ and $F_2(\widehat{x}_1, \widehat{x}_2)$ solutions of the linear system

$$\begin{aligned}
&\frac{\partial F_2}{\partial \widehat{x}_1} - \frac{\partial F_1}{\partial \widehat{x}_2} = 0,\\
&(\kappa_2 + \kappa_5\alpha_1)\frac{\partial F_1}{\partial \widehat{x}_1} + \kappa_3 \frac{\partial F_1}{\partial \widehat{x}_2} + (\kappa_4 + \kappa_5\alpha_2)\frac{\partial F_2}{\partial \widehat{x}_2} = 0.
\end{aligned} \tag{31}$$

The hypotheses of the Theorem 2 are verified, since the two operators Ξ_1 and Ξ_2 commute, and the variables $\widehat{x}_1$ and $\widehat{x}_2$ are invariants of both operators. By assuming that the dependent variables u_1 and u_2 depend upon the new independent variables $\widehat{x}_1$ and $\widehat{x}_2$, the system (27) reduces to the following linear form:

$$\begin{aligned}
&\frac{\partial u_2}{\partial \widehat{x}_1} - \frac{\partial u_1}{\partial \widehat{x}_2} = 0,\\
&(\kappa_2 + \kappa_5\alpha_1)\frac{\partial u_1}{\partial \widehat{x}_1} + \kappa_3 \frac{\partial u_1}{\partial \widehat{x}_2} + (\kappa_4 + \kappa_5\alpha_2)\frac{\partial u_2}{\partial \widehat{x}_2} + \kappa_5 = 0.
\end{aligned} \tag{32}$$

Finally, the system (32) is equivalent to the linear second order partial differential equation

$$(\kappa_2 + \kappa_5\alpha_1)\frac{\partial^2 u}{\partial \widehat{x}_1^2} + \kappa_3 \frac{\partial^2 u}{\partial \widehat{x}_1 \partial \widehat{x}_2} + (\kappa_4 + \kappa_5\alpha_2)\frac{\partial^2 u}{\partial \widehat{x}_2^2} + \kappa_5 = 0. \tag{33}$$

5 Inverse problem: Lie remarkable systems

Within the context of inverse problems of Lie group analysis of PDE's an interesting question may arise whether there exist non trivial equations which are in one-to-one correspondence with their invariance groups.

Let us suppose we have a system of PDE's

$$\widehat{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{34}$$

with $\widehat{\Delta}$ assigned function of its arguments, and determine the infinitesimal operators of the admitted Lie symmetries, say

$$\Xi_1,\ \Xi_2,\ \dots,\ \Xi_p. \tag{35}$$

In general, if we consider a generic system of PDE's

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{36}$$

with Δ unspecified function of its arguments, and impose the invariance with respect to the Lie symmetries generated by (35), we recover some restrictions of the functional dependence of Δ that rarely led to

$$\Delta\equiv\widehat{\Delta}. \tag{37}$$

When this occurs we call the system (34) a *Lie remarkable system of PDE's.*

3 Definition (Lie remarkable system of PDE's). The system of PDE's

$$\widehat{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{38}$$

with $\widehat{\Delta}$ assigned function of its arguments, is Lie remarkable if the request of invariance of the general system of PDE's

$$\Delta\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(r)}\right)=0, \tag{39}$$

with Δ unspecified function of its arguments, with respect to the Lie point symmetries of (38), leads to

$$\Delta\equiv\widehat{\Delta}. \qquad \blacksquare \tag{40}$$

The set of Lie remarkable systems of PDE's is certainly nonempty. In fact, if we consider a system of linear homogeneous equations

$$\mathcal{L}[\mathbf{x}]\mathbf{u}=0, \qquad \mathbf{x}\in R^n, \quad \mathbf{u}\in R^N, \tag{41}$$

where $\mathcal{L}[\mathbf{x}]$ is a linear differential operator depending on the independent variables $\mathbf{x}$, then it admits the infinitesimal operator

$$\Xi = \sum_{A=1}^{N} f_A \frac{\partial}{\partial u_A}, \tag{42}$$

where $\mathbf{f}(\mathbf{x})$ is an arbitrary solution of

$$\mathcal{L}[\mathbf{x}]\mathbf{f} = 0. \tag{43}$$

This operator uniquely characterizes system (41) and corresponds to the well known principle of linear superposition of solutions.

