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Relativistic mechanics, cosymplectic manifolds and symmetries

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Abstract. We consider the formulation by Janyška and Modugno of the phase space of relativistic mechanics in the framework of jets of 1-dimensional time-like submanifolds. Here, the gravitational and electromagnetic structures are encoded in a cosymplectic form. We derive the equation of motion of one relativistic particle in this framework, and prove that the Lagrangian of our model is non-degenerate. This makes the phase space a universal primary constraint. Finally, we show as all symmetries of the equation of motion (including higher or generalized symmetries) can be interpreted as distinguished vector fields on the phase space.

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1 Introduction

Relativistic mechanics is usually formulated, in a differential geometric context, as a theory on the tangent bundle of spacetime. Anyway, this formulation has some well-known drawbacks: first of all the Lagrangian (which, in the case of vanishing electromagnetic field, is the arc-length with respect to the given Lorentz metric) is degenerate, *i.e.* its Hessian is a singular matrix. The degeneracy is due to the fact that the Lagrangian is invariant with respect to the affine reparametrization of curves. Hence, to provide a Hamiltonian formulation it is necessary to use Dirac's constraints [7].

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The above problems are well-known; as a way to solve them, some authors tried to formulate mechanics on 'trajectories', or non-parametrized curves [2, 21, 24]. None of them, however, present a complete model for the relativistic mechanics.

Recently, a formulation of relativistic mechanics based on jets of submanifolds and cosymplectic forms has been presented [9–11]. Here the tangent bundle is replaced by the first jet space of time-like curves as the phase space of the theory. Time-like curves can be regarded as distinguished 1-dimensional submanifolds: indeed, the phase space is a subspace of the first jet of 1-dimensional submanifolds of spacetime.

The literature on jets of submanifolds (also known as 'contact elements', 'differentiable elements', 'first-order caps', 'extended jets' etc., see [18–20, 27] and references therein) is less spread with respect to the literature on jets of fibrings. Jets of 1-dimensional submanifolds of order r are equivalence classes of 1-dimensional submanifolds having a contact of order r. Clearly, two curves have a contact if and only if their tangent space is the same; hence this notion is independent of the parametrization. In particular, the first jet of 1-dimensional submanifolds of spacetime turns out to be the projectivized tangent bundle of spacetime. The language of jets of submanifolds is a natural and straightforward generalization of the language of tangent spaces, and shall not be regarded as a unnecessary mathematical construction. Indeed, being relativistic mechanics dependent on trajectories as submanifolds (irrespectively of any parametrization), jets of submanifolds are the most natural language that differential geometry provides to model such a physical theory. Other models with tangent spaces introduce one extra degree of freedom which has to be discarded with the mechanism of Dirac constraints.

In [9] a cosymplectic form (in the sense of [1, 5, 14]) naturally induced from the gravitational and the electromagnetic fields has been introduced on the first jet space of time-like curves. Such a form has a one-dimensional kernel (Reeb vector field) whose integral curves are the trajectories of the system. Then, a quantum theory for a scalar particle on the above background has been formulated by analogy with the geometric quantization of classical mechanical systems [28,29], and this leads to a covariant Klein–Gordon equation [8]. We are aware of the well known difficulties for physical interpretation of the probability current and for the formulation of the Hilbert stuff.

Anyway, the variational formulation of the equation of motion was absent in the above model. This article is aimed at filling this gap. Namely, after recalling preliminaries on jets in section 2 and the model by Janyška and Modugno in section 3, we derive the equation of motion using the variational calculus on jets of submanifolds first introduced by A. M. Vinogradov in the context of infinite-order jets [4,25–27] then specialized by us to finite order jets [15,18,19]. As a by-product, we show that a Lagrangian formulation of relativistic mechanics arise in a very natural way on jets of submanifolds, contrarily to what is stated in [24]. Moreover, we prove that the Lagrangian of our model is non-degenerate. This shows that our phase space is, indeed, a universal primary constraint. Hence, a Hamiltonian formulation could be achieved, but this will be the subject of future research (see also [12]).

Due to the 'cleaner' mathematical setting of the theory, we obtain a natural picture of symmetries of the equation of motion in our scheme, including symmetries depending on velocities (higher symmetries in the sense of [4, 27], generalized symmetries in the sense of [22]). We stress that the point symmetries of our equation (i.e, symmetries induced by spacetime transformations) recover in a much simpler way *projective collineations*, *i.e.* transformations bringing geodesics into geodesics even without preserving the parametrization. Usually such transformations are defined in a much more involved way (see [3, 6, 13]).

In future papers we hope to provide also the Hamiltonian formalism as well as computations on classical, projective and higher symmetries of the equations of motions in distinguished models of spacetime.

We end this introduction by some mathematical preliminaries.

