Note di Matematica **23,** n. 2, 2004/2005, 33–73.

PDEs reduction and λ-symmetries

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Received: 30/5/2004; accepted: 30/5/2004.

Abstract. So called λ-symmetries were introduced by Muriel and Romero, and geometrically characterized by Pucci and Saccomandi [8, 12], in the ODE case. We extend them to the PDE framework. In this context the central object is a horizontal one-form μ, and we speak of μ-prolongations of vector fields and μ-symmetries of PDEs. The latter are as good as standard symmetries in providing symmetry reduction of PDEs (or systems thereof) and explicit invariant solutions.

Keywords: Differential equations, Lie symmetries, invariant solutions, $\lambda-$symmetries

MSC 2000 classification: 58J70, 35A30

Introduction

It was recently pointed out by Muriel and Romero [8] that, beside standard symmetries, another class of transformations is equally useful in providing symmetry reduction for scalar ordinary differential equations (ODEs); these were christened $\mathcal{C}^\infty$ *symmetries*, or even λ-*symmetries*, as they depend on a smooth scalar function λ (see also [5, 9] for applications of λ-symmetries). Soon afterwards, Pucci and Saccomandi identified the most general class of transformations sharing the "useful" properties of standard symmetries for what concerns reduction of a scalar ODE [12].

In the present note we extend the concept of λ-symmetries to the case of partial differential equations (PDEs) or systems of PDEs for $u^a = u^a(x^1, \dots, x^p)$, with $a = 1, \dots, q$; see also [2, 4]. In this case the transformations of interest depend on a semibasic matrix-valued one form $\mu = \Lambda_i \mathrm{d}x^i$, the matrix functions Λ_i take value in $\mathcal{G}$, the Lie algebra of the group $GL(q)$, and are such to satisfy the compatibility condition $D_i\Lambda_j - D_j\Lambda_i + [\Lambda_i, \Lambda_j] = 0$ (this amounts to μ being the pullback of the canonical Maurer-Cartan form on $GL(q)$ up to contact forms, or equivalently the pullback of the horizontal Maurer-Cartan form, see [2]). The transformations of this class leaving invariant the solution manifold for an equation Δ will be said to be μ-symmetries, or Λ^1-symmetries, of Δ.

In order to obtain such an extension, we found it convenient to characterize λ-prolongations in $J^{(n)}M$, where (M, π, B) is the space of dependent and independent variables seen as a bundle over the space B of independent variables, in a geometrical way; once this characterization is obtained, it is promptly extended from the ODEs case of $B = \mathbf{R}$ to the PDEs case $B = \mathbf{R}^p$, and then to the case of systems of PDEs.

We will thus be able to obtain a sound definition of μ-prolongations and μ-symmetries of a PDE. We will also show that, in analogy with the ODE case, μ-symmetries are as useful as standard symmetries in what concerns the symmetry reduction, and the determination of invariant solutions, of PDEs. Our approach will suffer from the same limitations as the standard PDE symmetry reduction method.

In this note we deal first with the case of scalar PDEs, i.e. one equation for one dependent variable $u = u(x^1, \dots, x^p)$, in analogy with the theory of λ-symmetries for ODEs; in this case Λ_i reduce to scalar functions λ_i and the compatibility condition reduces to $D_i \lambda_j = D_j \lambda_i$. In a second time, we also deal with the case of several dependent variables.

Acknowledgements

We would like to thank G. Saccomandi and R. Vitolo for their invitation to report on our work in the Lecce conference on Symmetry of Differential Equations. The work of GG was supported by *"Fondazione CARIPLO per la ricerca scientifica"* under the program *"Teoria delle perturbazioni per sistemi con simmetria"* (2000-2003). Both authors are partially supported by *GNFM-INdAM* through the project *"Simmetrie e tecniche di riduzione per equazioni differenziali di interesse fisico-matematico"*.

1 Prolongations

In this section we fix our notation (mainly following the usual one in the field, see e.g. [10]) and recall some basic definition [3, 10, 13, 15].

Let us consider a space $M = B \times U$ with coordinates $x \in B \simeq \mathbf{R}^p$ and $u \in U \simeq \mathbf{R}^q$; when setting a differential equation in this space, we will think of the x as independent variables, and the u as dependent ones. Thus, more precisely, M will be the total space of a (trivial) linear bundle (M, π, B) over the base space B, with fiber $\pi^{-1}(x) = U$.

Given a bundle P, we will denote $\Gamma[P]$ the set of sections of this bundle, and by $\mathcal{X}[P]$ the set of vector fields in P.

1.1 Jet spaces and contact structure

The bundle M can be *prolonged* to the k-th **jet bundle** $(J^{(k)}M, \pi_k, B)$, with $J^{(0)}M \equiv M$; the total space of the jet bundle is also called the jet space, for short. This can be thought as the space $\mathbf{R}^{d(k)}$ ($d(k)$ a suitable integer) of the (x, u) and the x-derivatives of the u up to order k. The jet bundle has a multifibered structure, as each $J^{(k)}M$ is a bundle over $J^{(k-1)}M$, and therefore over each jet space of lower degree.

We can consider the x-derivatives of u as coordinates in $J^{(k)}M$ on the same footing as the (x, u), provided we introduce in the jet space $J^{(k)}M$ a canonical **contact structure**, i.e. the module generated by the set of canonical contact one-forms $\vartheta^a_J := \mathrm{d}u^a_J - u^a_{J,m}\mathrm{d}x^m$.

The contact structure in $J^{(k)}M$ defines a field of $(p+q)$-dimensional linear spaces in $J^{(k)}M \subset \mathrm{T}((J^{(k-1)}M)$, the contact distribution, corresponding to the tangent subspace spanned by vector fields $Y \in \mathcal{X}[J^{(k)}M]$ annihilated by the contact forms, i.e. such that $Y \lrcorner \theta = 0$ for any contact form θ.

The general form of such vector fields is, as well known,

$$Y \;=\; \sum \xi^i D_i^{(k)} \;+\; V.$$

Here D_i is the total derivative [3, 10, 13, 15] with respect to x^i, $D_i := (\partial/\partial x^i) + u^a_i(\partial/\partial u^a) + u^a_{ij}(\partial/\partial u^a_j) + \cdots$, and $D_i^{(k)}$ its truncation to the k-th jet space; and V is a generic vector field in $\mathcal{X}[J^{(k)}M]$, vertical for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$ (the latter will not appear if we work with infinite-order prolongations; it will however disappear when we deal with a given differential equations and symmetry vector fields for it). The operator $D_i^{(k)}$ reads

$$D_i^{(k)} \;:= (\partial/\partial x^i) \,+ \sum_{a=1}^{q} \sum_{|J|=1}^{k-1} \; u^a_{J,i}(\partial/\partial u^a_J).$$

In the following we will write, for ease of notation, simply D_i instead of $D_i^{(k)}$.

1.2 Functions, sections, prolongations

The function $u = f(x)$ corresponds to a section $\gamma_f \in \Gamma[M]$, i.e.

$$\gamma_f \;=\; \{(x, u) \;:\; u = f(x)\,\}.$$

Knowledge of the function $u = f(x)$ implies, of course, knowledge of all of its x-derivatives. Thus, the corresponding section $\gamma_f \in \Gamma[M]$ is uniquely prolonged to a section $\gamma_f^{(k)} \in \Gamma[J^{(k)}M]$: e.g.,

$$\gamma_f^{(2)} \;=\; \{(x, u, p, q) \;:\; u^a = f^a(x)\,,\, p^a_j = \partial_j f^a(x)\,,\, q^a_{ij} = \partial_{ij} f^a(x)\}.$$

This can be described in geometrical terms by means of the contact distribution introduced above: $\gamma_f^{(k)}$ is the unique lift of the curve γ_f in M to a curve in $J^{(k)}M$ which (i) projects down to γ_f in M, and (ii) is everywhere tangent to the field of contact linear spaces.

1.3 Vector fields and their prolongations

Consider a vector field $X \in \mathcal{X}[M]$; this can be written, in the (x,u) coordinates, as

$$X \ = \ \xi^i(x,u)\frac{\partial}{\partial x^i} \ + \ \varphi^a(x,u)\frac{\partial}{\partial u^a}.$$

This can be uniquely prolonged to a vector field $X^{(k)}$ in $J^{(k)}M$ by requiring it *preserves the contact structure* (the precise meaning of this will be defined in a moment). The *prolongation formula* [3, 10, 13, 15] is indeed expressing this condition.

The analytical meaning of the prolongation operation is, as well known, the following: if we consider the transformation of the x and u generated by the vector field, the transformation induced on the x-derivatives of the u will be the one generated by $X^{(k)}$ [3, 10, 13, 15].

When we act with a vector field X as above on M, we induce a local group of transformation acting in $\Gamma[M]$; at the infinitesimal level, under the map $\exp[\varepsilon X]$ the section $\gamma_f = \{(x,u) : u = f(x)\}$ is mapped into $\gamma_{\widehat{f}}$ with

$$\widehat{f}^a(x) \ = \ f^a(x) \ + \ \varepsilon[\varphi^a(x,u) - \xi^i(x,u)\partial_i f^a(x)]_{u=f(x)} \ + \ o(\varepsilon).$$

(We also say for ease of writing that X acts on the space of functions $u = f(x)$ by $X : f \to \widehat{f}$). The transformations undergone by the x-derivatives of the u can be read off by this formula, and define – for generic $u = f(x)$ – the prolonged vector field.

We write a vector field in $J^{(k)}M$ as

$$Y \ = \ X \ + \ \sum_{|J|=1}^{k} \Phi^a_J \frac{\partial}{\partial u^a_J}$$

where X is as above, $J = j_1, \dots, j_p$ is a multiindex, and the sum is over all multiindices of modulus $|J| = j_1 + \cdots + j_p$ up to the order of the jet space. We also write D_J for the total derivative $D^{j_1}_{x^1} \cdots D^{j_p}_{x^p}$, and u^a_J for $D_J u^a$; moreover $u_{J,i}$ will denote $D_i u_J$. Then Y is the prolongation of X if and only if the coefficients Φ^a_J satisfy the **prolongation formula**

$$\Phi^a_J \ = \ D_J\varphi^a \ - \ D_J(\xi^i u^a_i) \ + \ \xi^i D_J u^a_i. \tag{1}$$

It is well known, and of use for our discussion later on, that the prolongation formula is also easily recast in recursive form. We denote by $\widehat{J} = J + e_k$ the multiindex with entries $\widehat{j}_i = j_i + \delta_{ik}$, and for short $u_{J,k} := u_{J+e_k}$, $\Phi^a_{J,k} := \Phi^a_{J+e_k}$. Then (1) is equivalent to the recursive formula

$$\Phi^a_{J,k} \ = \ D_k \Phi^a_J \ - \ u^a_{J,m} D_k \xi^m \tag{2}$$

with $\Phi^a_0 = \varphi^a$ (see e.g. sect. 2.3 of [10]).

2 Prolongations and contact structure

In this section we discuss the geometrical aspects of the prolongation operation in terms of *contact structures*. This is by no means original (see e.g. the discussion in [13]), but we go into some detail as we need these for our subsequent generalization.

We equip $J^{(k)}M$ with a standard *contact structure* $\mathcal{E}$, i.e. the module over $C^\infty(W)$ generated by the set of standard contact one-forms

$$\vartheta^a_J \ := \ \mathrm{d}u^a_J - u^a_{J,m} \mathrm{d}x^m$$

with $a = 1, \ldots, q$, $|J| = 0, \ldots, k-1$. For any function $f : J^{(k-1)}M \to \mathbf{R}$ we can write

$$\mathrm{d}f \ = \ (D_i f)\mathrm{d}x^i \ + \ \widehat{\vartheta}[f] \tag{3}$$

where $\widehat{\vartheta}[f] \in \mathcal{E}$ is some contact form whose explicit expression (easy to compute) is irrelevant here.

1 Definition. Let Y be a vector field on $J^{(k)}M$. We say that Y *preserves the contact structure* if $\mathcal{L}_Y : \mathcal{E} \to \mathcal{E}$.

2 Proposition. *The vector field $Y \in \mathcal{X}[J^{(k)}M]$, projecting to a vector field $X \in \mathcal{X}[M]$ on M, is the prolongation of a vector field $X \in \mathcal{X}[M]$ if and only if it preserves the contact structure in $J^{(k)}M$.*

Proof. This is a classical result, see e.g. [13]. We give however a proof, both to fix notation and as the proof of theorems 1 and 2 below will be quite similar. We write a general vector field on $J^{(k)}M$ as $Y = \xi^i \partial_{x^i} + \Phi^a_J \partial_{u^a_J}$ (with $|J| = 0, \ldots, k$ and $\Phi^a_0 \equiv \varphi^a$). By standard computations,

$$\mathcal{L}_Y(\vartheta^a_J) \ = \ -\Phi^a_{J,m}\mathrm{d}x^m + \mathrm{d}\Phi^a_J - u^a_{J,m}\mathrm{d}\xi^m .$$

Hence, using (3), we have $\mathcal{L}_Y(\vartheta^a_J) = [-\Phi^a_{J,i} + D_i\Phi^a_J - u^a_{J,m}(D_i\xi^m)]\mathrm{d}x^i + \Theta$ with $\Theta \in \mathcal{E}$; we conclude that Y preserves the contact structure if and only if the coefficient of all the $\mathrm{d}x^i$ in the above vanish, i.e. if and only if $\Phi^a_{J,i} = D_i\Phi^a_J - u^a_{J,m}(D_i\xi^m)$. This, however, is just (2) above. $\boxed{QED}$

3 Lemma. *The vector field Y preserves the contact structure $\mathcal{E}$ if and only if, for any $\vartheta \in \mathcal{E}$ and any $i = 1, \dots p$, $([D_i, Y]) \lrcorner \vartheta \ = \ 0$.*

Proof. We write ϑ^a_J and Y as above, and note that $D_i = \partial_i + u^a_{J,i}(\partial/\partial u^a_J)$, where of course $\partial_i = \partial/\partial x^i$. With this notation and standard computations, we get

$$[D_i, Y] \ = \ (D_i \xi^m) \partial_m \ + \ (D_i \Phi^a_J - \Phi^a_{J,i})(\partial/\partial u^a_J) \ ; \tag{4}$$

hence we get

$$[D_i, Y] \lrcorner \vartheta^a_J \ = \ -\Phi^a_{J,i} \ + \ \left(D_i \Phi^a_J - u^a_{J,m}(D_i \xi^m) \right) \ ,$$

which vanishes if and only if the Φ^a_J satisfy (2). $\boxed{QED}$

4 Corollary. *The vector field Y preserves the contact structure $\mathcal{E}$ if and only if $[D_i, Y] = h^m_i D_m + V$ for some $h^m_i \in \Lambda^0(J^{(k)}M)$ and V a vertical vector field for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$.*

Proof. The vector fields D_m span the set of non-vertical vector fields (for the fibration $\pi_{k,k-1}$) in the annihilator of the contact forms. Alternatively, this follows at once from (3), with $h^m_i = D_i \xi^m$. $\boxed{QED}$

3 Scalar ODEs: λ-prolongations and λ-symmetries

In this section we will restrict to the case of scalar ODEs, i.e. to the case where the bundle (M, π, B) has $B = \mathbf{R}$ as base space, and a one-dimensional fiber, $\pi^{-1}(x) = \mathbf{R}$. We will characterize in geometrical terms the λ-prolongations introduced by Muriel and Romero [8] (see also [9] and [5]), and further studied by Pucci and Saccomandi [12]. These were defined only in the scalar case (a single equation for a single dependent variable u), and we also restrict to this setting. In section 4 we extend this to scalar PDEs, deferring treatment of systems to sections 6 and 7.