There exist also nonlinear examples of Lie remarkable systems. Let us consider a 2×2 quasilinear autonomous homogeneous system

$$\begin{aligned} a(u_1,u_2)\frac{\partial u_1}{\partial x_1} + b(u_1,u_2)\frac{\partial u_1}{\partial x_2} + c(u_1,u_2)\frac{\partial u_2}{\partial x_1} + d(u_1,u_2)\frac{\partial u_2}{\partial x_2} = 0,\\ p(u_1,u_2)\frac{\partial u_1}{\partial x_1} + q(u_1,u_2)\frac{\partial u_1}{\partial x_2} + r(u_1,u_2)\frac{\partial u_2}{\partial x_1} + s(u_1,u_2)\frac{\partial u_2}{\partial x_2} = 0; \end{aligned}$$

it is left invariant [6] by the Lie group with vector field

$$\Xi = F_1(u_1,u_2)\frac{\partial}{\partial x_1} + F_2(u_1,u_2)\frac{\partial}{\partial x_2}, \tag{44}$$

where F_1 and F_2 satisfy the linear system

$$\begin{aligned} d(u_1,u_2)\frac{\partial F_1}{\partial u_1} - b(u_1,u_2)\frac{\partial F_1}{\partial u_2} - c(u_1,u_2)\frac{\partial F_2}{\partial u_1} + a(u_1,u_2)\frac{\partial F_2}{\partial u_2} = 0,\\ s(u_1,u_2)\frac{\partial F_1}{\partial u_1} - q(u_1,u_2)\frac{\partial F_1}{\partial u_2} - r(u_1,u_2)\frac{\partial F_2}{\partial u_1} + p(u_1,u_2)\frac{\partial F_2}{\partial u_2} = 0. \end{aligned}$$

This vector field allows for the introduction of the *hodograph transformation* that linearizes 2×2 quasilinear autonomous and homogeneous systems.

There are, of course, examples of equations that are *not* Lie remarkable. For instance, the Korteweg-deVries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{45}$$

which indeed has many remarkable properties, admits only the Lie point symmetries generated by the vector fields

$$\begin{aligned} &\Xi_1 = \partial_t, \qquad \Xi_2 = \partial_x,\\ &\Xi_3 = t\partial_x + \partial_u, \qquad \Xi_4 = 3t\partial_t + x\partial_x - 2u\partial_u. \end{aligned}$$

But, if we consider a general third order equation, then the requirement of the invariance with respect to the infinitesimal operators of KdV equation is not sufficient to characterize uniquely the KdV equation.

6 Monge–Ampère equations

The homogeneous Monge–Ampère equation for the surface $u(x, y)$ with zero Gaussian curvature

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{46}$$

is a fully symmetric equation with respect to x, y, u.

It admits the 15–parameter group whose Lie algebra is spanned by the operators

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}\right), \tag{47}$$

$\forall a, b \in \{x, y, u\}$.

Rosenhaus [32, 33] proved that, starting with the general equation

$$\Delta(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{48}$$

and imposing the invariance condition

$$\Xi^{(2)}\Delta = 0\Big|_{\Delta=0}, \tag{49}$$

where $\Xi^{(2)}$ is the second prolongation of each of the operators (47), then (provided we exclude trivial cases) we are led to

$$\Delta = u_{xx}y_{yy} - u_{xy}^2 = 0. \tag{50}$$

The same can be done [32, 33] for the n–dimensional counterpart. Starting from the general equation

$$\Delta\left(x_i, u, u_{x_i}, u_{x_ix_j}\right) = 0, \tag{51}$$

and assuming the invariance with respect to the Lie group operators

$$\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial a}, \qquad a\frac{\partial}{\partial b}, \qquad a\left(\sum_{i=1}^{n} x_i\frac{\partial}{\partial x_i} + u\frac{\partial}{\partial u}\right), \tag{52}$$

$\{a, b\} \in \{x_1, \ldots, x_n, u\}$, *i.e.*,

$$\Xi^{(2)}\Delta = 0\Big|_{\Delta=0}, \tag{53}$$

where $\Xi^{(2)}$ is the second prolongation of each of the operators (52), we are easily led to

$$\Delta = \det\left\|u_{x_ix_j}\right\| = 0. \tag{54}$$

The Rosenhaus result can be extended to the most general second order completely exceptional equations (in 1, 2 and 3 space dimensions) and to the third order Monge–Ampère equation in 1 space dimension.