The theory of *unit space* has been developed in [9, 12] in order to make explicit the independence of classical and quantum mechanics from the choice of unit of measurements. Unit spaces have the same algebraic structure as $\mathbf{R}_+$, but no natural basis. We assume the (1-dimensional) unit spaces $\mathbb{T}$ (space of *time intervals*), $\mathbb{L}$ (space of *lengths*) and $\mathbb{M}$ (space of *masses*). We set $\mathbb{T}^{-1} \equiv \mathbb{T}^*$, and analogously for $\mathbb{L}$, $\mathbb{M}$. Tensor fields appearing in the theory will be usually *scaled*, *i.e.*, they will take values in unit spaces according to their physical interpretation. For example, the metric will take values in the space of area units $\mathbb{L}^2$.

We assume the following constant elements: the *light velocity* $c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ and the *Planck's constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$. Moreover, we say a *charge* to be an element $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbf{R}$.

We will assume coordinates to be dimensionless (*i.e.*, real valued). We assume manifolds and maps to be $\mathcal{C}^\infty$.

2 Jets of submanifolds: an overview

Here we will recall the basics about jets of 1-dimensional submanifolds. Our main sources are [15, 18–20], where the structures constructed below are given in the general case of jets of n-dimensional submanifolds.

Let $r \geq 0$. An r-jet of a 1-dimensional submanifold $s \subset E$ at $x \in E$ is defined to be the equivalence class of 1-dimensional submanifolds having a contact with s of order r at x. The equivalence class is denoted by $j_r s(x)$, and the quotient set by $J^r(E,1)$. Moreover, the image of the map $j_r s \colon s \to J^r(E,1)$ defines a natural lifting of the 1-dimensional submanifold s to $J^r(E,1)$. For $r > p$ we have the bundles $\pi^r_p : J^r(E,1) \to J^p(E,1)$. Note that the bundles π^{r+1}_r are affine bundles for $r \geq 1$ [19, 26]. Of course $J^0(E,1) = E$. A chart (x^λ) on E is said to be *adapted* to a 1-dimensional submanifold s if s can be expressed in coordinate as $(x^0, s^i(x^0))$, where s^i are local real functions. The set $J^r(E,1)$ has a natural manifold structure; a chart (x^0, x^i) on E induce the chart $(x^0, x^i, x^i_{\boldsymbol{\alpha}})$ on $J^r(E,1)$ such that $x^i_{\boldsymbol{\alpha}} \circ s = \partial^{|\boldsymbol{\alpha}|} s^i x^{\boldsymbol{\alpha}}$ (here $\boldsymbol{\alpha}$ is a multi-index of length r containing, in the 1-dimensional case, only the index 0 repeated r times).

We introduce the bundle $T^{r+1,r} \stackrel{\text{def}}{=} J^{r+1}(E,1) \times_{J^r(E,1)} TJ^r(E,1)$; the *pseudo-horizontal* subbundle $H^{r+1,r} \subset T^{r+1,r}$ [18, 19] is defined by

$$H^{r+1,r} \stackrel{\text{def}}{=} \left\{ (j_{r+1}s(x), \upsilon) \in T^{r+1,r} \mid \upsilon \in T_{j_r s(x)}(j_r s(s)) \right\}. \qquad (1)$$

It is easy to realize that $H^{r+1,r} = J^{r+1}(E,1) \times_{J^1(E,1)} H^{1,0}$. We also have the *pseudo-vertical* bundle $V^{r+1,r} \stackrel{\text{def}}{=} T^{r+1,r}/H^{r+1,r}$. The bundles $H^{1,0}$ and $V^{1,0}$ are strictly related with the horizontal and vertical bundle in the case of jets of fibrings. A local basis for the sections of the bundle $H^{r+1,r}$ is $D^{(r+1)}_0 = \partial_0 + x^i_{\boldsymbol{\alpha}0}\partial^{\boldsymbol{\alpha}}_i$, while a local basis for the sections of its dual $H^{r+1,r*}$ is $\overline{d}^0 \stackrel{\text{def}}{=} d^0|_{H^{r+1,r}}$ The inclusion $D^{(r+1)} \colon J^{r+1}(E,1) \to H^{r+1,r*} \otimes T^{r+1,r}$ is said to be the $(r+1)$-th order *contact structure* on $J^{r+1}(E,1)$. Its coordinate expression is $D^{(r+1)} = \overline{d}^0 \otimes D^{(r+1)}_0 = \overline{d}^0 \otimes (\partial_0 + x^i_{\boldsymbol{\alpha}0}\partial^{\boldsymbol{\alpha}}_i)$. The basis of the annihilators of $D^{(r+1)}_0$ in $T^{r+1,r*}$, or *contact forms*, is denoted by $\{\omega^i_{\boldsymbol{\alpha}}\}$, where $\omega^i_{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} d^i_{\boldsymbol{\alpha}} - x^i_{\boldsymbol{\alpha}0} d^0$. They generate a space of forms which is naturally isomorphic to $V^{r+1,r*}$. For deeper discussions on jets of submanifolds, see [15, 18–20] and references therein.