We simply write u_n for $D^n_x u$, and similarly for Ψ_n. The standard contact forms in $J^{(k)}M$ will be $\vartheta_n = \mathrm{d}u_n - u_{n+1}\mathrm{d}x$, with $n = 0, \dots, k-1$.

3.1 Basic definitions

Let us start by recalling the definition of λ-prolongations and λ-symmetries as given by Muriel and Romero, using an obvious notation for x-derivatives of the u.

5 Definition. Let $X = \xi(\partial/\partial x) + \varphi(\partial/\partial u)$ be a vector field on M, and $Y = X + \sum_{n=1}^k \Psi_n(\partial/\partial u_n)$ a vector field on $J^{(k)}M$. Let $\lambda : J^{(1)}M \to \mathbf{R}$ be a smooth function. We say that Y is the *λ-prolongation* of X if its coefficients satisfy the **λ-prolongation formula**

$$\Psi_{n+1} \;=\; [(D_x+\lambda)\Psi_n] \,-\, u_{n+1}[(D_x+\lambda)\xi] \tag{5}$$

for all $n = 0, \ldots, k-1$.

6 Definition. Let Δ be a k-th order ODE for $u = u(x)$, $u \in U = \mathbf{R}$, and let $(M = U \times B, \pi, B)$ be the corresponding variables bundle. Let the vector field Y in $J^{(k)}M$ be the λ-prolongation of the Lie-point vector field X in M. Then we say that X is a *λ-symmetry* of Δ if and only if Y is tangent to the solution manifold S_Δ, i.e. iff there is a smooth function Φ on $J^{(k)}M$ such that $Y(\Delta) = \Phi\Delta$.

7 Remark. We stress that in this note we take $\lambda : J^{(1)}M \to \mathbf{R}$, which guarantees that the λ prolongation of a Lie-point vector field in M is a proper vector field in each $J^{(n)}M$. One could also consider $\lambda : J^{(r)}M \to \mathbf{R}$, obtaining obvious generalizations of the results given here. In this case the λ-prolongations of X would be generalized vector fields in each $J^{(n)}M$ with $n > 0$ even if X is a Lie-point vector field. The same applies to the μ-prolongations to be considered in later sections.

We will not discuss here the relevance of λ-symmetries, referring to [8, 12]; we just recall that they are as useful as standard ones in that one can perform symmetry reduction to the same extent as for standard symmetries [8].

The basic property of λ-prolongations behind this feature was clearly pointed out by Pucci and Saccomandi [12], and can be expressed in terms of the characteristics of the vector fields Y which are λ-prolongations of X.

3.2 A geometrical characterization of λ-prolongations

Let us now equip $J^{(1)}M$, seen as the bundle $(J^{(1)}M, \pi, B)$ ($B = \mathbf{R}$ in the ODE case), with a distinguished smooth real function $\lambda(x, u, u_x)$. We note for later discussion that to this is associated a semi-basic one-form $\mu \in \Lambda^1(J^{(1)}M)$, i.e. the one-form $\mu = \lambda(x, u, u_x)\mathrm{d}x$

8 Definition. Let Y be a vector field on the contact manifold $(J^{(k)}M, \mathcal{E})$, and $\lambda \in \Lambda^0(J^{(1)}M)$ a smooth function on M. We say that Y *λ-preserves the contact structure* if, for any contact one-form $\theta \in \mathcal{E}$,

$$\mathcal{L}_Y(\theta) \;+\; (Y \,\lrcorner\, \theta)\,\lambda\mathrm{d}x \;=\; \widehat{\theta} \tag{6}$$

for some contact one-form $\widehat{\theta} \in \mathcal{E}$.

9 Theorem. *Let (M, π, B) be a bundle over the real line $B = \mathbf{R}$ with fiber $\pi^{-1}(x) = \mathbf{R}$, and let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$. Let Y be a* *vector field on the jet space $J^{(k)}M$, which projects to a vector field X on M. Then Y is the λ-prolongation of X if and only if it λ-preserves the contact structure.*

PROOF. We write a general vector field on $J^{(k)}M$ as

$$Y = \xi\partial_x + \sum_{m=0}^{k} \Psi_m(\partial/\partial u_m);$$

as the contact forms are $\vartheta_n = \mathrm{d}u_n - u_{n+1}\mathrm{d}x$ $(n = 0, \dots, k-1)$, we have by explicit computation

$$\mathcal{L}_Y(\vartheta_n) + (Y \lrcorner \vartheta_n)\lambda\mathrm{d}x = [-\Psi_{n+1} + D_x\Psi_n - u_{n+1}D_x\xi + \lambda(\Psi_n - u_{n+1}\xi)]\,\mathrm{d}x + \widehat{\theta}$$

with $\widehat{\theta}$ a contact form. Thus (6) is satisfied if and only if the Ψ_n satisfy the λ-prolongation formula (5). $\boxed{QED}$

We can also provide an alternative characterization of λ-prolonged vector fields in terms of their commutation properties with the total derivative operator D_x, similarly to what we did for standard prolongations in lemma 1.

Let Y be a vector field on the jet space $J^{(k)}M$, with (M, π, B) a vector bundle over the real line $B = \mathbf{R}$, and let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$. Then Y is the λ-prolongation of a vector field X on M, if and only if for any $\vartheta \in \mathcal{E}$

$$[D_x, Y] \lrcorner \vartheta \;=\; \lambda(Y \lrcorner \vartheta). \tag{7}$$

PROOF. Looking back at the proof of lemma 1, $[D_x, Y] \lrcorner \vartheta$ is given by (4) specialized to the case with $p = 1$: with the obvious notation $u_n := D_x^n u$ (and similarly for Ψ_n) we have $[D_x, Y] = -\Psi_{n+1} + (D_x\Psi_n - u_{n+1}D_x\xi)$; on the other hand, it is easy to check that $Y \lrcorner \vartheta_n \;=\; \Psi_n - u_{n+1}\xi$. Thus eq. (7) is equivalent to $\Psi_{n+1} = [(D_x + \lambda)\Psi_n] - u_{n+1}[(D_x + \lambda)\xi]$, i.e. to the λ-prolongation formula (5). $\boxed{QED}$

10 Corollary. *In the hypotheses of lemma 2, Y is the λ-prolongation of a vector field X on M, if and only if $[D_x, Y] = \lambda Y + hD_x + V$ with λ, h scalar functions on $J^{(1)}M$ and V a vertical vector field for the fibration $\pi_{k,k-1} : J^{(k)}M \to J^{(k-1)}M$.*

11 Remark. Theorem 1 shows that our geometrical formulation, i.e. definition 4, is equivalent to the standard (analytical) one, i.e. definition 2. The advantage of our formulation is twofold: on the one hand, we have a better geometrical understanding of λ-prolongations (λ-symmetries), further clarifying in which sense they generalize standard ones. On the other hand, our geometrical formulation is readily extended from the ODEs to the PDEs case. As we discuss later on, we can moreover generalize the standard symmetry reduction method for PDEs to an analogous λ-symmetry reduction. We will also show later on (sections 6 and 7) that this definition extends, suitably generalized, to the vector case; the same holds for the reduction procedure.

4 Scalar PDEs: μ-prolongations and μ-symmetries

As mentioned in the previous section, our approach to λ-prolongations and λ-symmetries is readily generalized to the PDEs case, i.e. to the case where $B = \mathbf{R}^q$ is not restricted to be one-dimensional.

The role of the scalar function λ will now be played by an array of p smooth functions $\lambda_i : J^{(1)}M \to \mathbf{R}$ (remark 1 holds also in this context), which will be the components of a semibasic form $\mu \in \Lambda^1(J^{(1)}M)$. The only additional ingredient required in the scalar PDE case is a compatibility condition between the semibasic form μ and the contact structure – this is eq. (10) below – automatically satisfied in the ODE case.

Actually, our formulation of λ-prolongations in the ODE case was such that the results, and even their proofs, are the same also in the PDE case – except of course for the appearance of new indices related to the independent variables.

In view of our geometric approach, however, it is convenient to focus on the form μ rather than on the q-ple of smooth functions λ_i. We will thus call the analogue of λ-prolongations and λ-symmetries in the PDE frame, μ-prolongations and μ-symmetries.

4.1 μ-prolongations

We equip $(J^{(1)}M, \pi, B)$ with a distinguished semi-basic one-form μ,

$$\mu \;=\; \lambda_i \, \mathrm{d}x^i. \tag{8}$$

We require that μ is *compatible* with the contact structure defined in $J^{(k)}M$, for $k \geq 2$, in the sense that

$$\mathrm{d}\mu \in \mathcal{J}(\mathcal{E}) \ , \tag{9}$$

where $\mathcal{J}(\mathcal{E})$ is the Cartan ideal generated by $\mathcal{E}$ (we recall that a two-form α is in $\mathcal{J}(\mathcal{E})$ if and only if $\alpha = \rho^J \wedge \vartheta_J$ for some one-forms ρ^J).

It should be noted that this condition does not appear when we deal with first order equations, i.e. with first order μ-prolongations.

We note also that for $p = 1$ eq. (9) is automatically satisfied: indeed, $\mathrm{d}\mu = (\partial\lambda/\partial u)\mathrm{d}u \wedge \mathrm{d}x + (\partial\lambda/\partial u_x)\mathrm{d}u_x \wedge \mathrm{d}x = (\partial\lambda/\partial u)\vartheta_0 \wedge \mathrm{d}x + (\partial\lambda/\partial u_x)\vartheta_1 \wedge \mathrm{d}x$.

12 Lemma. *Condition (9) is equivalent to*

$$D_i\lambda_j - D_j\lambda_i \;=\; 0. \tag{10}$$

Proof. As λ_i is a function on $J^{(1)}M$, we have $\mathrm{d}\mu = (\partial\lambda_j/\partial x^i)\mathrm{d}x^i \wedge \mathrm{d}x^j + (\partial\lambda_j/\partial u)\mathrm{d}u \wedge \mathrm{d}x^j + (\partial\lambda_j/\partial u_i)\mathrm{d}u_i \wedge \mathrm{d}x^j$, i.e.

$$\begin{aligned} \mathrm{d}\mu = \ & [(\partial\lambda_j/\partial x^i) + u_i(\partial\lambda_j/\partial u) + u_{ik}(\partial\lambda_j/\partial u_k)]\mathrm{d}x^i \wedge \mathrm{d}x^j + \\ & + (\partial\lambda_j/\partial u)\vartheta_0 \wedge \mathrm{d}x^j \ + \ (\partial\lambda_j/\partial u_i)\vartheta_i \wedge \mathrm{d}x^j . \end{aligned}$$

The two latter terms are of course in $\mathcal{J}(\mathcal{E})$, while no form $\mathrm{d}x^i \wedge \mathrm{d}x^j$ belongs to $\mathcal{J}(\mathcal{E})$; thus, (9) is satisfied if and only if the coefficients of all these terms vanish. This condition is precisely (10). (Note this extends to the case where μ is semibasic for $(J^{(n)}, \pi_n, B)$, see remark 1.) $\boxed{QED}$

13 Definition. Let Y be a vector field on the contact manifold $(J^{(k)}M, \mathcal{E})$, and μ a semibasic form on M compatible with $\mathcal{E}$. We say that Y *μ-preserves the contact structure* if, for any $\theta \in \mathcal{E}$, there is a form $\widehat{\theta} \in \mathcal{E}$ such that

$$\mathcal{L}_Y(\theta) \ + \ (Y \lrcorner \theta)\mu \ = \ \widehat{\theta}. \tag{11}$$

14 Definition. A vector field Y in $J^{(k)}M$ which projects to X in M and which μ-preserves the contact structure is said to be the *μ-prolongation* of order k, or the k-th μ-prolongation, of X.

15 Theorem. *Let Y be a vector field on the jet space $J^{(k)}M$, with (M, π, B) a vector bundle over $B = \mathbf{R}^p$, written in coordinates as*

$$Y \ = \ X + \sum_{|J|=1}^{k} \Psi_J \frac{\partial}{\partial u_J} \ ,$$

with $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ a vector field on M. Let $\mathcal{E}$ be the standard contact structure in $J^{(k)}M$, and $\mu = \lambda_i dx^i$ a semibasic one-form on $(J^{(1)}M, \pi, B)$, compatible with $\mathcal{E}$. Then Y is the μ-prolongation of X if and only if its coefficients (with $\Psi_0 = \varphi$) satisfy the **scalar μ-prolongation formula**

$$\Psi_{J,i} \ = \ (D_i + \lambda_i)\Psi_J \ - \ u_{J,m}\,(D_i + \lambda_i)\xi^m. \tag{12}$$

Proof. The standard contact forms in $J^{(k)}M$ are $\vartheta_J = \mathrm{d}u_J - u_{J,i}\mathrm{d}x^i$, with $|J| = 0, \ldots, k-1$. Thus, as already computed in the proof of proposition (1), $\mathcal{L}_Y(\vartheta_J) = (-\Psi_{J,i} + D_i\Psi_J - u_{J,m}D_i\xi^m)\mathrm{d}x^i + \Theta$ with Θ a contact form. On the other hand, it is easy to compute that $Y \lrcorner \vartheta_J = \Psi_J - u_{J,m}\xi^m$. Therefore,

$$\begin{aligned} & \mathcal{L}_Y(\vartheta_J) + (Y \lrcorner \vartheta_J)\mu \ = \\ & \quad = \ [(-\Psi_{J,i} + D_i\Psi_J - u_{J,m}D_i\xi^m) + \lambda_i(\Psi_J - u_{J,m}\xi^m)]\,\mathrm{d}x^i \ + \ \Theta. \end{aligned}$$

This is a contact form if and only if the coefficients of all the $\mathrm{d}x^i$ vanish, i.e. if and only if (12) is satisfied. $\boxed{QED}$

As for standard and λ-prolongations, μ-prolongations have a specific behaviour for what concerns their commutation with the total derivatives D_i.

16 Lemma. *If Y is the μ-prolongation of a Lie-point vector field X, with $\mu = \lambda_i dx^i$, then for any contact form ϑ,*

$$[D_i, Y] \,\lrcorner\, \vartheta \ = \ \lambda_i \, (Y \,\lrcorner\, \vartheta). \tag{13}$$

PROOF. In the proof of lemma 1 we have computed $[D_i, Y] \,\lrcorner\, \vartheta_J = -\Psi_{J,i} + D_i \Psi_J - u_{J,m} D_i \xi^m$; needless to say, $Y \,\lrcorner\, \vartheta_J = \Psi_J - u_{J,m}\xi^m$ and thus (13) is equivalent to the μ-prolongation formula (12). $\boxed{QED}$

17 Corollary. *In the hypotheses of lemma 4, Y is the μ-prolongation of a vector field X on M, if and only if $[D_i, Y] \ = \ \lambda_i Y + h_i^m D_m + V$ with λ_i, h_i^m scalar functions on $J^{(1)}M$ and V a vertical vector field for the fibration of $J^{(k)}M$ over $J^{(k-1)}M$.*

4.2 μ-prolongations versus ordinary ones

It is quite remarkable that a simple relation exists between the μ-prolongation of a vector field and its ordinary prolongation. In this section, in order to discuss such relation, we write $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ for the vector field in M, and denote its ordinary prolongations as $X^{(k)} = X + \Phi_J(\partial/\partial u_J)$, while its μ-prolongations are denoted as $Y = X + \Psi_J(\partial/\partial u_J)$. The form μ is written, as usual, $\mu = \lambda_i \mathrm{d}x^i$, and of course $\Psi_J = \Phi_J$ when all the λ_i (or at least all those for i such that $j_i \neq 0$) vanish.