7 Completely exceptional equations

Quasilinear and *a fortiori* nonlinear hyperbolic equations usually possess solutions which break down in a finite time. For example, for a single quasilinear equation of order n the jump in the higher order derivatives of the field u propagates along the characteristics with speeds depending on the field u and its derivatives up to the order $(n-1)$. The discontinuities in the first order derivatives of the characteristic velocities are responsible for the breakdown in the $(n-1)$–th order derivatives of the solution.

In certain circumstances it may occur that a speed of propagation depends on the field u and its derivatives in such a way that its first order derivatives are not discontinuous. A wave propagating with such a velocity is said exceptional in the sense of Lax and Boillat [34]. If all the admitted speeds of propagation have this property, the equation is said completely exceptional. A completely exceptional equation behaves in some sense like a linear equation.

A typical example of a nonlinear completely exceptional equation is the Monge–Ampère equation assumed to be hyperbolic. This fact has been proved by Boillat [35] in $(1+1)$ dimensions, by Ruggeri [36] in $(2+1)$ dimensions, by Donato et al. [37] in $(3+1)$ dimensions, and again by Boillat [38] in the general case of $(n+1)$ dimensions. What has been proved is that all the second order completely exceptional equations are given as a linear combination of all minors extracted from the Hessian matrix.

In [37] it has been shown that, under suitable conditions, there exists (in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions) a Bäcklund transformation of reciprocal type mapping the Monge–Ampère equations to a linear canonical form.

Moreover, the Monge–Ampère equations, written under the form of first order systems of PDE's, can be reduced to linear form by means of invertible point transformations suggested by the invariance with respect to Lie groups of point symmetries [31].

Also, the second order completely exceptional equations are Lie remarkable since they are uniquely characterized by their Lie point symmetries. Here we illustrate this results in the case of $(1+1)$ dimensions.

4 Theorem. *The* $(1+1)$*–dimensional Monge–Ampère equation*

$$\kappa_1(u_{tt}u_{xx}-u_{tx}^2)+\kappa_2 u_{tt}+\kappa_3 u_{tx}+\kappa_4 u_{xx}+\kappa_5=0, \tag{55}$$

where κ_i *(*$i=1,\ldots,5$*) are constant, is Lie remarkable.*

Proof. The equation is invariant with respect to the Lie groups generated by

$$
\begin{aligned}
\Xi_1 &= \partial_t, \qquad \Xi_2 = \partial_x, \qquad \Xi_3 = \partial_u, \\
\Xi_4 &= t\partial_u, \qquad \Xi_5 = x\partial_u, \\
\Xi_6 &= \kappa_1 t\partial_t - \kappa_1 x\partial_x + (\kappa_2 x^2 - \kappa_4 t^2)\partial_u, \\
\Xi_7 &= 2\kappa_1 t\partial_x + (\kappa_3 t^2 - 2\kappa_2 xt)\partial_u, \\
\Xi_8 &= 2\kappa_1 x\partial_x + (-\kappa_2 x^2 + \kappa_4 t^2 + 2\kappa_1 u)\partial_u, \\
\Xi_9 &= 2\kappa_1 x\partial_t + (\kappa_3 x^2 - 2\kappa_4 xt)\partial_u.
\end{aligned}
$$

Now let us prove that, by requiring the invariance of the generic equation

$$\Delta(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{56}$$

with respect to these operators, *i.e.*,

$$\Xi^{(2)}\Delta = 0\Big|_{\Delta=0}, \tag{57}$$

where $\Xi^{(2)}$ is the second prolongation of each admitted operator, we recover exactly equation (55).

The request of invariance of (56) with respect to the operators Ξ_1, Ξ_2 and Ξ_3 leads respectively to:

$$\frac{\partial\Delta}{\partial t} = 0, \qquad \frac{\partial\Delta}{\partial x} = 0, \qquad \frac{\partial\Delta}{\partial u} = 0. \tag{58}$$

The request of invariance with respect to the operators Ξ_4 and Ξ_5 leads respectively to:

$$\frac{\partial\Delta}{\partial u_t} = 0, \qquad \frac{\partial\Delta}{\partial u_x} = 0. \tag{59}$$