Jets of submanifolds are a natural environment for differential equations and the calculus of variations. An *ordinary differential equation of order* r is a submanifold $\mathcal{Y} \subset J^r(E,1)$. A *solution* is a 1-dimensional submanifold $s \subset E$ such that $j_r s(s) \subset \mathcal{Y}$.

Given a 1-form α on $J^1(E,1)$ and a 1-dimensional submanifold $s \subset E$ we have the *action* [19]:

$$A_U(s) = \int_U (j_1 s)^* \alpha,$$

where U is a regular oriented 1-dimensional submanifold of s with compact closure. The action is determined by α modulo contact forms: a contact form C fulfills $(j_1 s)^* C = 0$ for every 1-dimensional submanifold s. The *horizontalization* [18, 19] takes the 1-form α into a section $h(\alpha)$ of $H^{2,1*}$ which does not contain contact factors. If $\alpha = \alpha_0 d^0 + \alpha_i d^i + \alpha_i^0 d_0^i$ then we have the coordinate expression $h(\alpha) = (\alpha_0 + \alpha_i x_0^i + \alpha_i^0 x_{00}^i)\overline{d}^0$. For the above reasons, the action depends only on the 'horizontal' component of α, hence we have

$$A_U(L) = \int_U (\alpha_0 + \alpha_i x_0^i + \alpha_i^0 x_{00}^i)\overline{d}^0.$$

Note that here $\overline{d}^0$ transforms according to a more complex law with respect to the case of jets of fibrings.

Then, there is a natural differential operator, the Euler–Lagrange operator, bringing a Lagrangian into its Euler–Lagrange morphism. It can be defined in the framework of A. M. Vinogradov's $\mathcal{C}$-spectral sequence [25, 26] (see [18, 19] for its finite-order jet equivalent). We have

$$\mathcal{E}(h(\alpha)) \colon J^3(E, 1) \to V^{1,0^*} \otimes H^{1,0^*}.$$

The corresponding Euler–Lagrange equations are

$$\mathcal{E}(h(\alpha)) = \Big(\partial_i h(\alpha)_0 - D_0^{(3)}\left(\partial_i^0 h(\alpha)_0\right)\Big)\,\omega^i \otimes \overline{d}^0 = 0, \tag{2}$$

where we have set $h(\alpha)_0 \stackrel{\text{def}}{=} \alpha_0 + \alpha_i x_0^i + \alpha_i^0 x_{00}^i$. The Euler–Lagrange equation can be achieved in another way, using properties of the $\mathcal{C}$-spectral sequence. Namely, the horizontalization can be generalized to k-forms; it produces forms which always have a horizontal factor. Then, the following diagram commute:

$$\begin{tikzcd} \Lambda_1^1 \arrow[r, "d"] \arrow[d, "h"'] & \Lambda_1^2 \arrow[d, "h'"] \\ \bar{\Lambda}_1^1 \arrow[r, "\mathcal{E}"] & E_1^{1,1} \end{tikzcd} \tag{3}$$

where Λ_1^k denotes the space of k-forms on $J^1(E, n)$, h is the horizontalization, h' is horizontalization followed by factorization of total divergencies [18, 19], d is the standard differential and $\mathcal{E}$ is the Euler–Lagrange operator between the space of Lagrangians $\bar{\Lambda}_1^1$ and the space of Euler–Lagrange morphisms $E_1^{1,1}$. So, we can compute the Euler–Lagrange morphism through the equality

$$\mathcal{E}(h(\alpha)) = h'(d\alpha). \tag{4}$$

3 General relativistic phase space

In this section we summarize the model of general relativistic phase space given by Janyška and Modugno [9].

Spacetime. We assume the *spacetime* to be a manifold E, with $\dim E = 4$, endowed with a scaled Lorentz metric $g : E \to \mathbb{L}^2 \otimes T^*E \otimes_E T^*E$ whose signature is $(-+++)$. Moreover, we assume E to be oriented and time-like oriented.

In what follows, Latin indexes $i, j, \ldots$ will label space-like coordinates and run from 1 to 3, Greek indexes $\lambda, \mu, \varphi, \ldots$ will label spacetime coordinates and run from 0 to 3. Charts on E are denoted by (x^φ); the corresponding bases of vector fields and 1-forms are denoted, respectively, by ∂_φ and d^φ. An element $u_0 \in \mathbb{T}$, or, equivalently, its dual $u^0 \in \mathbb{T}^{-1}$, is said to be a *time unit of measurement*. We have the coordinate expressions $g = g_{\varphi\psi} d^\varphi \otimes d^\psi$, where $g_{\varphi\psi} \colon E \to \mathbb{L}^2 \otimes \mathbf{R}$. We will use charts such that ∂_0 is time-like and time-like oriented, and ∂_i are space-like.