The vector field X can be cast in evolutionary form [3, 10, 13, 15] as

$$X \ = \ Q \frac{\partial}{\partial u} \quad , \quad Q \ := \ \varphi - u_i \xi^i .$$

The equations $D_J Q = 0$, with $|J| = 0, \ldots, k-1$ identify the X-invariant space $\mathcal{I}_X \subset J^{(k)}M$. We denote by $\mathcal{F}$ the module over $\mathcal{C}^\infty(J^{(k)}M)$ generated by the $D_J Q$, i.e. the set of functions F which can be written as $F = c^J D_J Q$ for some smooth functions $c^J : J^{(k)}M \to \mathbf{R}$, and by $\mathcal{F}^{(m)} \subseteq \mathcal{F} \equiv \mathcal{F}^{(k)}$ those which depend only on variables $(x, u^{(m)})$, $m \leq k$. Needless to say, $D_i : \mathcal{F}^{(m-1)} \to \mathcal{F}^{(m)}$.

18 Theorem. *Let X, Y, μ be as above. Write the coefficients Ψ_J as*

$$\Psi_J \ = \ \Phi_J + F_J .$$

Then the functions F_J satisfy the recursion relation (with $F_0 = 0$)

$$F_{J,i} \ = \ (D_i + \lambda_i) F_J \ + \ \lambda_i D_J Q. \tag{14}$$

Proof. In order to show that the statement of the theorem holds at all orders, we proceed recursively: we suppose (14) holds for all $|J| < h$, and wish to prove that it holds also for $|J| = h$. Any $\widehat{J}$ of order h can be written as $J + e_i$ for some i and some J of order $h-1$; formula (14) holds for Ψ_J. Thus, by the μ-prolongation formula,

$$\begin{aligned} \Psi_{J,i} &= (D_i + \lambda_i)\Phi_J - u_{J,m}(D_i + \lambda_i)\xi^m + (D_i + \lambda_i)F_J \\ &= [D_i\Phi_J - u_{J,m}D_i\xi^m] + \lambda_i[\Phi_J - u_{J,m}\xi^m] + (D_i + \lambda_i)F_J. \end{aligned}$$

The first term is just $\Phi_{J,i}$, and the last is already in the form appearing in (14); so we have to look only at the second one.

Take an s such that $j_s \neq 0$ in the multiindex J, and write $K = J - e_s$. Then, using the standard prolongation formula,

$$\Phi_J - u_{J,m}\xi^m = [D_s\Phi_K - u_{K,m}D_s\xi^m] - (D_s u_{K,m})\xi^m = D_s\left(\Phi_K - u_{K,m}\xi^m\right).$$

We can then repeat the procedure on any index q such that $K = J - e_s$ has a nonzero q entry, and so on. In the end, recalling that $\Psi_0 = \varphi$, we have

$$\Phi_J - u_{J,m}\xi^m \ = \ D_J\left(\varphi - u_m\xi^m\right) \ = \ D_J Q \ ;$$

this also follows from the formula for prolongation of the evolutionary representative of X.

Going back to our computation, we have thus shown that

$$\Psi_{J,i} \ = \ \Phi_{J,i} \ + \ \lambda_i D_J Q \ + \ (D_i + \lambda_i)F_J.$$

This shows that if (14) is satisfied at order $h-1$, it is also satisfied at order h.

It is easy to check that (14) holds at order one, i.e. for $|J| = 0$: indeed, by the μ-prolongation formula (12) and the ordinary prolongation formula (1),

$$\begin{aligned} \Psi_i &= (D_i + \lambda_i)\varphi - u_m(D_i + \lambda_i)\xi^m \\ &= (D_i\varphi - u_m D_i\xi^m) \ + \ \lambda_i\left(\varphi - u_m\xi^m\right) \\ &= \Phi_i + \lambda_i Q. \end{aligned}$$

We conclude that (14) holds at all orders. $\boxed{QED}$

This theorem provides an economic way of computing μ-prolongations of X if we already know its ordinary prolongations. Theorem 3 also has a rather obvious consequence, which will be relevant in the following.

19 Lemma. *Let X be a vector field on M, $\mathcal{E}$ the standard contact structure on $J^{(k)}M$, and μ any semibasic form on M compatible with $\mathcal{E}$. Then:* (i) *the μ-prolongation Y of X coincides with the ordinary prolongation $X^{(k)}$ on the invariant space $\mathcal{I}_X$;* (ii) *the space $\mathcal{I}_X \subset J^{(k)}M$ is invariant under the μ-prolongations of X, for any semibasic form μ compatible with $\mathcal{E}$.*

PROOF. By definitions, any function $F \in \mathcal{F}$ vanishes identically on $\mathcal{I}_X$. Thus (14) guarantees that $\Psi_J = \Phi_J$ on $\mathcal{I}_X$, i.e. proves point *(i)*.

As for *(ii)*, this is a known property of standard prolongations, easily checked by using the evolutionary representative of X, $X_Q := Q(\partial/\partial u)$. Its prolongation is $X_Q^{(k)} = (D_J Q)(\partial/\partial u_J)$, where the sum is over all multiindices with $|J| \le k$, and $X^{(k)} = X_Q^{(k)} + \xi^i D_i$. Thus, $X^{(k)}$ reduces to $W = \xi^i D_i$ on $\mathcal{I}_X$; and W is obviously tangent to $\mathcal{I}_X$. $\boxed{QED}$

4.3 μ-symmetries

One can define μ-symmetries of a partial differential equation as Lie-point vector fields whose μ-prolongation is a symmetry of the equation.

20 Definition. Let X be a vector field on M, let $\mu = \lambda_i \mathrm{d}x^i$ satisfy condition (10), and let $Y \in \mathcal{X}[J^{(k)}M]$ be the μ-prolongation of order k of X. Let Δ be a differential equation of order k in M, $\Delta := F(x, u^{(k)}) = 0$, and $\mathcal{S} \subset J^{(k)}M$ be the solution manifold for Δ. If $Y : \mathcal{S} \to \mathrm{T}\mathcal{S}$, we say that X is a *μ-symmetry* for Δ. If Y leaves invariant each level manifold for F, we say that X is a *strong μ-symmetry* for Δ.

The relevant point is that **μ-symmetries can be used to obtain group-invariant solutions**, i.e. one can introduce μ-symmetry reductions of PDEs and obtain invariant solutions to the original PDE from these, by the same method as for standard symmetries.

Note that in this way we parallel again the ODE case, where it was proven by Muriel and Romero and by Pucci and Saccomandi that λ-symmetries are as good as standard ones for reduction of the equation.

5 The μ-symmetry reduction method for scalar PDEs

As well known, symmetry reduction for PDEs is conceptually different from symmetry reduction for ODEs: while in the latter case it yields a reduced equation whose solutions provide, together with an integration, the most general solution to the original ODE, in the PDE case the reduced equation provides only the symmetry-invariant solutions to the original PDE.

5.1 The PDE reduction method

In this subsection we briefly recall (using the notation introduced so far) symmetry reduction for scalar PDEs in the case of standard symmetries; this is discussed in detail in a number of textbooks and research papers, see e.g.

[3, 10, 13, 15]. We will just discuss reduction under a single vector field, rather than a general (i.e. higher dimensional) Lie algebra.

Consider a PDE Δ of order k, which we may think in the form $F(x, u^{(k)}) = 0$ with $F : J^{(k)}M \to \mathbf{R}$ a smooth scalar function. Let the Lie-point vector field X in M, with prolongation $X^{(k)}$ in $J^{(k)}M$, be a (standard) symmetry for Δ. Then we proceed as follows, following Olver. (For more details, see e.g. the discussion in chapter 3 of [10]).

First of all we pass to symmetry-adapted coordinates in M. In practice, we have to determine a set of p independent invariants for X in M, which we will denote as $(y^1, \dots, y^{p-1}, v)$: these will be our X-invariant coordinates, and essentially identify the G-orbits, while the remaining coordinate σ will be acted upon by G.

In other words, G-orbits will correspond to fixed value of (y, v) coordinates and to σ taking values in a certain subset of the real line (thus (y, v) are coordinates on the *orbit space* $\Omega = M/G$, see [10])

The invariants will be given by some functions $y^i = \eta^i(x, u)$ $(i = 1, \dots, p-1)$ and $v = \zeta(x, u)$ of $x^1, \dots, x^p$ and u. If X acts transversally, we can invert these for x and u as functions of $(y, v; \sigma)$, i.e. write $x^i = \chi^i(y, v; \sigma)$ $(i = 1, \dots, p)$ and $u = \beta(y, v; \sigma)$.

If now we decide to see the $(y; \sigma)$ as independent variables and the v as the dependent one, we can use the chain rule to express x-derivatives of u as σ and y-derivatives of v. Using these, we can finally write Δ in terms of the $(y, v; \sigma)$ coordinates and derivatives of v in the y and σ; this will turn out to be an equation which, when subject to the side condition $\partial v/\partial\sigma = 0$, is independent of σ.

The condition $\partial v/\partial\sigma = 0$ expresses the fact that the solutions are required to be invariant under X, i.e. the equation obtained in this way represents the restriction of Δ to the space of G-invariant functions, and therefore it is sometimes also denoted as Δ/G.

Suppose we are able to determine some solution $v = \Phi(y)$ to the reduced equation; we can write this in terms of the (x, u) coordinates as $\zeta(x, u) = \Phi[\eta(x, u)]$, which yields implicitly $u = f(x)$: this is the corresponding X-invariant solution to the original equation Δ in the original coordinates.

21 Remark. The symmetry reduction method for PDEs can also be seen in a slightly different way: if we look for X-invariant solution $u = f(x)$ to Δ, we determine the characteristic $Q = \varphi - u_i\xi^i$ of the vector field X, and supplement Δ with the equations $E_J := D_J Q = 0$ with $|J| = 0, \dots, k-1$. The equation E_0 requires that the evolutionary representative $X_Q = Q(\partial/\partial u)$ vanish on γ_f, i.e. that u is X-invariant, and all the equations with $|J| > 0$ are just differential consequences of this. The X-invariant solutions to Δ are in one to one correspondence with the solutions to the system $\Delta_{(X)} := \{\Delta; E_J\}$. See e.g. [15] for details, and for how this approach is used in a more general context.

5.2 On the justification of the method

The method described above is rigorously justified in chapter 3 of [10], to which we refer for details. In this subsection we just recall what is the key step in the proof, as we will have to prove a similar property also holds for μ-prolongations in order to justify the extension of this method to μ-symmetries.

As discussed in sect.1, a function $u = f(x)$ is invariant under the action of the vector field X in M if and only if $\widehat{Q}(x) := Q[x, f(x)] = 0$ (with Q the characteristic of X).

We consider the equation $E_0 := Q = 0$ and all of its differential consequences $E_J := D_J Q = 0$ for $|J| < k$; this identifies the invariant manifold $\mathcal{I}_X \subset J^{(k)}M$. Passing to the evolutionary representative $X_Q = Q(\partial/\partial u)$ of X, it is obvious that X_Q and its prolongations vanish on $\mathcal{I}_X$.

The X-invariant solutions to Δ will be the solutions to the system $\Delta_{(X)}$ made of Δ and of the invariance condition:

$$\begin{cases} F(x, u^{(k)}) = 0, & \\ D_J Q = 0 & (|J| = 0, \ldots, k-1). \end{cases} \tag{15}$$

We denote the solution manifold to this system as $\mathcal{S}_X \subset \mathcal{I}_X \subset J^{(k)}M$. The invariance of $\mathcal{S}_X$, as discussed by Olver [10], guarantees that the method recalled above is justified.

Recall now that the prolongations of X and X_Q satisfy

$$X^{(k)} \ = \ X_Q^{(k)} \ + \ \xi^i D_i^{(k)}. \tag{16}$$

22 Lemma. *The (standard) prolongation $X^{(k)}$ of X reduces to $\xi^i D_i$ on $\mathcal{I}_X$, and is tangent to $\mathcal{S}_X$.*

PROOF. The field $X_Q^{(k)}$ vanishes on $\mathcal{I}_X$ because of the equations E_J, and the D_i are symmetries of any system, as the differential consequences of any equation of the system are satisfied by solutions to the system. By (16), this proves the claim. $\boxed{QED}$

5.3 Reduction of PDEs under μ-symmetries

In the case of μ-symmetries of PDEs, we can proceed exactly in the same way as for standard symmetries in order to determine G-invariant solutions.

Note that the step consisting in the introduction of symmetry-adapted coordinates is exactly the same; the difference lies of course in the step connected to the prolongation structure.

We describe here how the standard symmetry reduction method is formulated to deal with μ-symmetries. We suppose that X is a μ-symmetry of Δ, acting transversally for the fibration (M, π, B), and denote the μ-prolongation of X as $Y \in \mathcal{X}[J^{(k)}M]$.

First of all we pass to symmetry-adapted coordinates $(y, v; \sigma)$ in M, as in the standard case. We retain the notation introduced in subsection 1. We further proceed as there, i.e. use the chain rule to express x-derivatives of the u as σ and y-derivatives of the v. Using these, we can finally write Δ in terms of the $(y, v; \sigma)$ coordinates and their derivatives.

Again, looking for X-invariant solutions means supplementing the equation with the side condition $\partial v/\partial\sigma = 0$, or with the conditions $D_J Q = 0$, see eq. (15) above, in the original coordinates.

Now the point is that *if* the equation thus obtained is independent of σ, we have indeed obtained a symmetry reduction of the original equation. In this case solutions $v = \Phi(y)$ to the reduced equation can be written in terms of the (x, u) coordinates as $\zeta(x, u) = \Phi[\eta(x, u)]$ and yield implicitly $u = f(x)$, the corresponding X-invariant solution to the original equation.

However, the vector field Y is not the ordinary prolongation of X, and thus we are not *a priori* guaranteed it leaves $\mathcal{S}_X$ or $\mathcal{I}_X$ invariant. Thus, in order to justify the method sketched above – i.e. in order to prove that the standard PDE reduction method still applies in the case of μ-symmetries – we have to prove the following theorem 4. Note that the only difference with respect to the standard case will be that it is the vector field Y, and not the ordinary prolongation $X^{(k)}$ of X, to be tangent to the solution manifold of Δ in $J^{(k)}M$.

23 Theorem. *Let Δ be a scalar PDE of order k for $u = u(x^1, \dots, x^p)$. Let $X = \xi^i(\partial/\partial x^i) + \varphi(\partial/\partial u)$ be a vector field on M, with characteristic $Q := \varphi - u_i\xi^i$, and let Y be the μ-prolongation of order k of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \to \mathrm{T}\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(k)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_J Q = 0$ for all J with $|J| = 0, \dots, k-1$.*

Proof. Recall that $\mathcal{S}_X$ is the intersection of the solution manifold $\mathcal{S}_0$ to Δ with the X-invariant set $\mathcal{I}_X$ (see remark 3 above, or [15]). The former is Y-invariant by assumption, as X is a μ-symmetry of Δ; the Y-invariance of $\mathcal{I}_X$ is guaranteed by lemma 5 above. Therefore the proof for the standard case [10] extends to the present setting. $\boxed{QED}$

24 Remark. The property $Y : \mathcal{I}_X \to \mathrm{T}\mathcal{I}_X$ can be shown in a alternative way without resorting to comparison with the standard case, i.e. using the geometrical characterization of μ-prolonged vector fields, as follows.