Therefore, at this stage we reduced to the equation

$$\Delta(u_{tt}, u_{tx}, u_{xx}) = 0. \tag{60}$$

Now, let us impose the invariance with respect to the remaining operators. What we get is:

$$
\begin{aligned}
&(\kappa_1 u_{xx} + \kappa_2)\frac{\partial\Delta}{\partial u_{xx}} - (\kappa_1 u_{tt} + \kappa_4)\frac{\partial\Delta}{\partial u_{tt}} = 0\Big|_{\Delta=0}, \\
&(\kappa_1 u_{xx} + \kappa_2)\frac{\partial\Delta}{\partial u_{tx}} + (2\kappa_1 u_{tx} - \kappa_3)\frac{\partial\Delta}{\partial u_{tt}} = 0\Big|_{\Delta=0}, \\
&(2\kappa_1 u_{tx} - \kappa_3)\frac{\partial\Delta}{\partial u_{xx}} + (\kappa_1 u_{tt} + \kappa_4)\frac{\partial\Delta}{\partial u_{tx}} = 0\Big|_{\Delta=0}.
\end{aligned}
\tag{61}
$$

Let us impose the constraint $\Delta = 0$ by introducing 3 Lagrange multipliers, and, further, let us assume that equation (60) can be solved with respect to u_{tt}, *i.e.*,

$$\Delta = \Delta_1 \,(u_{xx}) u_{tt} + \Delta_2(u_{tx}, u_{xx})\,, \tag{62}$$

where Δ_1 and Δ_2 depend upon the indicated arguments.

The use of (62) into Equations (61) leads us to obtain

$$\Delta = \kappa_1(u_{tt}u_{xx} - u_{tx}^2) + \kappa_2 u_{tt} + \kappa_3 u_{tx} + \kappa_4 u_{xx} + \kappa_5, \tag{63}$$

whereupon, setting

$$\Delta = 0, \tag{64}$$

the Monge–Ampère equation (55) is recovered. ■

Similar results are valid for the multidimensional versions of second order completely exceptional equations.

8 Third order completely exceptional equations

Consider a general third order PDE

$$\Delta \left(x_\alpha, u, u_{x_\alpha}, u_{x_\alpha x_\beta}, u_{x_\alpha x_\beta x_\gamma}\right) = 0, \tag{65}$$

with Δ regular function of its arguments; the unknown field u depend on the variables x_α with $\alpha = 0, \dots, n$; x_0 is the time variable, whereas if $1 \leq \alpha \leq n$ x_α's are the space variables; moreover, α, β and γ are such that $\alpha \leq \beta \leq \gamma$.

Let us suppose the equation (65) to be hyperbolic and consider the weak discontinuity waves across a hypersurface Σ of equation $\phi(x_\alpha) = 0$. The Cauchy problem associated with the equation (65) is defined in terms of the fourth order quasilinear equation obtained by taking the derivative with respect to one independent variable, say x_0:

$$\frac{d\Delta}{dx_0} = 0, \tag{66}$$

so that we recover a fourth order quasilinear equation.

By the standard positions

$$\frac{\partial}{\partial x_0} \to -\lambda\delta, \qquad \frac{\partial}{\partial x_i} \to \nu_i\delta, \quad (i = 1, \dots, n), \tag{67}$$

where

$$\lambda = -\frac{\phi_t}{|\nabla\phi|}, \qquad \nu_i = \frac{\phi_{x_i}}{|\nabla\phi|}, \qquad \delta = \left.\frac{\partial}{\partial\phi}\right|_{\phi=0^-} - \left.\frac{\partial}{\partial\phi}\right|_{\phi=0^+}, \tag{68}$$

we obtain the root $\lambda = 0$, which we discard since it is artificially introduced by the time derivative, and the characteristic polynomial:

$$P(\lambda) = \lambda^3 \frac{\partial \Delta}{\partial u_{ttt}} - \lambda^2 \nu_i \frac{\partial \Delta}{\partial u_{ttx_i}} + \lambda \nu_i \nu_j \frac{\partial \Delta}{\partial u_{tx_ix_j}} - \nu_i \nu_j \nu_k \frac{\partial \Delta}{\partial u_{x_ix_jx_k}} = 0 \quad \text{on } \Sigma; \tag{69}$$

here and in the sequel the Einstein convention of sum over repeated indices is assumed.

The relation (69) characterizes the possible normal speeds of propagation λ of the discontinuities in the fourth order derivatives of u across Σ, in terms of ν_i and the derivatives of Δ with respect to the third order derivatives of u.