Kinematics. A time-like 1-dimensional submanifold $s \subset E$ is said to be a *motion*, whose *velocity* is $j_1 s$. The open subbundle $U^1 E \subset J^1(E, 1)$ of velocities of motions is said to be the (general relativistic) *phase space*.

In an analogous way we introduce the subbundles $U^r E \subset J^r(E, 1)$. Note that the bundles π^{r+1}_r are affine if $r \geq 1$. By a restriction we have the natural bundle structure $\pi^1_0 : U^1 E \to E$. A section $o \colon E \to U^1 E$ is said to be an *observer*. A typical chart (x^0, x^i) on E induces a local fibred chart $(x^0, x^i; x^i_0)$ on $U^1 E$ such that, if $x^i|_s = s^i \circ x^0|_s$, then

$$x^i_0 \circ j_1 s = \partial s^i x^0.$$

The restriction of the pseudo-horizontal bundle $H^{1,0}$ to $U^1 E$ admits the following global trivialization

$$H^{1,0} \to U^1 E \times (\mathbb{T} \otimes \mathbf{R}),\ (j_1 s(x), \upsilon) \to \left(j_1 s(x), \pm \frac{\|\upsilon\|}{c} \right),$$

where the sign depends on the time orientation of υ. This enables us to use a normalized version of the contact structure

$$д_1 \stackrel{\text{def}}{=} c \frac{D^{(1)}}{\|D^{(1)}\|} : U^1 E \to \mathbb{T}^* \otimes TE,$$

with coordinate expression $д_1 = c\alpha^0 \left(\partial_0 + x^i_0 \partial_i\right)$, where

$$\alpha^0 = |g_{00} + 2g_{0j} x^j_0 + g_{ij} x^i_0 x^j_0|^{-1/2} \in \mathbb{L}^{-1} \otimes \mathbf{R}.$$

We have $g\circ(д_1, д_1) = -c^2$, hence $U^1 E$ can be regarded as a non-linear subbundle $U^1 E \subset \mathbb{T}^* \otimes TE$, whose fibres are diffeomorphic to $\mathbf{R}^3$. We also have the dual counterpart of $д_1$

$$\tau \stackrel{\text{def}}{=} -c^{-2}\, g^\flat \circ д_1 \colon U^1 E \to \mathbb{T} \otimes T^*E, \tag{5}$$

with coordinate expression $\tau = \tau_\lambda \, d^\lambda = -c^{-1}\alpha^0(g_{0\lambda} + g_{i\lambda}x_0^i)d^\lambda$.

The metric g yields an orthogonal splitting of the tangent space TE on each $x \in E$ on which a time-like direction has been assigned. In other words, we have the splitting [9]

$$U^1E \times_E TE = H^{1,0} \oplus_{U^1E} V_g^{1,0}, \tag{6}$$

where $V_g^{1,0}$ is naturally isomorphic to $V^{1,0}$. The projection $\pi^\perp$ on $V_g^{1,0}$ is denoted by θ. Let us set $\breve{g}_{0\lambda} \stackrel{\text{def}}{=} g(D_0^{(1)}, \partial_\lambda) = g_{0\lambda} + g_{i\lambda}x_0^i$. Then we have the coordinate expression $\theta = d^\lambda \otimes \partial_\lambda + (\alpha^0)^2 \breve{g}_{0\lambda} d^\lambda \otimes (\partial_0 + x_0^i \partial_i)$, and the local basis $b_i \stackrel{\text{def}}{=} \partial_i - c\alpha^0 \tau_i(\partial_0 + x_0^i \partial_i)$ of $V_g^{1,0}$. According to the above splitting we have the following decomposition of the lift of g on $T^{1,0}$

$$g \circ \pi_0^1 = g_\parallel + g_\perp, \tag{7}$$

where $g_\parallel$ is a metric on $H^{1,0}$ and $g_\perp$ is a metric on $V_g^{1,0}$. The coordinate expression of the above metric and of their contravariant form $\overline{g}_\parallel$ and $\overline{g}_\perp$ are

$$g_{\parallel 00} = -\frac{1}{(\alpha^0)^2} \qquad \overline{g}_\parallel{}^{00} = -(\alpha^0)^2$$
$$g_{\perp ij} = g_{ij} + c^2\tau_i\tau_j, \qquad \overline{g}_\perp{}^{ij} = g^{ij} - g^{i0}x_0^j - g^{0j}x_0^i + g^{00}x_0^i x_0^j$$

The vertical derivative $V\text{д}_1$ induces the linear fibred isomorphism

$$v : VU^1E \to \mathbb{T}^* \otimes V_g^{1,0} \tag{8}$$

over U^1E, with coordinate expression $v = c\alpha^0 \, d_0^i \otimes b_i$.