Denote by $\mathcal{I}_X^{(m)} \subset J^{(k)}M$ the set of points identified by E_J for $|J| \le m$. We first show that if $\mathcal{I}_X^{(m)}$ is invariant under Y, then $\mathcal{I}_X^{(m+1)}$ is also Y-invariant (for $m = 0, \ldots, k-2$). Note that Y-invariance of $\mathcal{I}_X^{(m)}$ means that for all $|J| \le m$ there are functions β^K such that $Y(D_J Q) = \sum_{|K|=0}^m \beta^K D_K Q$.

We have $Y[D_i(D_J Q)] = [Y, D_i](D_J Q) - D_i(Y(D_J Q))$; from the corollary to lemma 4 this reads

$$\lambda_i Y(D_J Q) + h_i^s D_s(D_J Q) - D_i(Y(D_J Q)) + V(D_J Q)$$

with $V = \sum_{|K|=k} \ell^K (\partial/\partial u_K)$. The first term is in $\mathcal{I}_X^{(m)}$ by hypothesis, while the second and third ones are by definition in $\mathcal{I}_X^{(m+1)}$. The last term vanishes since $D_J Q$ does not contain u derivatives of order greater than $m+1$, and $m \le k-2$.

The proof of Y-invariance of $\mathcal{I}_X$ is hence reduced to proving Y-invariance of $\mathcal{I}_X^{(0)}$, i.e. of the manifold identified by $Q = 0$; as for X a Lie-point vector field Q depends only on first order derivatives, it suffices to consider the first μ-prolongation of X, which is just $X^{(1)} + \lambda_i Q \partial_{u_i}$. It is well known that $Q = 0$ is invariant under the ordinary prolongation $X^{(1)}$, and of course the other term vanishes on $Q = 0$.

This proves Y-invariance of $\mathcal{I}_X^{(0)}$ and hence, by the recursive argument given above, of all the $\mathcal{I}_X^{(m)}$ with $m = 0, 1, \ldots, k-1$.

The recursive property considered here can be seen as a sort of counterpart in the PDE case of the recursive property discussed by Pucci and Saccomandi as characterizing the λ-prolongations as telescopic vector fields in the ODE case [2].

6 μ-prolongations in vector framework

In this section we extend μ-prolongations to the case of $q > 1$ dependent variables. We will assume that the independent variables u take value in the vector space $U = \mathbf{R}^q$. It will be natural in this context to consider differential forms taking values in $\mathcal{G}$. Thus we will deal with *matrix-valued differential forms* (or more generally Lie-algebra valued differential forms), see e.g. [14].

From now on, the form μ will be written in local coordinates as

$$\mu \ := \ (\Lambda_i)^a_b \, \mathrm{d}x^i \tag{17}$$

where $\Lambda_i : J^{(1)}M \to \mathcal{G}$ are smooth q-dimensional real matrix functions; we recall that $\mathcal{G}$ is the Lie algebra of $GL(q)$.

In the case of vector structures, we generalize condition (1.7) to the following (19); we will then define μ-prolongation in the same way as in the scalar case.

Theorem 5 below provides the μ-prolongation formula in the vector case, and theorem 6 provides the expression of μ-prolongations in terms of ordinary ones.

In the vector case, the contact structure Θ (we use a different symbol than in the scalar case to emphasize we deal with vector-valued forms) will be spanned by vector-valued one-forms $\vartheta_J = (\vartheta^1_J, \ldots, \vartheta^q_J) \in \mathbf{R}^q \otimes \Lambda^1(M)$, where

$$\vartheta^a_J \ = \ \mathrm{d}u^a_J \ - \ u^a_{J,m}\mathrm{d}x^m. \tag{18}$$

25 Definition. We say that Y *μ-preserves the contact structure* Θ, with μ given by (17), if for any vector-valued contact forms $\vartheta \in \Theta$, there is a vector-valued contact forms $\widehat{\vartheta} \in \Theta$ such that

$$\mathcal{L}_Y(\vartheta^a) \ + \Big(Y \lrcorner \left[(\Lambda_i)^a_b \vartheta^b \right] \Big) \ \mathrm{d}x^i \ = \ \widehat{\vartheta}^a. \tag{19}$$

26 Definition. A vector field Y in $J^{(k)}M$ which projects to X in M and which μ-preserves the contact structure is said to be the *μ-prolongation* of order k, or the k-th μ-prolongation, of X.

In order to discuss vector fields in $J^{(k)}M$ which are μ-prolongations of vector fields in M, it will be convenient to agree on a general notation. That is, we write a general vector field in $J^{(k)}M$ in the form

$$Y \ = \ \xi^i \frac{\partial}{\partial x^i} \ + \ \psi^a_J \frac{\partial}{\partial u^a_J} \ . \tag{20}$$

27 Theorem. *The vector field Y μ-preserves the standard contact structure Θ if and only if its coefficients satisfy the* **vector μ-prolongation formula**

$$\Psi^a_{J,i} \ = \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ \Psi^b_J \ - \ u^b_{J,k} \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ \xi^k. \tag{21}$$

Proof. The standard contact structure is generated by the forms ϑ^a_J ($a = 1, \ldots, q$; $|J| = 0, \ldots, n-1$), see (18). By trivial computations we have in full generality, for Y of the form (20),

$$\mathcal{L}_Y(\vartheta^a_J) \ = \ \mathrm{d}\Psi^a_J - u^a_{J,k}\mathrm{d}\xi^k - \Psi^a_{J,i}\mathrm{d}x^i. \tag{22}$$

The first term on the right hand side is easily computed to be

$$\mathrm{d}\Psi^a_J \ = \ (D_i \Psi^a_J)\, \mathrm{d}x^i \ + \sum_{|M|=0}^{|J|} \frac{\partial \Psi^a_J}{\partial u^b_M} \vartheta^b_M .$$

(It is of course also possible to extend the sum up to $|M| = n$, since Ψ^a_J will not depend on u^b_M for $|M| > |J|$.) Note that we have rewritten forms which are vertical for the fibration of $J^{(n)}M$ over $J^{(n-1)}M$ in terms of contact forms; thus we have no vertical forms, but only horizontal and contact ones.

Similarly, for $\xi = \xi(x,u)$ we have

$$\mathrm{d}\xi^k \ = \ (D_i \xi^k)\,\mathrm{d}x^i \ + \ \frac{\partial \xi^k}{\partial u^b}\,\vartheta^b_0 .$$

Combining these, we have shown that, in full generality,

$$\mathcal{L}_Y(\vartheta^a_J) \ = \ - \left[\Psi^a_J \ - \ \left(D_i \Psi^a_J - u^a_{J,k} D_i \xi^k \right) \right] \,\mathrm{d}x^i \ + \ \widehat{\vartheta} \tag{23}$$

for some $\widehat{\vartheta} \in \Theta$.

When the coefficients of Y satisfy (21), we have therefore

$$\begin{array}{rcl} \mathcal{L}_Y(\vartheta^a_J)\|_{\mathrm{mod}(\Theta)} & = & - \left[(\Lambda_i)^a_b \left(\Psi^b_J - u^b_{J,k}\xi^k \right) \right] \,\mathrm{d}x^i \\ & = & - \left(Y \lrcorner [(\Lambda_i)^a_b \vartheta^b_J] \right) \,\mathrm{d}x^i \ ; \end{array}$$

in other words, if the coefficients of Y satisfy (21), then Θ is μ-preserved by Y.

Conversely, if we ask that Y μ-preserves the contact structure, we obtain from (23) that the coefficients Ψ^a_J of Y necessarily satisfy (21). This completes the proof. $\boxed{QED}$

28 Theorem. *Let $X = \xi^i(\partial/\partial x^i) + \varphi^a(\partial/\partial u^a)$ be a vector field in M. Let $\mu = (\Lambda_i)^a_b dx^i$. Then the coefficients Ψ^a_J of the μ-prolongation Y of X are expressed in terms of the coefficients Φ^a_J of the ordinary prolongation of the same vector field X as*

$$\Psi^a_J \ = \ \Phi^a_J \ + \ F^a_J \tag{24}$$

where the difference terms F^a_J satisfy the recursion relation (with $F^a_0 = 0$)

$$F^a_{J,i} \ = \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ F^b_J \ + \ (\Lambda_i)^a_b \, D_J Q^b. \tag{25}$$

Proof. We denote as usual by Q^a the characteristic vector of X, i.e. $Q^a := \varphi^a - u^a_k \xi^k$. Let us consider a general Ψ^a_J, which we write in the form (24); again by (21) and simple algebra, we obtain

$$\Psi^a_{J,i} \ = \ \left(D_i \Phi^a_J - u^a_{J,k} D_i \xi^k \right) \ + \ \left[D_i F^a_J \ + \ (\Lambda_i)^a_b \left(F^b_J + \Phi^b_J - u^b_{J,k}\xi^k \right) \right] .$$

The first term on r.h.s. is just $\Phi^a_{J,i}$: if we write $\Psi^a_{J,i}$ again in the form (24), this computation shows that $F^a_{J,i}$ is given by the term in square brackets. We have thus shown that

$$F^a_{J,i} \ = \ [\delta^a_b D_i \ + \ (\Lambda_i)^a_b] \ F^b_J \ + \ (\Lambda_i)^a_b \left(\Phi^a_J - \xi^k u^b_{J,k} \right) . \tag{26}$$

The term in the last bracket above can be rewritten as

$$\Phi^a_J - \xi^k u^b_{J,k} \ = \ D_J Q^b \ ,$$

see [4] or proceed directly by using the relation between the prolongations of X and of its evolutionary representative [3, 10, 13].

With this, it is immediate to recognize that (26) coincides with (25). This completes the proof. $\boxed{QED}$

Similarly to what happens for the λ_i in the scalar case, the (matrix) coefficients Λ_i of the form μ are not completely arbitrary. Consider the multiindices $J = (j_1, \ldots, j_p)$ and $M = (m_1, \ldots, m_p)$ with $m_s = j_s + \delta_{i,s} + \delta_{k,s}$; the coefficient Ψ^a_M can be obtained from Ψ^a_J by applying twice formula (21), but we can proceed in two different ways, i.e. pass first from Ψ^a_J to $\Psi^a_{J,i}$ and then to Ψ^a_M, or pass first from Ψ^a_J to $\Psi^a_{J,k}$ and then to Ψ^a_M. Needless to say, the result must be the same in the two cases, and this is the **compatibility condition** for the Λ_i.

29 Theorem. *The compatibility condition for the matrix coefficients Λ_i of the matrix valued form $\mu = (\Lambda_i)^a_b dx^i$ reads*

$$D_i \, \Lambda_j \ - \ D_j \, \Lambda_i \ + \ [\, \Lambda_i \, , \, \Lambda_j \,] \ = \ 0 \qquad \forall i, j = 1, \ldots, p. \tag{27}$$

Proof. We will use the shorthand notation $(\nabla_i)^a_b = [\delta^a_b D_i + (\Lambda_i)^a_b]$. With this, the vector μ-prolongation formula (21) reads

$$\Psi^a_{J,k} \ = \ (\nabla_k)^a_b \Psi^b_J - u^b_{J,m} (\nabla_k)^a_b \xi^m \ ;$$

applying this twice, we get

$$\Psi^a_{J,k,i} \ = \ \Big[(\nabla_i \nabla_k)^a_b \Psi^b_J - u^b_{J,m} (\nabla_i \nabla_k)^a_b \xi^m \Big] \ - \ \Big[u^b_{J,i,m} (\nabla_k)^a_b + u^b_{J,k,m} (\nabla_i)^a_b \Big] \, \xi^m \ ;$$

note the second square bracket is symmetric in the indices i, k. Thus

$$\left(\Psi^a_{J,k,i} \ - \ \Psi^a_{J,i,k} \right) \ = \ [\nabla_i, \nabla_k]^a_b \Psi^b_J - u^b_{J,m} [\nabla_i, \nabla_k]^a_b \xi^m .$$

As for the commutator $[\nabla_i, \nabla_k]$, this is easily computed to be

$$[\nabla_i, \nabla_k] \ = \ D_i \Lambda_k \ - \ D_k \Lambda_i \ + \ [\Lambda_i, \Lambda_k] \ ,$$

i.e. the expression given in the statement, see (27). $\boxed{QED}$

30 Remark. If the Λ_i are of the form $\Lambda_i = \lambda_i(x, u^{(1)}) L$ for some constant matrix L and the λ_i smooth functions on $J^{(1)}M$, then the compatibility condition reduces to $D_i \lambda_j - D_j \lambda_i = 0$; i.e. the same as for the scalar case [4]. The same holds for $\Lambda_i = \lambda^m_i L_m$ with L_m $(m = 1, \ldots, r)$ constant matrices spanning an abelian Lie algebra, i.e. we have the set of conditions $D_i \lambda^m_j - D_j \lambda^m_i = 0$ for all $m = 1, \ldots, r$.

31 Remark. One could consider cases in which the matrices Λ_i belong to a gauged non abelian Lie algebra. By this we mean that

$$\Lambda_i \ = \ \lambda_i^k(x, u^{(1)}) \ L_k$$

with $\lambda_i : J^{(1)}M \to \mathbf{R}$ smooth functions and where the L_k $(k = 1, \ldots, r)$ are generators of a (matrix) Lie algebra $\mathcal{G}$, so that $[L_i, L_j] = c_{ij}^k L_k$. In this case the compatibility condition reads

$$\left[(D_i \lambda_j^k - D_j \lambda_i^k) \ + \ c_{ab}^k \, \lambda_i^a \lambda_j^b \right] \ L_k \ = \ 0 \ ;$$

needless to say, this means that the term in square brackets must vanish for each k. For an abelian algebra, we recover the condition mentioned in the previous remark.

7 μ-symmetries and reduction of PDE systems

We start by a definition of μ-symmetries, analogous to the one given in the scalar case; this says that X is a μ-symmetry of a given PDEs system if its μ-prolongation is tangent to the solution manifold of the PDEs.

32 Definition. Let (M, π, B) be a vector fiber bundle over the p-dimensional manifold B, with fiber $\pi^{-1}(x) = U = \mathbf{R}^q$. Let $\Delta = \{\Delta_1, \ldots, \Delta_r\}$ be a system of PDEs of order n for $u^a = u^a(x)$, $a = 1, \ldots, q$, $x = (x^1, \ldots, x^p) \in B$, with solution manifold $S_\Delta \subset J^{(n)}M$. Let X be a vector field in M, and μ a $\mathcal{G}$-valued semibasic one-form on M satisfying condition (27). Let Y be the μ-prolongation of order n of X by this form. If $Y : S_\Delta \to \mathrm{T}S_\Delta$, we say that X is a *μ-symmetry* of Δ.

The relevance of μ-symmetries for scalar equations lied in that they can be used to obtain invariant solutions, exactly like standard symmetries. The same happens for the vector case.

We will assume without further mention that X satisfy the **transversality condition** in the bundle (M, π, B) [10].

33 Remark. We recall that the transversality condition is generically (but not always) satisfied when $q = 1$, but a relevant assumption for $q > 1$. The standard PDE symmetry reduction method, extended to μ-symmetries for scalar equations in [4], is valid only under the transversality assumption, and the same will of course be true here, where we consider the vector case.