The condition of complete exceptionality reads

$$\delta\lambda = -\lambda^3 \frac{\partial \lambda}{\partial u_{ttt}} + \lambda^2 \nu_i \frac{\partial \lambda}{\partial u_{ttx_i}} - \lambda \nu_i \nu_j \frac{\partial \lambda}{\partial u_{tx_ix_j}} + \nu_i \nu_j \nu_k \frac{\partial \lambda}{\partial u_{x_ix_jx_k}} = 0, \quad \text{on } \Sigma, \tag{70}$$

that must be verified for all roots of the characteristic polynomial

$$P(\lambda) = \lambda^3 \frac{\partial \Delta}{\partial u_{ttt}} - \lambda^2 \nu_i \frac{\partial \Delta}{\partial u_{ttx_i}} + \lambda \nu_i \nu_j \frac{\partial \Delta}{\partial u_{tx_ix_j}} - \nu_i \nu_j \nu_k \frac{\partial \Delta}{\partial u_{x_ix_jx_k}} = 0 \quad \text{on } \Sigma. \tag{71}$$

By some straightforward (even if cumbersome) calculations we arrive at the conditions ensuring the complete exceptionality. We obtain 3 differential constraints in the $(1+1)$–dimensional case, 18 differential constraints in the $(2+1)$–dimensional case, 64 differential constraints in the $(3+1)$–dimensional case, to be satisfied by the function Δ.

The $(1+1)$–dimensional case has been considered in [39] within the framework of Backlünd transformations of reciprocal type. They proved that the most general third order completely exceptional equation has the form:

$$\begin{aligned} &\kappa_1(u_{ttt}u_{txx} - u_{ttx}^2) + \kappa_2(u_{ttt}u_{xx} - u_{ttx}u_{txx}) + \kappa_3(u_{ttx}u_{xxx} - u_{txx}^2) \\ &\quad + \kappa_4 u_{ttt} + \kappa_5 u_{ttx} + \kappa_6 u_{txx} + \kappa_7 u_{xxx} + \kappa_8 = 0, \end{aligned} \tag{72}$$

where κ_i $(i = 1, \ldots, 8)$ are functions of the independent variables (t and x), the unknown field u together its first and second order partial derivatives (the so called inessential variables, since they are not involved in the complete exceptionality condition).

Now we prove that equation (72), when we assume the coefficients κ_i to be constant, is Lie remarkable.

5 Theorem. *The third order Monge–Ampère equation*

$$\begin{aligned} &\kappa_1(u_{ttt}u_{txx} - u_{ttx}^2) + \kappa_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \kappa_3(u_{ttx}u_{xxx} - u_{txx}^2) \\ &\quad + \kappa_4 u_{ttt} + \kappa_5 u_{ttx} + \kappa_6 u_{txx} + \kappa_7 u_{xxx} + \kappa_8 = 0, \end{aligned} \tag{73}$$

where κ_i $(i = 1, \ldots, 8)$ are constant, is Lie remarkable.

Proof.

The third order Monge–Ampère equation is left invariant by the group of Lie point symmetries whose Lie algebra (10–dimensional) is spanned by the operators:

$$
\begin{aligned}
&\Xi_1 = \partial_t, \qquad \Xi_2 = \partial_x, \qquad \Xi_3 = \partial_u,\\
&\Xi_4 = t\partial_u, \qquad \Xi_5 = x\partial_u,\\
&\Xi_6 = t^2\partial_u, \qquad \Xi_7 = tx\partial_u, \qquad \Xi_8 = x^2\partial_u,\\
&\Xi_9 = (2\kappa_3 t - 2\kappa_2 x)\partial_x + (2\kappa_2 t - 2\kappa_1 x)\partial_t +\\
&\qquad +(\kappa_3\kappa_7 t^3 - \kappa_3\kappa_6 t^2 x + \kappa_3\kappa_5 t x^2 + \kappa_4 x^3)\partial_u,\\
&\Xi_{10} = (-2\kappa_3 t + 4\kappa_2 x)\partial_x + 2\kappa_1 x\partial_t +\\
&\qquad +(6\kappa_2 u - \kappa_3\kappa_7 t^3 + \kappa_3\kappa_6 t^2 x - \kappa_3\kappa_5 t x^2 - \kappa_4 x^3)\partial_u.
\end{aligned}
\tag{74}
$$

Now, let us consider the general equation

$$
\Delta\left(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx}\right) = 0, \tag{75}
$$

and impose the invariance conditions with respect to each of the operators (74).