Gravitational and electromagnetic forms. The Levi-Civita connection $K^\natural$ on $TE \to E$ induces naturally a (non linear) connection $\Gamma^\natural$ on $U^1E \to E$ [9], which is expressed by a section $\Gamma^\natural : U^1E \to T^*E \otimes_{U^1E} TU^1E$, and has the coordinate expression

$$\Gamma^\natural = d^\varphi \otimes (\partial_\varphi + \Gamma^\natural{}_{\varphi 0}^{\ i} \, \partial_i^0), \tag{9}$$

with $\Gamma^\natural{}_{\varphi 0}^{\ i} = K_\varphi{}^i{}_j x_0^j + K_\varphi{}^i{}_0 - x_0^i(K_\varphi{}^0{}_j x_0^j + K_\varphi{}^0{}_0)$. The connections $K^\natural$ and $\Gamma^\natural$ are said to be *gravitational*.

Let m be a mass. Then, the gravitational connection $\Gamma^\natural$ and the metric g induce the form on U^1E

$$\Omega^\natural \stackrel{\text{def}}{=} \frac{m}{\hbar} \, g_\perp \lrcorner (v \circ \nu[\Gamma^\natural]) \wedge \theta : U^1E \to \bigwedge^2 T^*U^1E, \tag{10}$$

where $\nu[\Gamma^\natural] = \mathrm{id}[TU^1E] - \Gamma^\natural$ is the vertical projection associated with $\Gamma^\natural$, v is the isomorphism (8), θ is the vertical projection of the splitting (6) and the factor $m/\hbar$ is put in order to obtain a non-scaled object. It can be proved that

- $\Omega^\natural = -\frac{mc^2}{\hbar} d\tau$, hence $\Omega^\natural$ is an exact form;
- $\tau \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural$ is a volume form on $U^1 E$, hence Ω is non degenerate.

It turns out that $\Omega^\natural$ is a *contact form* (see [14]), and it is said to be the *gravitational form* of our model. Note that, being $\dim U^1 E = 7$, there exist no symplectic forms on $U^1 E$. We have the coordinate expression

$$\Omega^\natural = \frac{m}{\hbar} c\alpha^0 g_{\perp ij}(d_0^i - \Gamma^\natural{}_{\varphi 0}^{\,i}\, d^\varphi) \wedge \omega^j,$$

Now, we assume the *electromagnetic field* to be a closed scaled 2-form on E

$$F : E \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \wedge^2 T^* E. \tag{11}$$

We have the coordinate expression $F = 2F_{0j} d^0 \wedge d^j + F_{ij} d^i \wedge d^j$. We denote a local potential of F with $A : E \to T^*E$, according to $2dA = F$.

Given a charge q, the normalized electromagnetic field $q/\hbar\, F$ can be incorporated into the geometrical structure of the phase space, *i.e.* the gravitational form. Namely, we define the *total form*

$$\Omega \stackrel{\text{def}}{=} \Omega^\natural + \frac{q}{2\hbar} F : U^1 E \to \wedge^2 T^* U^1 E. \tag{12}$$

Of course $d\Omega = 0$ but Ω is exact if and only if F is exact. A local potential of Ω is

$$\Omega = d\left(-\frac{mc^2}{\hbar}\tau + \frac{q}{\hbar} A\right). \tag{13}$$

Moreover we have

$$\tau \wedge \Omega \wedge \Omega \wedge \Omega = \tau \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural,$$

so Ω is non degenerate. Hence, Ω is a *cosymplectic form* (in the sense of [1, 5]) encoding the gravitational and electromagnetic (classical) structures. Note that, even locally, Ω cannot be regarded as a contact form, because we have

$$\left(-\frac{mc^2}{\hbar}\tau + \frac{q}{\hbar} A\right) \wedge \Omega \wedge \Omega \wedge \Omega = \left(-\frac{mc^2}{\hbar}\tau + \frac{q}{\hbar} A\right) \wedge \Omega^\natural \wedge \Omega^\natural \wedge \Omega^\natural,$$

but the form $-mc^2/\hbar\,\tau + q/\hbar\, A$ may vanish at some point.

We recall that a unique connection Γ on $U^1 E \to E$ can be characterized through Ω [9].