When transversality fails, one should in general resort to the more general approach of Anderson, Fels and Torre [1]; in the case of *partial transversality* a simpler approach is also possible, see [6]. We will not discuss μ-symmetries in these frames.

In the case of scalar equations, the possibility of using μ-symmetries to perform symmetry reduction for PDEs relied ultimately on the fact that the space $\mathcal{I}_X \subset J^{(n)}M$ of X-invariant functions, identified by $Q = 0$ and its differential consequences, is invariant under vector fields Y in $J^{(n)}M$ which are μ-prolongations of X [4]. Moreover, the standard and the μ-prolongations of a vector field X coincide in $\mathcal{I}_X$; so the results valid for reduction of an equation Δ on $\mathcal{I}_X$ under standard symmetries also hold for μ-symmetries. We want now to prove that the same holds in the vector case.

34 Definition. Let $X = \xi^i(\partial/\partial x^i) + \varphi^a(\partial/\partial u^a)$ be a vector field in M; denote by $Q^a := \varphi^a - u^a_i \xi^i$ its characteristic vector. Then the *X-invariant manifold* in $J^{(n)}M$ is the subset $\mathcal{I}_X \subset J^{(n)}M$ identified by $D_J Q^a = 0$ for all $a = 1, \ldots, q$ and all multiindices J with $0 \le |J| \le n-1$.

35 Theorem. *Let $\mu = \Lambda_i dx^i$ be a $\mathcal{G}$-valued horizontal form satisfying (27). Let Y be the μ-prolongation of the vector field X. Then Y coincides with the standard prolongation of the same vector field X on $\mathcal{I}_X$.*

PROOF. This follows from theorem 6 and the definition of $\mathcal{I}_X$. Indeed, write Ψ^a_J in the form (24), and suppose that for $|J| = k$ the difference term F^a_J is written as a combination of the $D_J Q^b$, i.e. $F^a_J = (\Gamma^J)^a_b D_J Q^b$. Then from (25) we have

$$F^a_{J,i} \;=\; \delta^a_b [D_i(\Gamma^J)^b_c](D_J Q^c) + (\Lambda_i)^a_b \left[(\Gamma^J)^b_c (D_J Q^c) + D_J Q^b \right] \ ;$$

this is again a combination of terms of the form $D_J Q^b$. Thus if the F^a_J vanish on $\mathcal{I}_X$ for $|J| = k$, the F^a_J with $|J| \ge k$ also vanish on $\mathcal{I}_X$.

Note that $F^a_i = (\Lambda_i)^a_b Q^b$, so that the condition is satisfied for $|J| = 1$, and the proof of the theorem follows by the recursive computation above. $\boxed{QED}$

8 Examples – Scalar equations

In this section we consider examples of scalar equations (systems are considered in the next section) having nontrivial μ-symmetries; one easily determines the corresponding reduced equations. Example 1 will be discussed in full detail, while for the other ones we just give the relevant results.

We will only consider equations of first or second order in two independent variables, which we denote as (x, t). That is, we have an equation for $u = u(x, t)$. A vector field in M will be written as

$$X \;=\; \xi \partial_x + \tau \partial_t + \varphi \partial_u \ ,$$

and its characteristic will thus be $Q = \varphi - u_x \xi - u_t \tau$.

The μ-prolonged vector field will be

$$Y = X + \Psi^x \frac{\partial}{\partial u_x} + \Psi^t \frac{\partial}{\partial u_t} + \Psi^{xx} \frac{\partial}{\partial u_{xx}} + \Psi^{xt} \frac{\partial}{\partial u_{xt}} + \Psi^{tt} \frac{\partial}{\partial u_{tt}}. \qquad (28)$$

The symmetry-adapted variables will be (σ, y, v), where (y, v) correspond to invariants.

We always compute the most general second order equation Δ which admit a given vector field X as a strong μ-symmetry for a certain $\mu = \lambda \mathrm{d}x + \nu \mathrm{d}t$; this can be determined from the characteristic equation for the action of Y in $J^{(2)}M$. In particular, Δ will be written as

$$F[y, v; \zeta_1, \zeta_2; \eta_1, \eta_2, \eta_3] = 0$$

where F is an arbitrary smooth function of invariants y, v of order zero, ζ_1, ζ_2 of order one, and η_1, η_2, η_3 of order two, for Y in $J^{(2)}M$.

The reduced equation will simply be obtained by restricting the invariants to $\mathcal{I}_X$; this is obtained by resolving the equations $Q = D_x Q = D_t Q = 0$ for three of the variables in $J^{(2)}M$; needless to say, only four of the seven invariants will be independent on $\mathcal{I}_X$.

Example 1.

As a first example, to be dealt with in detail, we will consider μ-prolongations of the scaling vector field

$$X = x\partial_x + 2t\partial_t + u\partial_u. \qquad (29)$$

The invariant coordinates (y, v) and the parametric coordinate σ in $M = \{(x, t, u)\}$ can be chosen as

$$\sigma = x \ , \ y = x^2/t \ , \ v = u/x \ ;$$

the corresponding inverse change of variables is

$$x = \sigma \ , \ t = \sigma^2/y \ , \ u = \sigma v.$$

It follows easily that in the symmetry-adapted coordinates,

$$X = \sigma \partial_\sigma \ ;$$

hence the function $v = v(\sigma, y)$ is X-invariant if and only if $v_\sigma = 0$, as required by the general method (indeed, by the very definition of symmetry-adapted coordinates).

The partial derivatives of u are expressed in terms of the partial derivatives of $v = v(\sigma, y)$ by applying the procedure described in sect.5 as

$$u_x = v + 2yv_y + \sigma v_\sigma \ ; \ u_t = -(y^2/\sigma)v_y.$$

The above can be inverted to give $v_\sigma = (1/x)[u_x + 2(t/x)u_t - u/x] = -Q/x^2$, $v_y = -[t^2/x^3]u_t$. Similarly, at second order we get

$$\begin{aligned}
u_{xx} &= 2v_\sigma + 6(y/\sigma)v_y + \sigma v_{\sigma\sigma} + 4yv_{\sigma y} + 4(y^2/\sigma)v_{yy} \ , \\
u_{xt} &= -3(y^2/\sigma^2)v_y - (y^2/\sigma)v_{\sigma y} - 2(y^3/\sigma^2)v_{yy} \ , \\
u_{tt} &= 2(y^3/\sigma^3)v_y + (y^4/\sigma^3)v_{yy}.
\end{aligned}$$

Let us now come to the second μ-prolongations (standard prolongations will be obtained by setting $\lambda = \nu = 0$); we write these (at order two) in the form (28). With standard computations, i.e. using either (12) or (14), we get (recall in this case $Q = u - xu_x - 2tu_t$)

$$\begin{aligned}
\Psi^x &= \lambda Q \\
\Psi^t &= -u_t + \nu Q \\
\Psi^{xx} &= -u_{xx} + 2\lambda(D_x Q) + [\lambda^2 + (D_x\lambda)]Q \\
\Psi^{xt} &= -2u_{xt} + [\lambda(D_t Q) + \nu(D_x Q)] + (1/2)[2\lambda\nu + (D_t\lambda) + (D_x\nu)]Q \\
\Psi^{tt} &= -3u_{tt} + 2\nu(D_t Q) + [\nu^2 + (D_t\nu)]Q.
\end{aligned}$$

Note that, as ensured by our general results, these reduce to the coefficients of ordinary prolongations on the invariant space $\mathcal{I}_X$ (identified by $Q = 0$, $D_x Q = 0$, and $D_t Q = 0$).

We will now consider the simplest nontrivial choice for μ, i.e. $\mu = \lambda \mathrm{d}x$, with λ a real constant. In this case the explicit expression of Y is readily obtained by our previous general formulae:

$$\begin{aligned}
\Psi^x &= \lambda\,(u - xu_x - 2tu_t) \ , \\
\Psi^t &= -u_t \ , \\
\Psi^{xx} &= -u_{xx} - 2\lambda(xu_{xx} + 2tu_{xt}) + \lambda^2(u - xu_x - 2tu_t) \ , \\
\Psi^{xt} &= -2u_{xt} - \lambda(xu_{xt} + 2tu_{tt} + u_t) \ , \\
\Psi^{tt} &= -3u_{tt}.
\end{aligned}$$

We want to discuss equations which admit X as a μ-symmetry (with this choice), and their μ-symmetry reduction. It suffices to consider equations admitting X as a strong μ-symmetry. We define the following functions:

$$\begin{aligned}
& y := (x^2/t) \ , \quad v := (u/x) \ ; \\
& \zeta_1 := xu_t \ , \quad \zeta_2 := (u/x - 2tu_t/x - u_x)e^{\lambda x} \\
& \eta_1 := xtu_{tt} \ , \quad \eta_2 := \left(xu_t + 2xtu_{tt} + x^2u_{xt}\right)e^{\lambda x} \ , \\
& \eta_3 := (1/x)\left[(1-\lambda x)(u - xu_x) + 2\lambda xtu_t + x^2u_{xx} + 4xtu_{xt} + 4t^2u_{tt}\right]e^{2\lambda x}.
\end{aligned}$$

36 Proposition. *Consider the equation* $\Delta := F[y, v, \zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3] = 0$ *with* F *an arbitrary smooth function of its arguments. Let* λ *be a real constant. Then:*
(i) *The equation* Δ *admits the vector field* X *given in (29) as a* μ*-symmetry, with* $\mu = \lambda dx$*.*
(ii) *For* $(\partial F/\partial\zeta_2)^2 + (\partial F/\partial\eta_2)^2 + (\partial F/\partial\eta_3)^2 \neq 0$*,* X *is not an ordinary symmetry of* Δ*.*
(iii) *The* μ*-symmetry reduced equation, providing* X*-invariant solutions to* Δ*, is given by*

$$H[y, v, \zeta_1, \eta_1] \ := \ F[y, v, \zeta_1, 0, \eta_1, 0, 0] \ = \ 0. \tag{30}$$

Proof. The equation $\Delta : F = 0$ admits X as a strong μ-symmetry if and only if F can be written in terms of the invariants of the μ-prolongation of X, i.e. of Y given above (invariants for the Y action in $J^{(2)}M = \mathbf{R}^8$ are built with the method of characteristics).

Let us first consider the case of first order equations. Two independent "first order" invariants (more precisely, invariant functions on $J^{(1)}M$) for Y are easily seen to be the ζ_1 and ζ_2 defined above. Note that ζ_1 is also an invariant for the ordinary prolongation, while $\zeta_2 = -(Q/x)e^{\lambda x}$ is neither an invariant nor a relative invariant for it (due to the exponential term). Thus, any first order equation of the form $F[y, v, \zeta_1, \zeta_2] = 0$ (or equivalent to such an equation [3, 10, 13, 15]) admits X as a μ-symmetry (with $\mu = \lambda \mathrm{d}x$); if $\partial F/\partial\zeta_2 \neq 0$, then X is not an ordinary symmetry of the equation.

We can easily check that F (i.e. ζ_1 and ζ_2) will not depend on σ when restricted to the invariant space $v_\sigma = 0$. Indeed, using the above formulas for u_x and u_t, it results $\zeta_1 = -y^2 v_y$ and $\zeta_2 = \sigma v_\sigma e^{\lambda\sigma}$. We recall again that, as clear from $X = \sigma\partial_\sigma$ (or from $Q = -\sigma^2 v_\sigma$), the invariant space $\mathcal{I}_X$ is identified by $v_\sigma = 0$ and the differential consequences of this.

This discussion extends to second order equations once one checks that the η_1, η_2, η_3 given above are indeed independent "second order" invariants (more precisely, invariant functions on $J^{(2)}M$) for Y, obviously also independent of the ζ_1, ζ_2. This also completes the proof of point (i).

Note now that η_1 is also a second order differential invariant for the ordinary prolongation $X^{(2)}$ of X, while η_2 and η_3 are not. As ζ_2, η_2, η_3 are independent, the only way in which F of the form considered here can admit X as a standard symmetry is by not depending on these. This proves point (ii).

Finally, note that restriction to $\mathcal{I}_X$ does not change the form of ζ_1 and η_1. From the above explicit expression of the invariants we have that $\zeta_2 = e^{\lambda x}Q/x$, $\eta_2 = -xe^{\lambda x}(D_t Q)$, and $\eta_3 = (e^{2\lambda x}/x)[(1-\lambda x)Q - 2t(D_t Q) - x(D_x Q)]$. Therefore, $\zeta_2 = \eta_2 = \eta_3 = 0$ on $\mathcal{I}_X$, and (30) represents indeed the reduced equation for Δ

in the space of X-invariant functions. This proves point (iii), hence completes the proof. $\boxed{QED}$

We would like to note that, as claimed by (30), the reduced equation corresponds to the one which would be obtained by the standard symmetry reduction (under X) of the equation $F[y, v, \zeta_1, \widehat{\zeta_2}, \eta_1, \widehat{\eta_2}, \widehat{\eta_3}] = 0$ to $\mathcal{I}_X$, where $\widehat{\zeta_2}$ is obtained from ζ_2 by setting $\lambda = 0$, and the like for $\widehat{\eta_2}$ and $\widehat{\eta_3}$.

Finally, let us discuss a completely concrete example, albeit the procedure here is completely analogous to the one to be followed for standard symmetries. Consider the equation $\Delta := \eta_1 - \zeta_2 = 0$. This is written as

$$xt\, u_{tt} \ + \ (u_x + (2t/x)u_t - u/x)\ e^{\lambda x} \ = \ 0$$

in the original coordinates; in the adapted ones it reads

$$y^3 v_{yy} + 2y^2 v_y + \sigma e^{\lambda\sigma} v_\sigma \ = \ 0.$$

The corresponding reduced equation is

$$y^2\ [y v_{yy} + 2 v_y] \ = \ 0\ ;$$

the general solution to this is

$$v(y) \ = \ c_1 \ + \ c_2/y$$

where c_i are real constants. Going back to the original coordinates, the corresponding solutions are

$$u(x,t) \ = \ \frac{c_1\, x^2 \ + \ c_2\, t}{x}.$$

Example 2.

Consider equations given by $F[y, v; \zeta_1, \zeta_2; \eta_1, \eta_2, \eta_3] = 0$ with F a smooth function of its arguments, where y, v are as above, and

$$\begin{array}{l} \zeta_1 = u_x\ ; \quad \zeta_2 = u_t/x - (u/x - u_x)/(2t)\ ; \\ \eta_1 = x u_{xx}\ , \quad \eta_2 = u_{xt} + x u_{xx}/(2t)\ , \\ \eta_3 = (x u_{xx} - 3u_x)/(4t^2) + u_{xt}/t + u_{tt}/x - u_t/(xt) + 3u/(4xt^2). \end{array}$$

Then the vector field (29) is a μ-symmetry of the equation $F = 0$, with $\mu = -(1/t)\mathrm{d}t$.

Indeed, y and v are invariants as before, and the coefficients of the second μ-prolongation are now

$$\begin{array}{l} \Psi_x = 0\ , \quad \Psi_t = u_t - u/t + (x/t)u_x \\ \Psi_{xx} = -u_{xx}\ , \quad \Psi_{xt} = (x/t)u_{xx} \\ \Psi_{tt} = u_{tt} - (2/t)\left[(u_t - x u_{xt}) - (1/t)(u - x u_x)\right]. \end{array}$$

One can check that ζ_1, ζ_2 and η_1, η_2, η_3 are invariant. Again, ζ_1 is also an invariant for the ordinary prolongation, while ζ_2 is not. Note that in adapted coordinates, it results $\zeta_1 = v + \sigma v_\sigma + 2yv_y$ and $\zeta_2 = [y/(2\sigma)]v_\sigma$; hence on $\mathcal{I}_X$, given again by $v_\sigma = 0$ and differential consequences, we have $\zeta_2 = 0$ and $\zeta_1 = v + 2yv_y$. Similarly η_1 is invariant also for the ordinary prolongation, while η_2 and η_3 are not. We omit expressions for the η_i in adapted coordinates.