The invariance with respect to the operators Ξ_1, Ξ_2 and Ξ_3 implies that

$$
\frac{\partial\Delta}{\partial t} = 0, \qquad \frac{\partial\Delta}{\partial x} = 0, \qquad \frac{\partial\Delta}{\partial u} = 0, \tag{76}
$$

whereas the invariance with respect to Ξ_4 and Ξ_5 requires that

$$
\frac{\partial\Delta}{\partial u_t} = 0, \qquad \frac{\partial\Delta}{\partial u_x} = 0. \tag{77}
$$

Moreover, the invariance with respect to Ξ_6, Ξ_7 and Ξ_8 leads to

$$
\frac{\partial\Delta}{\partial u_{tt}} = 0, \qquad \frac{\partial\Delta}{\partial u_{xt}} = 0, \qquad \frac{\partial\Delta}{\partial u_{xx}} = 0. \tag{78}
$$

Therefore the function Δ depend at most on the third order derivatives of u. Finally, by requiring the invariance with respect to the two remaining operators

Ξ_9 and Ξ_{10} we obtains the conditions

$$\begin{aligned}
&(3\kappa_2 u_{xxx} + 3\kappa_1 u_{xxt} + 3\kappa_4)\,\frac{\partial \Delta}{\partial u_{xxx}} + \\
&\quad + (-\kappa_3 u_{xxx} + \kappa_2 u_{xxt} + \kappa_1 u_{xtt} + \kappa_3\kappa_5)\,\frac{\partial \Delta}{\partial u_{xxt}} + \\
&\quad + (-2\kappa_3 u_{xxt} - \kappa_2 u_{xtt} + \kappa_1 u_{ttt} - \kappa_3\kappa_6)\,\frac{\partial \Delta}{\partial u_{xtt}} + \\
&\quad (-3\kappa_3 u_{xtt} - 3\kappa_2 u_{ttt} + 3\kappa_3\kappa_7)\,\frac{\partial \Delta}{\partial u_{ttt}} = 0\Big|_{\Delta=0}, \qquad (79)\\
&(-3\kappa_2 u_{xxx} - 3\kappa_1 u_{xxt} - 3\kappa_4)\,\frac{\partial \Delta}{\partial u_{xxx}} + \\
&\quad + (\kappa_3 u_{xxx} - \kappa_2 u_{xxt} - 2\kappa_1 u_{xtt} - \kappa_3\kappa_5)\,\frac{\partial \Delta}{\partial u_{xxt}} + \\
&\quad (2\kappa_3 u_{xxt} + \kappa_2 u_{xtt} - \kappa_1 u_{ttt} + \kappa_3\kappa_6)\,\frac{\partial \Delta}{\partial u_{xtt}} + \\
&\quad (3\kappa_3 u_{xtt} + 3\kappa_2 u_{ttt} - 3\kappa_3\kappa_7)\,\frac{\partial \Delta}{\partial u_{ttt}} = 0\Big|_{\Delta=0}. \qquad (80)
\end{aligned}$$

Also in this case let us impose the constraint $\Delta = 0$ through the introduction of Lagrange multipliers, and assume that equation (75) can be solved with respect to u_{ttt}, *i.e.*,

$$\Delta(u_{ttt}, u_{ttx}, u_{txx}, u_{xxx}) = \Delta_1(u_{txx}, u_{xxx})u_{ttt} + \Delta_2(u_{ttx}, u_{txx}, u_{xxx}), \qquad (81)$$

where Δ_1 and Δ_2 are regular functions of the indicated arguments.

By inserting (81) into equations (79) and (80) and performing some straightforward algebra we finally get (without any assumptions on the constants κ_i):

$$\begin{aligned}
\Delta &= \kappa_1(u_{ttt}u_{txx} - u_{ttx}^2) + \kappa_2(u_{ttt}u_{xxx} - u_{ttx}u_{txx}) + \kappa_3(u_{ttx}u_{xxx} - u_{txx}^2) + \\
&+ \kappa_4 u_{ttt} + \kappa_5 u_{ttx} + \kappa_6 u_{txx} + \kappa_7 u_{xxx} + \kappa_8,
\end{aligned}$$

whereupon, equating to zero, we have the third order Monge–Ampère equation. ■

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