Equations of motion. We define a *second order connection*[1] $\gamma^\natural$ [9,23] on spacetime as the map

$$\gamma^\natural \overset{\text{def}}{=} д_1 \lrcorner \Gamma \colon U^1E \to U^2E \overset{д_2}{\hookrightarrow} \mathbb{T}^* \otimes TU^1E. \tag{14}$$

1 Remark. The map $\gamma^\natural$ can be regarded both as a section of π^2_1 and as a scaled vector field on the phase space because it takes its values in the image of the inclusion $д_2$. We will drop this inclusion, being clear from the context what is the representation of $\gamma^\natural$ that we are using. □

We have the coordinate expressions

$$\gamma^{\natural i}_{00} \overset{\text{def}}{=} x^i_{00} \circ \gamma^\natural = \gamma^{\natural i}_{00} = \Gamma_0{}^i{}_0 + \Gamma_j{}^i{}_0 x^j_0, \qquad \gamma^\natural = c\alpha^0(\partial_0 + x^i_0 \partial_i + \gamma^{\natural i}_{00} \partial^0_i).$$

As a scaled vector field on U^1E, $\gamma^\natural$ fulfills

$$\gamma^\natural \lrcorner \Omega^\natural = 0, \qquad \gamma^\natural \lrcorner \tau = 1,$$

hence it is the Reeb vector field associated with the contact form $\Omega^\natural$ [14].

It is natural to ask about analogous properties of the total form Ω. It can be easily proved that there exists a second order connection γ such that $\gamma \lrcorner \Omega = 0$ and $\gamma \lrcorner \tau = 1$. Such a connection takes the form $\gamma = \gamma^\natural + \gamma^e$, where

$$\gamma^e \colon U^1E \to \mathbb{T}^* \otimes VU^1E. \tag{15}$$

Note that the above sum is performed in U^2E, as $\mathbb{T}^* \otimes VU^1E$ is the associated fibre bundle [19]. We have the coordinate expression

$$\gamma^e = -\frac{q}{m} \bar{g}_\perp{}^{ik}(F_{0k} + F_{jk} x^j_0)\partial^0_i.$$

Of course, γ^e is the *Lorentz force* associated with F.

The *equation of motion* is the submanifold of U^2E defined as follows

$$\nabla[\gamma] = j_2 s - \gamma \circ j_1 s = 0. \tag{16}$$

In other words, the equation of motion is the image of the section γ. It is equivalent to $j_2 s - \gamma^\natural \circ j_1 s = \gamma^e \circ j_1 s$. We have the coordinate expression

$$\begin{aligned} x^i_{00} - K_0{}^i{}_0 - 2K_0{}^i{}_j x^j_0 + K_0{}^0{}_0 x^i_0 + 2K_0{}^0{}_j x^i_0 x^j_0 - K_j{}^i{}_k x^j_0 x^k_0 + K_j{}^0{}_k x^j_0 x^k_0 x^i_0 = \\ = -\frac{q}{m} \bar{g}_\perp{}^{ik}(F_{0k} + F_{jk} x^j_0). \end{aligned} \tag{17}$$

[1] The map $\gamma^\natural$ plays here a role analogous to that of the geodesic spray in the formulation of mechanics on tangent spaces.

2 Remark. The above system of equations is different from the usual equations of general relativistic mechanics. The main difference is that the above equations are on *unparametrized trajectories* rather than on *parametrized trajectories*. Indeed, it is well-known that general relativistic mechanics is invariant with respect to reparametrizations [7]. The above approach allows us to discard the extra degree of freedom constituted by the parameter of motions. Hence, the equations of motions are just 3 instead of the standard 4. □

4 Variational formulation of the equations of motion

Now, we derive the equation of motion from the variational viewpoint. To do this we use the geometric theory of calculus of variations on jets of submanifolds by A. M. Vinogradov. He formulated the theory on infinite order jets, but to the purposes of the analysis of our model it is more suitable to use the finite order analogue developed in [15, 18, 19].

3 Theorem. *We have $h(\Omega) = \eta$, where*

$$\eta \colon U^2 E \to V^{1,0^*} \otimes H^{1,0^*}$$

with coordinate expression

$$\eta = \frac{m}{\hbar}\Big(c\alpha^0 g_{\perp ij}(x^i_{00} - \gamma^{\natural i}_{00}) - \frac{q}{m}(F_{0j} + F_{ij}x^i_0)\Big)\,\omega^j \otimes \overline{d}^0.$$

Proof. The form Ω can be locally written as follows:

$$\begin{aligned}
\Omega &= \frac{m}{\hbar} c\alpha^0 g_{\perp ij}(d^i_0 - \gamma^{\natural i}_{00} d^0 - \Gamma_h{}^i_0 \omega^h) \wedge \omega^j \\
&\qquad\qquad\qquad + \frac{q}{2\hbar}(2F_{0j} d^0 \wedge d^j + F_{ij} d^i \wedge d^j) \\
&= \frac{m}{\hbar} c\alpha^0 g_{\perp ij}(\omega^i_0 + (x^i_{00} - \gamma^{\natural i}_{00}) d^0 - \Gamma_h{}^i_0 \omega^h) \wedge \omega^j \\
&\qquad\qquad\qquad + \frac{q}{\hbar}(F_{0j} + F_{ij} x^i_0) d^0 \wedge \omega^j + \frac{q}{2\hbar} F_{ij}\, \omega^i \wedge \omega^j \\
&= \eta + C,
\end{aligned}$$

having set

$$C = \frac{m}{\hbar} c\alpha^0 g_{\perp ij}(\omega^i_0 - \Gamma_h{}^i_0 \omega^h) \wedge \omega^j + \frac{q}{2\hbar} F_{ij}\, \omega^i \wedge \omega^j$$