Hence, any equation given by $F[y, v, \zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3] = 0$ will admit X as a (strong) μ-symmetry, and if $(\partial F/\partial\zeta_2)^2 + (\partial F/\partial\eta_2)^2 + (\partial F/\partial\eta_3)^2 \neq 0$, X is not an ordinary symmetry.

The invariant subset $\mathcal{I}_X \subset J^{(2)}M$ is in this case identified by

$$u_t = (u - xu_x)/(2t) \ , \ u_{xt} = -xu_{xx}/(2t) \ , \ u_{tt} = (x^2 u_{xx} + xu_x - u)/(4t^2).$$

The restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1 = u_x$, $\zeta_2 = 0$, and $\eta_1 = xu_{xx}$, $\eta_2 = \eta_3 = 0$; hence the reduced equation will be simply

$$H[y, v, \zeta_1, \eta_1] \ := \ F[y, v; \zeta_1, 0; \eta_1, 0, 0] \ = \ 0.$$

Example 3.

Any equation of the form $F[y, v, \zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3] = 0$, where F is a smooth function of its arguments and we have defined

$$\begin{array}{l}
y = \sqrt{x^2 + t^2} \ , \quad v = u \ , \\
\zeta_1 = u_x/x \ , \quad \zeta_2 = u_t - (t/x)u_x \ , \\
\eta_1 = (r^2/x^3)(xu_{xx} - u_x) \ , \quad \eta_2 = (r/x^3)(xtu_{xx} - x^2 u_{xt} - tu_x) \ , \\
\eta_3 = u_{tt} + (t/x^3)(xtu_{xx} - 2x^2 u_{xt} - tu_x) \ ,
\end{array}$$

admits the vector field

$$X \ = \ x\partial_t - t\partial_x$$

as a μ-symmetry, with $\mu = -(1/x)\mathrm{d}x$.

The coefficients defining the μ-prolongation are in this case

$$\begin{array}{l}
\Psi^x = -(t/x)u_x \ , \quad \Psi^t = u_x \ , \\
\Psi^{xx} = (2t/x^2)(u_x - xu_{xx}) \ , \\
\Psi^{xt} = u_{xx} - (1/x)(u_x + tu_{xt}) \ , \quad \Psi^{tt} = 2u_{xt} \ ;
\end{array}$$

one checks easily that y, v and the functions ζ_i, η_j given above are independent invariants for Y.

The functions y, v provide invariants in M, and we can select $\sigma = \mathrm{arctg}(t/x)$; the inverse change of coordinates is given by $x = y\cos(\sigma)$, $t = y\sin(\sigma)$, $u = v$. The vector field X is then expressed as $X = \partial_\sigma$

The invariant subset $\mathcal{I}_X$ is in this case identified by $u_t = (t/x)u_x$, $u_{xt} = (xu_{xx}-u_x)(t/x^2)$, $u_{tt} = [xt^2u_{xx}+(x^2-t^2)u_x]/x^3$. The restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1 = u_x/x$, $\zeta_2 = 0$, $\eta_1 = (r^2/x^3)(xu_{xx} - u_x)$, $\eta_2 = 0$, $\eta_3 = u_x/x$ (note $\eta_3 = \zeta_1$); hence the reduced equation will be simply

$$H[y, v, \zeta_1, \eta_1] \ := \ F[y, v; \zeta_1, 0; \eta_1, 0, \zeta_1] \ = \ 0.$$

Example 4.

Consider equations of the form $F[y, v, \zeta_1, \zeta_2, \eta_1, \eta_2, \eta_3] = 0$, where F is a smooth function of its arguments and (with $k = x^2/(4t)$)

$$\begin{aligned}
& y = t \quad , \quad v = ue^k \ ; \\
& \zeta_1 \ = \ [u_x + xu/(2t)]\, e^{(k+\lambda x)} \ ; \\
& \zeta_2 \ = \ \big[u_t - (k/t)u - (\lambda t)^{-1}[u_x + xu/(2t)]\big] \ e^k \ ; \\
& \eta_1 \ = \ \big[(\lambda u_x + u_{xx}) + (k/t)u + (2t)^{-1}(u + \lambda xu + 2xu_x)\big] \ e^{(k+2\lambda x)} \ , \\
& \eta_2 \ = \ [-u_{xt} + k(2\lambda t^2)^{-1}(2 + \lambda x)u + (2\lambda t)^{-1}(2\lambda u_x + 2u_{xx} - \lambda xu_t) + \\
& \quad +(4\lambda t^2)^{-1}[x(4 + \lambda x)u_x + (2 + 4\lambda x)u]]\, e^{(k+\lambda x)} \ , \\
& \eta_3 \ = \ [16\lambda^2t^4u_{tt} - 32\lambda t^3u_{xt} + 8t^2(2u_{xx} + 2\lambda u_x - 2\lambda xu_t - \lambda^2x^2u_t) + \\
& \quad +8t\big(x(2 + \lambda x)u_x + (1 + 3\lambda x + \lambda^2x^2)u\big) + x^2(2 + \lambda x)^2u]\,(16\lambda^2t^4)^{-1}\, e^k.
\end{aligned}$$

These admit as μ-symmetry, with $\mu = \lambda \mathrm{d}x$ (λ a real constant) the vector field

$$X \ = \ 2t\,\partial_x \ - \ xu\,\partial_u. \tag{31}$$

The functions given above are indeed invariant under Y, the second μ-prolongation of X; this is identified by the coefficients

$$\begin{aligned}
& \Psi^x = -u - xu_x - \lambda xu - 2\lambda tu_x \ , \quad \Psi^t = -xu_t - 2u_x \ ; \\
& \Psi^{xx} = -2u_x - xu_{xx} - 2\lambda(u - xu_x - 2tu_{xx}) - \lambda^2(xu + 2tu_x) \ , \\
& \Psi^{xt} = -u_t - 2u_{xx} - xu_{xt} - \lambda(2u_x + 2tu_{xt} + xu_t) \ , \quad \Psi^{tt} = -4u_{xt} - xu_{tt}.
\end{aligned}$$

The invariant subset $\mathcal{I}_X$ is identified by $u_x = -xu/(2t)$, $u_{xt} = x(u - tu_t)/(2t^2)$, $u_{xx} = (x^2 - 2t)u/(4t^2)$. Restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1 = 0$, $\zeta_2 = \zeta_2^0 := (u_t - ku/t)e^k$, $\eta_1 = 0$, $\eta_2 = 0$, $\eta_3 = \eta_3^0 := [u_{tt} - (x^2/(2t^2))u_t + ((x^4 + 8tx^2)/(16t^4))u]e^k$. The reduced equation will be simply

$$H[y, v, \zeta_2^0, \eta_3^0] \ := \ F[y, v; 0, \zeta_2^0; 0, 0, \eta_3^0] \ = \ 0.$$

Example 5.

Any equation of the form $F(y,v;\zeta_1,\zeta_2;\eta_1,\eta_2,\eta_3)=0$ with F a smooth function of its arguments, which are (with $k=x^2/(4t)$)

$$
\begin{aligned}
& y=t\ ,\quad v=ue^k\ ;\\
& \zeta_1=(u_x/x)e^k\ ,\quad \zeta_2=u_t/u+(1/t)\log(u)\ ;\\
& \eta_1=(e^k/x^3)\,(xu_{xx}-u_x)\ ,\quad \eta_2=e^k\,[u_{xt}/x-xu_x/(4t^2)]\ ,\\
& \eta_3=-(u_{tt}/u)+(1/t^2)\log(u)\,[2(1-t[u_t/u+(1/t)\log(u)])+\log(u)]\ ,
\end{aligned}
$$

admits the same vector field (31) as a (strong) μ-symmetry, with $\mu=-[(1/x)\mathrm{d}x+(1/t)\mathrm{d}t]$.

In this case the coefficients of the vector field Y are given by

$$
\begin{aligned}
& \Psi_x=(-x+2t/x)u_x\ ,\\
& \Psi_t=-xu_t+(x/t)u\ ,\\
& \Psi_{xx}=(-x+4t/x)u_{xx}-(4t/x^2)u_x\ ,\\
& \Psi_{xt}=(-x+2t/x)u_{xt}+(x/t)u_x\ ,\\
& \Psi_{tt}=-xu_{tt}+(2x/t)u_t-(2x/t^2)u.
\end{aligned}
$$

The invariant set $\mathcal{I}_X$ is identified by $u_x=-xu/(2t)$, $u_{xt}=x(u-tu_t)/(2t^2)$, $u_{xx}=(x^2-2t)u/(4t^2)$. Restriction of the invariant functions to $\mathcal{I}_X$ yields

$$
\begin{aligned}
& \zeta_1=\zeta_1^0:=-ue^k/(2t)\ ,\quad \zeta_2=u_t/u+[\log(u)]/t\ ;\\
& \eta_1=\eta_1^0:=ue^k/(4t^2)\ ,\quad \eta_2=\eta_2^0:=-[4t^2u_te^k-(x^2+4t)ue^k]/(8t^3)\ ,\\
& \eta_3=\eta_3^0:=-[t^2u_{tt}+2(tu_t-u)\log(u)+u(\log(u))^2]/(t^2u).
\end{aligned}
$$

Note that these are functionally dependent:

$$
\begin{aligned}
\zeta_1^0 &= g_1(y,v) &&:= -v/(2y)\ ,\\
\eta_1^0 &= g_2(y,v) &&:= v/(4y^2)\ ,\\
\eta_2^0 &= g_3(y,v,\zeta_2) &&:= [v/(2y^2)]\ [1+\log(v)-y\,\zeta_2].
\end{aligned}
$$

The reduced equation will be simply

$$
H[y,v,\zeta_2,\eta_3^0]\ :=\ F[y,v;g_1(y,v),\zeta_2;g_2(y,v),g_3(y,v,\zeta_2),\eta_3]\ =\ 0.
$$

Example 6.

In previous examples the functions λ and ν in $\mu=\lambda\mathrm{d}x+\nu\mathrm{d}t$ were always depending only on x and t; in this last example they will depend on first order derivatives of the u.

Any equation of the form $F(y,v;\zeta_1,\zeta_2;\eta_1,\eta_2,\eta_3)=0$ with F a smooth function of its arguments, which are

$$\begin{array}{l}
y \ = \ t \ , \quad v \ = \ u/x \ ; \\
\zeta_1 = u + \log[1-u/(xu_x)] \ , \quad \zeta_2 = -(uu_t)/(x^2u_x) \ ; \\
\eta_1 \ = \ -e^{2u} \left[u/(x^2u_x^3)\right] \left[u^2(u_x^2-u_{xx}) + xu_x^3 - (1+xu_x)uu_x^2\right] \ , \\
\eta_2 \ = \ e^u \left[u^2/(x^3u_x^3)\right] \left[xu_xu_{xt} - (u_x + xu_{xx})u_t\right] \ , \\
\eta_3 \ = \ \left[u/(x^3u_x^3)\right] \left[xu_x^2u_{tt} - 2xu_xu_tu_{xt} + (2u_x + xu_{xx})u_t^2\right] \ ,
\end{array}$$

admits the vector field

$$X \ = x\,\partial_x \ + \ u\,\partial_u$$

as a (strong) μ-symmetry, with $\mu = u_x\mathrm{d}x + u_t\mathrm{d}t$.

In this case the coefficients of the vector field Y are given by

$$\begin{array}{l}
\Psi_x = uu_x - xu_x^2 \ , \quad \Psi_t = u_t(1+u-xu_x) \ , \\
\Psi_{xx} = u_{xx}(u-1-3xu_x) + u_x^2(u-xu_x) \ , \\
\Psi_{xt} = u_{xt}(u-2xu_x) - xu_tu_{xx} + u_tu_x(1+u-xu_x) \ , \\
\Psi_{tt} = u_{tt}(1+u-xu_x) - 2xu_tu_{xt} + (2+u-xu_x)u_t^2.
\end{array}$$

The invariant set $\mathcal{I}_X$ is given by $u_x = u/x$, $u_{xt} = u_t/x$, $u_{xx}=0$. Restriction of the invariant functions to $\mathcal{I}_X$ yields $\zeta_1 = \zeta_1^0 := u/x = v$, $\zeta_2 = \zeta_2^0 := -u_t/x$, $\eta_1 = \eta_2 = 0$, $\eta_3 = \eta_3^0 := u_{tt}/x$. We stress that this expression for ζ_1 is not obtained by a direct substitution: indeed now $Q=0$ means $u_x = u/x$; the general expression for ζ_1 given above becomes singular, but the expression for Ψ^x guarantees that u_x is constant, and actually equal to $u/x = v$ on $Q=0$.

The reduced equation will be simply

$$H[y,v,\zeta_2^0,\eta_3^0] \ := \ F[y,v;v,\zeta_2^0;0,0,\eta_3^0] \ = \ 0.$$

9 Examples – Systems of PDEs

In this final section we present examples illustrating the result discussed in sections 6 and 7 above, with systems of two second order equations for functions (u,v) of two independent variables (x,y). In the first two examples (7 and 8) the matrices Λ_i are actually constant multiples of the identity; in example 9 this is not the case, but Λ_i are still constant, and in example 10 the Λ_i are neither constant nor multiples of the identity. In example 11 we briefly consider a "gauged" μ-symmetry.

It should be noted that here we build examples of systems of equations for which the considered vector field X is a *strong* μ-symmetry, i.e. each equation of the system admits X as a strong μ-symmetry: $Y(\Delta_i)=0$ for $i=1,\dots,r$.

Needless to say, one could – starting from the formulas we provide – build examples of equations admitting the same vector fields X as "ordinary" μ-symmetries with standard techniques. That is, let A be a $r \times r$ matrix smooth function, $A : J^{(n)}M \to \mathrm{Mat}(r)$, and consider a system of equations $\widehat{\Delta}_i = 0$, $i = 1, \dots, r$ (solution manifold $\widehat{S}$) with $\widehat{\Delta}_i := A_{ij}\Delta_j$. Note that A will in general depend on the Δ_i as well as on the coordinate along the Y characteristic, say w. We assume $||A||$ is nowhere zero, so this has the same set of solutions as the "old" system $\Delta_i = 0$ (solution manifold S). Applying Y to the "new" system, we have $Y(\widehat{\Delta}_i) := Y(A_{ij})\Delta_j + A_{ij}Y(\Delta_j)$; the last term vanishes by hypothesis, and we are left with $Y(\widehat{\Delta}_i) := Y(A_{ij})\Delta_j = Y(A_{ij})A^{-1}_{jk}\widehat{\Delta}_k$, which vanishes on S, i.e. – as $\widehat{S} = S$ – on solutions to the system.

Example 7.

We consider first a very simple example of μ-symmetric PDEs in two independent and two dependent variables, which we denote as (x, y) and (u, v) respectively, with matrices Λ which are actually multiples of the identity matrix; this is quite related to example 1 above.