Of course, C is annihilated by the horizontalization because it has only contact factors and no horizontal factors $\overline{d}^0$. $\boxed{QED}$

We observe that the map η is an Euler–Lagrange type morphism: it is divergence-free because it takes values in a space of divergence-free forms [19]. Now, the following corollary provides the obvious link between η and the equation of motion.

4 Corollary. *We have the equality*

$$\eta = \overline{g}_\perp(\nabla[\gamma]),$$

where $\overline{g}_\perp$ is the metric on $V_g^{1,0*}$ induced by g (7). Hence, the Euler–Lagrange type equation $\eta \circ j_2 s = 0$ and the equation of motion are the same submanifold of U^2E.

It is natural to ask if η comes from a Lagrangian: the answer, of course, is positive.

5 Theorem. *The relation $\Omega = d(-mc^2/\hbar\,\tau + q/\hbar\,A)$ implies that η admits the (local) Lagrangian*

$$\mathcal{L}\colon U^1E \to H^{1,0*}, \quad \mathcal{L} = h\left(-\frac{mc^2}{\hbar}\tau + \frac{q}{\hbar}\,A\right), \tag{18}$$

with coordinate expression

$$\mathcal{L} = L_0\,\overline{d}^0 = \Big(-\frac{mc}{\hbar}|g_{00} + 2g_{0j}x_0^j + g_{ij}x_0^i x_0^j|^{1/2} + \frac{q}{\hbar}\,(A_0 + x_0^i A_i)\Big)\,\overline{d}^0.$$

Proof. It comes from the commutative diagram (3), or, equivalently, the identity (4) (see [18, 19]). $\boxed{QED}$

Of course, $\mathcal{L}$ is defined on the same domain as A, and it is global if and only if F is exact (including the distinguished case $F = 0$).

A simple computation proves the following corollary.

6 Corollary. *The form $-mc^2/\hbar\,\tau + q/\hbar\,A$ turns out to be the Poincaré–Cartan form associated with the Lagrangian of the above theorem.*

5 Regularity of the equation of motion

Let us denote by Hess($\mathcal{L}$) the Hessian of $\mathcal{L}$, *i.e.*, the second differential of $\mathcal{L}$ along the fibres of $\pi_{1,0}$. We have

$$\mathrm{Hess}(\mathcal{L})\colon U^1E \to V^*U^1E \otimes V^*U^1E \otimes H^{1,0*},$$
$$\mathrm{Hess}(\mathcal{L}) = \frac{\partial^2 L_0}{\partial x_0^i \partial x_0^j}\, d_0^i \otimes d_0^j \otimes \overline{d}^0.$$

7 Theorem. *The Hessian of $\mathcal{L}$ is a non-singular matrix.*

Proof. By a direct computation we have

$$\frac{\partial^2 L}{\partial x_0^i \partial x_0^j} = \frac{mc}{\hbar}\,\frac{\partial^2 (\alpha^0)^{-1}}{\partial x_0^i \partial x_0^j} = -\frac{mc}{\hbar}\,\alpha^0 g_{\perp ij} \tag{19}$$

$\boxed{QED}$

8 Remark. In the standard relativistic mechanics the arc-length Lagrangian $\tilde{\mathcal{L}} = \sqrt{g_{AB}\dot{y}^A\dot{y}^B}$ on TE has an Hessian matrix whose rank is 3, hence it is singular. □

The above theorem shows that our model for the relativistic mechanics admits a Hamiltonian formalism which is related with the above Lagrangian formalism via a hyperregular transformation. Hence, we have the following consequence.

9 Corollary. *The injective morphism*

$$д_1 : U^1E \to \mathbb{T}^* \otimes TE$$

provides a primary constraint for any choice of a time scale $u_0 \in \mathbb{T}$. *Hence,* U^1E *can be regarded as a universal primary constraint.*

6 Symmetries of the equation of motion

We have seen, in the construction of the pseudo-horizontal bundle, that to any point $j_{r+1}s(x) \in J^{r+1}(E,1)$ there corresponds the 1-dimensional line $T_{j_rs(x)}(j_rs(s))$. The span $\mathcal{C}_{j_rs(x)}$ of such lines obtained by varying the point $j_{r+1}s(x)$ on the fibre of π^{r+1}_r is called a *Cartan space* at $j_rs(x)$. Then we obtain the *Cartan distribution* $\mathcal{C}$ on $J^r(E,1)$. On any differential equation $\mathcal{Y} \subset J^r(E,1)$ we have the *induced Cartan distribution* $T\mathcal{Y}\cap\mathcal{C}$. A vector field on $J^r(E,1)$ which preserves the Cartan distribution is called a *classical external* symmetry of $\mathcal{Y}$ if it is tangent to $\mathcal{Y}$. A vector field on $\mathcal{Y}$ which preserves the induced Cartan distribution is called a *classical internal* symmetry.

Now let us consider the set of the first differential consequences of $\mathcal{Y}$

$$\mathcal{Y}^{(1)} = \{j_{r+1}s(x) \in J^{r+1}(E,1) \mid j_rs(x) \in \mathcal{Y},\ T_{j_rs(x)}(j_rs(s)) \subset T_{j_rs(x)}\mathcal{Y}\}.$$

We can define $\mathcal{Y}^{(m)}$ by iteration. The differential equation $\mathcal{Y}$ is said *formally integrable* if $\pi^{r+m}_{r+m-1}|_{\mathcal{Y}^{(m)}} : \mathcal{Y}^{(m)} \to \mathcal{Y}^{(m-1)}$ are smooth fibre bundles. In this case the space $\mathcal{Y}^{(\infty)}$ can be introduced as the projective limit of the sequence $\cdots \to \mathcal{Y}^{(m)} \to \mathcal{Y}^{(m-1)} \to \cdots$ A *higher internal* symmetry of $\mathcal{Y}$ is a vector field on $\mathcal{Y}^{(\infty)}$ which preserves the induced Cartan distribution on $\mathcal{Y}^{(\infty)}$.

We denote by R^γ the 1-dimensional distribution on U^1E associated to γ.

The results exposed below are analogous to the results obtained in [16] in the case of the geodesic equation for a Riemannian manifold.

10 Proposition. *All the prolongations of the equation of motion* $\nabla[\gamma] = 0$ *are naturally diffeomorphic to* U^1E *and the induced Cartan distribution is isomorphic to* R^γ.

Proof. For any point $\theta \in U^1E$ we define $\gamma^{(1)}$ by $\gamma^{(1)}(\theta) = (\gamma(\theta))^{(1)}$. Also, we have

$$(\gamma(\theta))^{(1)} \equiv \big(\gamma(\theta), T_\theta\gamma(R^\gamma_\theta)\big)\,.$$

Then it is easy to realize that $T_\theta\gamma^{(1)}(R^\gamma_\theta)$ is the induced Cartan plane on the first prolongation of the equation at the point $(\gamma(\theta))^{(1)}$ and it is isomorphic to R^γ_θ. The proposition follows by iterating this reasoning. $\boxed{QED}$

11 Corollary. *Higher internal symmetries are vector fields on* U^1E *preserving the distribution* R^γ, *and coincide with the classical internal symmetries.*

Of course, vector fields which are contained in the distribution R^γ are symmetries of our equation, that we call *trivial symmetries*. We are interested in symmetries which are defined up to trivial ones. Such symmetries are of the form

$$\ni_\varphi = \overline{D}^{(2)}_{\boldsymbol{\sigma}}(\varphi^i)\partial_{x^i_{\boldsymbol{\sigma}}}\,, \quad |\boldsymbol{\sigma}| \leq 1\,, \quad 1 \leq i \leq 3$$

where $\overline{}$ denotes the restriction to the equation $\nabla[\gamma] = 0$ and $\varphi = (\varphi^i)$ is a section of $V^{1,0}$ restricted to U^1E.

Such a section has to satisfy the k differential equations

$$\frac{\partial\nabla[\gamma]^k}{\partial x^i_{\boldsymbol{\sigma}}}\overline{D}^{(2)}_{\boldsymbol{\sigma}}\varphi^i = 0, \quad |\boldsymbol{\sigma}| \leq 2, \tag{20}$$

where $\overline{D}^{(2)}_{00} = \overline{D}^{(2)}_0 \circ \overline{D}^{(2)}_0$.

Then we realize that the correspondence which associates with a section φ satisfying (20) the symmetry $\ni_\varphi$ is bijective. We call this φ a *generating section* of an (higher) internal symmetry.

For what concerns classical external symmetries, by virtue of Lie-Bäcklund theorem [4, 27], they are just lifting of vector fields defined on spacetime. Such symmetries coincide with the *projective* vector fields of spacetime, as they send unparametrized solutions into unparametrized solutions.

We would like to stress that any classical external symmetry restricts to an internal one, but an internal symmetry can not be always prolonged to an external one. Anyway in [4, 15] it is proved that, for the equation of motion, classical external symmetries are a subalgebra of classical internal ones.

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