Let us consider the vector field

$$X \ = \ x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$$

and the form

$$\mu \ = \ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mathrm{d}x$$

with λ a real constant; this corresponds to matrices Λ_i given by

$$\Lambda_{(x)} = \lambda I \ , \ \Lambda_{(y)} = 0.$$

By applying the vector μ-prolongation formula (21), or using theorem 6 and (24), (25), we determine the second μ-prolongation Y of X; with the notation introduced above, the result is the following:

$$\begin{aligned}
&\Psi^1_x = \lambda\,(u - u_x\,x - 2\,u_y\,y)\\
&\Psi^1_y = -u_y\\
&\Psi^1_{xx} = -u_{xx} + \lambda\,(-2\,u_{xx}\,x - 4\,u_{xy}\,y) + \lambda^2\,(u - u_x\,x - 2\,u_y\,y)\\
&\Psi^1_{xy} = -2\,u_{xy} + \lambda\,(-u_y - u_{xy}\,x - 2\,u_{yy}\,y)\\
&\Psi^1_{yy} = -3\,u_{yy}\\
&\Psi^2_x = v_x + \lambda\,(2\,v - v_x\,x - 2\,v_y\,y)\\
&\Psi^2_y = 0\\
&\Psi^2_{xx} = \lambda\,(2\,v_x - 2\,v_{xx}\,x - 4\,v_{xy}\,y) + \lambda^2\,(2\,v - v_x\,x - 2\,v_y\,y)\\
&\Psi^2_{xy} = -v_{xy} + \lambda\,(-(v_{xy}\,x) - 2\,v_{yy}\,y)\\
&\Psi^2_{yy} = -2\,v_{yy}.
\end{aligned}$$

We can then solve the characteristic equation for the flow of Y in $J^{(2)}M$, and determine a basis of Y-invariant functions. Such a basis, obtained in this way, is provided by the following set of functions:

$$\begin{array}{l}
\rho = x^2/y \ , \\
w_1 := u/x \ , \\
w_2 := v/x^2 \ ; \\
\zeta_1 := y u_y / x \ , \\
\zeta_2 := (u_x + 2 y u_y / x - u/x) e^{\lambda x} \ , \\
\zeta_3 := v_y \ , \\
\zeta_4 := (v_x/x - 2v/x^2 + 2 y v_y / x^2) e^{\lambda x} \ ; \\
\eta_1 := x^2 v_{yy} \ , \\
\eta_2 := x^3 u_{yy} \ , \\
\eta_3 := (x v_{xy} + 2 y v_{yy}) e^{\lambda x} \ , \\
\eta_4 := (x^2 u_{xy} + x u_y + 2 x y u_{yy}) e^{\lambda x} \ , \\
\eta_5 := [v_{xx} + 4v/x^2 - 3 v_x / y + 4 y u_{xy} - 4 y v_y / x^2 + 4 y u_y / x + \\
\quad -4 y^2 v_{yy} / x^2 + 8 y^2 u_{yy} / x + \lambda (v_x - 2v/x + 2 y v_y / x)] e^{2 \lambda x} \ , \\
\eta_6 := [-u_x + u/x + x u_{xx} + 4 y u_{xy} + 4 y^2 u_{yy} / x + \\
\quad +\lambda(-u + x u_x + 2 y u_y)] e^{2\lambda x}.
\end{array} \tag{32}$$

Any (system of) second order equation of the form

$$F^i \left[y, w_1, w_2; \zeta_1, \ldots, \zeta_4; \eta_1, \ldots, \eta_6 \right] = 0 \tag{33}$$

with F^i $(i = 1, \ldots, n)$ a smooth function of its arguments, admits X as a (strong) μ-symmetry.

In order to consider the μ-symmetry reduced equation, it suffices to consider the restriction of the functions ζ_i, η_j on $\mathcal{I}_X$. (Note that on $\mathcal{I}_X$ the μ-prolongation and the ordinary prolongation coincide, see theorem 8; hence also the invariant of these two vector fields in $J^{(n)}M$ coincide when restricted to $\mathcal{I}_X$.)

The manifold $\mathcal{I}_X$ is identified by $Q = D_x Q = D_y Q = 0$; in the present case these read

$$\begin{array}{ll}
Q^1 & := u - x u_x - 2 y u_y = 0 \\
Q^2 & := 2v - x v_x - 2 y v_y = 0 \\
D_x Q^1 & := -x u_{xx} - 2 y u_{xy} = 0 \\
D_x Q^2 & := v_x - x v_{xx} - 2 y v_{xy} = 0 \\
D_y Q^1 & := -u_y - x u_{xy} - 2 y u_{yy} = 0 \\
D_y Q^2 & := -x v_{xy} - 2 y v_{yy} = 0
\end{array}$$

and the solution is provided by

$$\begin{array}{lll}
u_y = (u - x u_x)/(2y) \ , & u_{xy} = -(x u_x x)/(2y) \ , & u_{yy} = -(u_y + x u_{xy})/(2y) \ ; \\
v_y = (2v - x v_x)/(2y) \ , & v_{xy} = (v_x - x v_{xx})/(2y) \ , & v_{yy} = -(x v_{xy})/(2y).
\end{array}$$

Substituting these into (32) above, we obtain the expressions for the reduction of first and second order Y-invariants restricted to $\mathcal{I}_X$; these are

$$\begin{array}{l}
\zeta_1^0 = (u - u_x\, x) \,/\, (2\, x) \ , \ \zeta_2^0 = 0 \ , \\
\zeta_3^0 = (2v - x v_x) \,/\, (2y) \ , \ \zeta_4^0 = 0 \ ; \\
\eta_1^0 = \left(x^3 \,(-v_x + x v_{xx}) \right) / \left(4y^2 \right) \ , \\
\eta_2^0 = \left(x^3 \,(-u + x\,(u_x + x u_{xx})) \right) / \left(4y^2 \right) \ , \\
\eta_3^0 = 0 \ , \ \eta_4^0 = 0 \ , \ \eta_5^0 = 0 \ , \ \eta_6^0 = 0.
\end{array}$$

Thus, the X-invariant solutions of (33) are obtained as solution of the reduced (system of) equations

$$H^i[y, w_1, w_2; \zeta_1^0, \zeta_3^0, \eta_1^0, \eta_2^0] \ := \ F^i \left[y, w_1, w_2; \zeta_1^0, 0, \zeta_3^0, 0; \eta_1^0, \eta_2^0, 0, 0, 0, 0 \right] = 0.$$

Example 8.

As a second example we consider the vector field

$$X \ = \ -x\, \partial_x \ + \ u\, \partial_u \tag{34}$$

and the form

$$\mu \ = \ \begin{pmatrix} \mathrm{d}x - \mathrm{d}y & 0 \\ 0 & \mathrm{d}x - \mathrm{d}y \end{pmatrix} \ ;$$

this corresponds to matrices Λ_i given by

$$\Lambda_{(x)} = I \ , \ \Lambda_{(y)} = -I.$$

The second μ-prolongation Y of X is now identified by:

$$\begin{array}{l}
\Psi_x^1 = u + u_x - u_x\,(-1 - x) \ , \\
\Psi_x^2 = -(v_x\,(-1 - x)) \ , \\
\Psi_y^1 = -u + u_y - u_x\, x \ , \\
\Psi_y^2 = -(v_x\, x) \ , \\
\Psi_{xx}^1 = u + 3\, u_x - u_x\,(-1 - x) - u_{xx}\,(-1 - x) + u_{xx}\,(2 + x) \ , \\
\Psi_{xx}^2 = v_x - v_x\,(-1 - x) - v_{xx}\,(-1 - x) + v_{xx}\,(1 + x) \ , \\
\Psi_{xy}^1 = -u - u_x + u_y + u_x\,(-1 - x) - u_{xx}\, x + u_{xy}\,(2 + x) \ , \\
\Psi_{xy}^2 = v_x\,(-1 - x) - v_{xx}\, x + v_{xy}\,(1 + x) \ , \\
\Psi_{yy}^1 = u - 2\, u_y + u_{yy} + u_x\, x - 2\, u_{xy}\, x \ , \\
\Psi_{yy}^2 = v_x\, x - 2\, v_{xy}\, x.
\end{array}$$

We can then solve the characteristic equation for the flow of Y in $J^{(2)}M$, and determine a basis of Y-invariant functions. Such a basis is now given by:

$$
\begin{array}{l}
\rho = y \ , \\
w_1 = xu \ , \\
w_2 = v \ ; \\
\zeta_1 = e^x \left(xu + x^2 u_x \right) \ , \\
\zeta_2 = e^x x v_x \ , \\
\zeta_3 = v_y - e^x x v_x \mathrm{Ei}(-x) \ , \\
\zeta_4 = x u_y - e^x (xu + x^2 u_x) \mathrm{Ei}(-x) \ ; \\
\eta_1 = e^x \left[-2 e^x x u + e^x x^3 u_{xx} + 3 e^x x (u + x u_x) + e^x x^2 (u + x u_x) \right] \ , \\
\eta_2 = e^x [e^x x v_x + e^x x^2 v_x + e^x x^2 v_{xx}] \ , \\
\eta_3 = -e^x x \{ -(u_y + x u_{xy}) + e^x [u(1+x) + x(3u_x + x u_x + x u_{xx})] \mathrm{Ei}(-x) \} \ , \\
\eta_4 = e^x x [v_{xy} - e^x (v_x + x v_x + x v_{xx}) \mathrm{Ei}(-x)] \ , \\
\eta_5 = x u_{yy} + e^x (xu + x^2 u_x) \mathrm{Ei}(-x) + \\
\quad - e^x [-2 e^x x u + e^x x^3 u_{xx} + 3 e^x (xu + x^2 u_x) + e^x x (xu + x^2 u_x)] \left(\mathrm{Ei}(-x) \right)^2 + \\
\quad + 2 e^x x \mathrm{Ei}(-x) \{ -u_y - x u_{xy} + e^x [u(1+x) + x(3u_x + x u_x + x u_{xx})] \mathrm{Ei}(-x) \} \ , \\
\eta_6 = x \, \mathrm{Ei}(-x) - e^x (e^x x v_x + e^x x^2 v_x + e^x x^2 v_{xx}) \left(\mathrm{Ei}(-x) \right)^2 + \\
\quad - 2 e^x x \mathrm{Ei}(-x) \left[v_{xy} - e^x (v_x + x v_x + x v_{xx}) \mathrm{Ei}(-x) \right].
\end{array} \tag{35}
$$

Here the function Ei is the gaussian integral

$$\mathrm{Ei}(z) \ := \ - \int_{-z}^{\infty} t^{-1} \, e^{-t} \, dt.$$

Any (system of) second order equation of the form

$$F^i \left[y, w_1, w_2; \zeta_1, \ldots, \zeta_4; \eta_1, \ldots, \eta_6 \right] = 0 \tag{36}$$

with F^i a smooth function of its arguments, admits X as a (strong) μ-symmetry.

Let us consider the restriction of the functions ζ_i, η_j on $\mathcal{I}_X$. In the present case, in order to identify $\mathcal{I}_X$ we have to solve

$$
\begin{array}{lll}
u + x u_x = 0 \ , & 2u_x + x u_{xx} = 0 \ , & u_y + x u_{xy} = 0 \ ; \\
x v_x = 0 \ , & v_x + x v_{xx} = 0 \ , & x v_{xy} = 0 \ ;
\end{array}
$$

and the manifold $\mathcal{I}_X$ is hence given by

$$
\begin{array}{lll}
u_x = -u/x \ , & u_{xy} = -u_y/x \ , & u_{xx} = 2u/x^2 \ , \\
v_x = 0 \ , & v_{xy} = 0 \ , & v_{xx} = 0.
\end{array}
$$

Substituting these into (35) above, we obtain the expressions for the reduction of first and second order Y-invariants to $\mathcal{I}_X$; these are

$$
\begin{array}{l}
\zeta_1^0 = 0 \ , \ \zeta_2^0 = 0 \ , \ \zeta_3^0 = v_y \ , \ \zeta_4^0 = x u_y \ ; \\
\eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0 \ ; \eta_5^0 = x u_{yy} \ , \ \eta_6^0 = v_{yy}.
\end{array}
$$

Thus, the X-invariant solutions of (36) are obtained as solution of the reduced (system of) equations

$$H^i[y, w_1, w_2; v_y, xu_y, xu_{yy}, v_{yy}] := F^i\left[y, w_1, w_2; 0, 0, \zeta_3^0, \zeta_4^0; 0, 0, 0, 0, \eta_5^0, \eta_6^0\right] = 0.$$

Example 9

Let us now consider again the vector field (34), but now with the form

$$\mu = \begin{pmatrix} \lambda_1 \mathrm{d}x & -\lambda_2 \mathrm{d}y \\ \lambda_2 \mathrm{d}y & \lambda_1 \mathrm{d}x \end{pmatrix} ,$$

with λ_i real constants; this corresponds to matrices Λ_i given by

$$\Lambda_{(x)} = \lambda_1 I \ , \ \Lambda_{(y)} = \lambda_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The second μ-prolongation Y of X is identified by:

$$\begin{aligned}
&\Psi^1_x = 2u_x + \lambda_1(u + xu_x) \ , \\
&\Psi^2_x = v_x(1 + \lambda_1 x) \ , \\
&\Psi^1_y = u_y - \lambda_2 x v_x \ , \\
&\Psi^2_y = \lambda_2(u + xu_x) \ , \\
&\Psi^1_{xx} = 3u_{xx} + 2\lambda_1(xu_{xx} + 2u_x) + \lambda_1^2(xu_x + u) \ , \\
&\Psi^2_{xx} = 2v_{xx} + 2\lambda_1(xv_{xx} + v_x) + \lambda_1^2 x v_x \ , \\
&\Psi^1_{xy} = 2u_{xy} + \lambda_1(u_y + xu_{xy}) - \lambda_2(v_x + xv_{xx}) - \lambda_1\lambda_2 x v_x \ , \\
&\Psi^2_{xy} = v_{xy} + \lambda_1 x v_{xy} + \lambda_2(2u_x + xu_{xx}) + \lambda_1\lambda_2(u + xu_x) \ , \\
&\Psi^1_{yy} = u_{yy} - 2\lambda_2 x v_{xy} - \lambda_2^2(u + xu_x) \ , \\
&\Psi^2_{yy} = -\lambda_2 x v_x + 2\lambda_2^2(u_y + xu_{xy}).
\end{aligned}$$

We can then solve the characteristic equation for the flow of Y in $J^{(2)}M$, and determine a basis of Y-invariant functions.

A basis of Y-invariant functions in $J^{(2)}M$ is now given by:

$$
\begin{aligned}
&\rho = y \ , \\
&w_1 = xu \ , \\
&w_2 = v \ ; \\
&\zeta_1 = e^{\lambda_1 x}\, x\,(u + u_x\, x) \ , \\
&\zeta_2 = e^{\lambda_1 x}\, v_x\, x \ , \\
&\zeta_3 = v_y - \lambda_2\,(u + u_x\, x) - e^{\lambda_1 x}\,\lambda_1\,\lambda_2\, x\,(u + u_x\, x)\,\mathrm{Ei}(-(\lambda_1\, x)) \ , \\
&\zeta_4 = (u_y + (\lambda_2\, v_x)\,/\,(\lambda_1))\, x \ ; \\
&\eta_1 = e^{2\lambda_1 x}\,(u\, x + 3\, u_x\, x^2 + u_{xx}\, x^3 + \lambda_1\,(u\, x^2 + u_x\, x^3)) \ , \\
&\eta_2 = e^{2\,\lambda_1\, x}\, x\,(v_x + \lambda_1\, v_x\, x + v_{xx}\, x) \ , \\
&\eta_3 = e^{\lambda_1 x}\,[(u_y + x u_{xy}) + (\lambda_2/\lambda_1)(2v_x + x v_{xx})] \ , \\
&\eta_4 = e^{\lambda_1 x}[x v_{xy} - \lambda_2(2x u_x + x^2 u_{xx}) - \lambda_1\lambda_2(xu + x^2 u_x)] + \\
&\quad - e^{2\lambda_1 x}[\lambda_1\lambda_2(xu + 3x^2 u_x + x^3 u_{xx}) + \lambda_1^2\lambda_2(x^2 u + x^3 u_x)][\mathrm{Ei}(-\lambda_1 x)] \ , \\
&\eta_5 = x u_{yy} + 2(\lambda_2/\lambda_1) x v_{xy} - \lambda_2^2[2x(u + x u_x) + (2/\lambda_1)(2x u_x + x^2 u_{xx})] + \\
&\quad + e^{\lambda_1 x}[\lambda_2^2(xu + x^2 u_x)]\mathrm{Ei}(-\lambda_1 x) - e^{2\lambda_1 x}[\lambda_2^2(4(xu + 3x^2 u_x + x^3 u_{xx}) + \\
&\quad + \lambda_1(x^2 u + x^3 u_x))]\mathrm{Ei}(-2\lambda_1 x) \ , \\
&\eta_6 = -2\,\lambda_2\, v_{xy} + v_{yy} + {\lambda_2}^2\,(2\,\lambda_1\,(u + u_x\, x) + 2\,(2\, u_x + u_{xx}\, x + v_x\, x) + \\
&\quad + (2\,(2\, v_x + v_{xx}\, x))\,/\,(\lambda_1)) \ + \\
&\quad + \ 2\, e^{2\,\lambda_1\, x}\,{\lambda_2}^2\, x\,(2\,(v_x + \lambda_1\, v_x\, x + v_{xx}\, x)\,\mathrm{Ei}(-2\,\lambda_1\, x) + \\
&\quad + {\lambda_1}^2\,(u + \lambda_1\, u\, x + x\,(3\, u_x + \lambda_1\, u_x\, x + u_{xx}\, x))\,\mathrm{Ei}(-(\lambda_1\, x))^2) + \\
&\quad + e^{\lambda_1\, x}\,\lambda_2\,(3\,\lambda_2\, v_x\, x + 4\,{\lambda_1}^2\,\lambda_2\, x\,(u + u_x\, x) + 2\,\lambda_1\,(-(v_{xy}\, x) + \\
&\quad + \lambda_2\,(u + 5\, u_x\, x + 2\, u_{xx}\, x^2)))\,\mathrm{Ei}(-(\lambda_1\, x)).
\end{aligned}
\tag{37}
$$

As usual, any (system of) second order equation of the form

$$F^i\,[y, w_1, w_2; \zeta_1, \ldots, \zeta_4; \eta_1, \ldots, \eta_6] = 0 \tag{38}$$

with F^i a smooth function of its arguments, admits X as a (strong) μ-symmetry.

Let us consider the restriction of the functions ζ_i, η_j on $\mathcal{I}_X$. In the present case $\mathcal{I}_X$ is given by

$$
\begin{array}{lll}
u_x = -u/x \ , & u_{xy} = -u_y/x \ , & u_{xx} = 2u/x^2 \ , \\
v_x = 0 \ , & v_{xy} = 0 \ , & v_{xx} = 0.
\end{array}
$$

Substituting these into (37) above, we obtain the expressions for the reduction of first and second order Y-invariants restricted to $\mathcal{I}_X$; these are

$$
\begin{array}{l}
\zeta_1^0 = 0 \ , \ \zeta_2^0 = 0 \ , \ \zeta_3^0 = v_y \ , \ \zeta_4^0 = x u_y \ ; \\
\eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0 \ ; \eta_5^0 = x u_{yy} \ , \ \eta_6^0 = v_{yy}.
\end{array}
$$

Thus, the X-invariant solutions of (38) are obtained as solution of the reduced (system of) equations

$$H^i[y, w_1, w_2; v_y, x u_y, x u_{yy}, v_{yy}] := F^i\left[y, w_1, w_2; 0, 0, \zeta_3^0, \zeta_4^0; 0, 0, 0, 0, \eta_5^0, \eta_6^0\right] = 0.$$

Example 10

We consider next the elementary vector field generating a scaling in x,

$$X = x\partial_x$$

and the form

$$\mu = \begin{pmatrix} -(x/y)\mathrm{d}y & -y\mathrm{d}x + (x^2 - x)\mathrm{d}y \\ -(1/y^2)\mathrm{d}y & (x/y)\mathrm{d}y \end{pmatrix} ;$$

this corresponds to matrices

$$\Lambda_x = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix} , \ \Lambda_y = \frac{1}{y^2} \begin{pmatrix} -xy & (x^2 - x)y^2 \\ 1 & xy \end{pmatrix} .$$

The coefficients of the second μ-prolongation Y are given by

$$\begin{array}{l}
\Psi^1_x = -u_x + xyv_x \ , \\
\Psi^2_x = -v_x \ , \\
\Psi^1_y = (x^2 - x^3)v_x + (x^2/y)u_x \ , \\
\Psi^2_y = (x/y^2)u_x - (x^2/y)v_x \ ; \\
\Psi^1_{xx} = -2u_{xx} + 2yv_x + 2xyv_{xx} \ , \\
\Psi^2_{xx} = -2v_{xx} \ , \\
\Psi^1_{xy} = -u_{xy} + xv_x + (x - x^2)v_x + (x^2 - x^3)v_{xx} + (x^2/y)u_{xx} + xyv_{xy} + \\
\quad +(x/y)(u_x - xyv_x) \ , \\
\Psi^2_{xy} = -v_{xy} + (x/y^2)u_{xx} - (x/y)v_x - (x^2/y)v_{xx} + (1/y^2)(u_x - xyv_x) \ , \\
\Psi^1_{yy} = 2x(x - x^2)v_{xy} - (x - x^2)[(x/y^2)u_x - (x^2/y)v_x] - (x^2/y^2)u_x + \\
\quad +2(x^2/y)u_{xy} - [(x^2/y)(x - x^2)v_x + (x^3/y^2)u_x] \ , \\
\Psi^2_{yy} = (1/y)\{-2(x/y^3)u_x + 2(x/y^2)u_{xy} + (x^2/y^2)v_x + \\
\quad -(1/y^2)[x(x - x^2)v_x + (x^2/y)u_x] - 2(x^2/y)v_{xy} + (x^2/ys)u_x - (1/(xy))v_x\}.
\end{array}$$

A basis for the Y-invariant functions in $J^{(2)}M$ is provided by:

$$\begin{array}{l}
\rho = y \ , \ w_1 = u \ , \ w_2 = v \ ; \\
\zeta_1 = xu_x - x^2yv_x \ , \\
\zeta_2 = xv_x \ , \\
\zeta_3 = (1/y)[yu_y - x^2yv_x - x(xu_x - x^2yv_x)] \ , \\
\zeta_4 = v_y - (1/y^2)\,(xu_x - x^2yv_x)\,\log(x) \ ;
\end{array}$$

$$
\begin{aligned}
&\eta_1 = x^2 u_{xx} - 2x^2 y v_x - 2x^3 y v_{xx} \ , \\
&\eta_2 = x^2 v_{xx} \ , \\
&\eta_3 = x[u_{xy} - x(2v_x + x v_{xx} + y v_{xy})] \ , \\
&\eta_4 = -(1/y^2)\{xy(x v_x + x^2 v_{xx} - y v_{xy}) + \\
&\quad + x[u_x + x(u_{xx} - 3y v_x - 2xy v_{xx})] \log(x)\} \ , \\
&\eta_5 = (1/y^2)[4x^2 u_x + 2x^3 u_{xx} + y(-2x^2 u_{xy} - 4x^3 v_x - 2x^4 v_{xx} + \\
&\quad + y u_{yy} - 2x^2 y v_{xy} + 2x^3 y v_{xy})] \ , \\
&\eta_6 = (1/y^3)\{-2x^2 y v_x - 2x^3 y v_{xx} + y^3 v_{yy} + \\
&\quad + 2x[u_x + y(x v_x + x^2 v_{xx} + xy v_{xy} - u_{xy})] \log(x)\}.
\end{aligned}
$$

As usual, any (system of) second order equation of the form

$$F^i\,[y, w_1, w_2; \zeta_1, \ldots, \zeta_4; \eta_1, \ldots, \eta_6] = 0 \ ,$$

F^i a smooth function of its arguments, admits X as a (strong) μ-symmetry.

The system identifying $\mathcal{I}_X$ is now given by

$$
\begin{aligned}
&x u_x = 0, \quad u_x + x u_{xx} = 0, \quad x u_{xy} = 0, \\
&x v_x = 0, \quad v_x + x v_{xx} = 0, \quad x v_{xy} = 0 \ ;
\end{aligned}
$$

thus $\mathcal{I}_X$ is the linear space on which

$$u_x = u_{xx} = u_{xy} = 0 \ ; v_x = v_{xx} = v_{xy} = 0.$$

Restriction of the invariant functions given above to this space yields

$$
\begin{aligned}
&\zeta_1^0 = \zeta_2^0 = 0 \ , \ \zeta_3^0 = u_y \ , \ \zeta_4^0 = v_y \ ; \\
&\eta_1^0 = \eta_2^0 = \eta_3^0 = \eta_4^0 = 0 \ , \ \eta_5^0 = u_{yy} \ , \ \eta_6^0 = v_{yy}.
\end{aligned}
$$

The reduced system is therefore

$$H^i[y, u, v; u_y, v_y, u_{yy}, v_{yy}] \ = \ F^i[y, u, v; 0, 0, u_y, v_y; 0, 0, 0, 0, u_{yy}, v_{yy}] \ = \ 0.$$

Example 11

We consider now a case in the frame mentioned in remark 5 above, i.e. with Λ_i belonging to a gauged (abelian) matrix algebra. The simplest occurrence is provided by $\mathcal{G} = so(2)$, with only one generator L, i.e.

$$L \ = \ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will consider one such example, keeping to first order equations and to a very simple vector field X, i.e.

$$X \ = \ \partial_y.$$

Also, to avoid dealing with exceedingly complicate formulas, we will select a special form for the functions λ_i $(i = 1, 2)$:

$$\lambda_i \ = \ D_i[xy + \alpha(x)] \ ; \ \Lambda_i \ = \ \lambda_i \, L.$$

In this way we get

$$\mu \ = \ \begin{pmatrix} 0 & -[(y + \alpha'(x))\mathrm{d}x + x\mathrm{d}y] \\ [(y + \alpha'(x))\mathrm{d}x + x\mathrm{d}y] & 0 \end{pmatrix} .$$

The coefficients of the first μ-prolongation of X are given by:

$$\begin{array}{ll} \Psi^1_x = \left(y^2 + y\alpha'(x)\right) v_y \ , & \Psi^1_y = -u_y + xyv_y \ , \\ \Psi^2_x = -\left(y^2 + y\alpha'(x)\right) u_y \ , & \Psi^2_y = -v_y - xyu_y. \end{array}$$

A basis of invariant functions for this is given by

$$\begin{array}{l} \rho = x \ , \ w_1 = u \ , \ w_2 = v \ ; \\ \zeta_1 = (1/x) \left[xu_x - yu_y - xy \,\mathrm{Ci}(xy) \ (\cos(xy)v_y + \sin(xy)u_y)\right) \alpha'(x) \\ \quad + xy \,\mathrm{Si}(xy) \ (\cos(xy)u_y - \sin(xy)v_y) \ \alpha'(x)] \ , \\ \zeta_2 = (1/x) \left[xv_x - yv_y + xy \,\mathrm{Ci}(xy) \ (\cos(xy)u_y - \sin(xy)v_y) \ \alpha'(x) \right. \\ \quad \left. + xy \,\mathrm{Si}(xy) \ (\cos(xy)v_y + \sin(xy)u_y) \ \alpha'(x)\right] \ , \\ \zeta_3 = -y \sqrt{(u_y^2 + v_y^2)} \ , \\ \zeta_4 = xy - \arccos\left[-u_y/\sqrt{(u_y^2 + v_y^2)}\right] . \end{array}$$

Here $\mathrm{Ci}(z)$ and $\mathrm{Si}(z)$ are the integral cosine and sine functions, defined by

$$\mathrm{Ci}(z) := -\int_z^\infty \frac{\cos(t)}{t} \, dt \ , \ \mathrm{Si}(z) := \int_0^z \frac{\sin(t)}{t} \, dt.$$

The invariant manifold $\mathcal{I}_X$ is obviously identified by $u_y = v_y = 0$. On this the invariant function ζ_4 given above is singular, while the other reduce to

$$\zeta_1 = u_x \ , \ \zeta_2 = v_x \ , \ \zeta_3 = 0.$$

We can still state that any system of equations of the form

$$F^a[x, u, v, \zeta_1, \zeta_2, \zeta_3] \ = \ 0$$

admits X as a (strong) μ-symmetry, and its X-invariant solutions are given by the system

$$H^a[x, u, v, \zeta_1, \zeta_2, \zeta_3] \ := \ F^a[x, u, v, u_x, v_x, 0] \ = \ 0.$$

10 Conclusions and outlook

In this paper, we have given a geometrical characterization of the λ-symmetry introduced by Muriel and Romero [8] and further studied by Pucci and Saccomandi [12]. This geometrical characterization is readily extended to the case of several independent variables, i.e. of PDEs. The central object in this frame is a horizontal one-form $\mu = \lambda_i(x, u, u_x)\mathrm{d}x^i$, which must satisfy some compatibility conditions. In the scalar case (one dependent variable) these are simply $D_i\lambda_j = D_j\lambda_i$, which means μ is locally the total derivative of a function $P(x, u)$, $\mu = DP = (D_iP)\mathrm{d}x^i$.

With this setting, it is also possible to deal with systems of PDEs (say q dependent variables); in this case the form μ is matrix-valued, $\mu = \Lambda_i(x, u, u_x)\mathrm{d}x^i$ with Λ_i nonsingular q-dimensional matrices. The compatibility conditions are now written as $D_i\Lambda_j - D_j\Lambda_i + [\Lambda_i, \Lambda_j] = 0$.

In this note we have not fully investigated the geometrical meaning of these; this will be discussed in a forthcoming paper [2], and is related to the Maurer-Cartan equation.

It should also be mentioned that in order to consider μ-symmetries of a given differential equation (or system) Δ, it suffices that μ satisfies the compatibility conditions on $S_\Delta \subset J^{(n)}M$ rather than on the full jet space $J^{(n)}M$. This corresponds to considering "internal μ-symmetries" rather than "external" ones [7].

We have also shown that μ-symmetries are as useful as standard ones in finding explicit (invariant) solutions of differential equations.

Let us now mention some direction of further development of our approach to μ-symmetries.

- We have shown by explicit examples that it is easy to build equations with nontrivial μ-symmetries; as for the problem of determining the μ-symmetries of a given equation, we refer to the discussion given in [4].

- In a forthcoming paper [2] we will also extend the concept of μ-symmetries to conditional and partial symmetries, and show again how μ-partial symmetries can lead to determination of explicit solutions to (systems of) PDEs.

- Finally, we also mention that μ-symmetries are related to nonlocal standard symmetries of exponential type; we refer again to [2, 4] for this and for explicit examples.

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