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Symmetries and symmetry-invariant solutions of differential equations

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Abstract. One of the typical applications of symmetry methods in the study of differential equations is the searching for symmetry-invariant solutions. I present here a review of some of the principal techniques related to this idea, together with a comparison between the various approaches, paying special attention to the notion of conditional symmetry (with a careful distinction between some different definitions), and to the concept of λ-symmetry. The close relationship between λ-symmetries for PDE's and standard symmetries is also pointed out.

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Introduction

This paper is devoted to a review and a revisitation of one of the most relevant applications of symmetry methods to the study of differential equations, namely the introduction of various techniques aimed at finding symmetry-invariant solutions of the given equation (both ordinary, ODE, and partial, PDE).

Starting from the simplest and standard case of equations admitting an "exact" symmetry (we will consider only Lie point-symmetries in the classical meaning), we will consider several generalizations, which include first of all the case of conditional symmetries (we will show that some care is needed in the introduction of this notion: indeed, different notions of conditional symmetries must be distinguished). The relationship between conditional symmetries and the more recent concept of partial symmetry will be also briefly pointed out. Great attention will be also devoted to the other recently introduced notion of λ-symmetries; their role in the study of ODEs (where they have been originally introduced) turns out to be completely different from their role in the context

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of PDEs. We will point out the close relationship existing, in the case of PDEs, between λ-symmetries and standard symmetries.

1 Exact symmetries

As already remarked, in the study of symmetry properties of a differential problem, and in the searching for its symmetry-invariant solutions, the simplest situation occurs clearly when the given equation admits an "exact" (to be distinguished from conditional, partial, and so on) symmetry. A popular example is provided by rotation-invariant equations: symmetry-invariant solutions are radial ones, as well known.

For the sake of concreteness, we shall consider here only "geometrical" or Lie point-symmetries, i.e. transformations generated by vector fields X of the form (sum over repeated indices is always understood)

$$X \,=\, \xi_i(x,u)\frac{\partial}{\partial x_i} + \varphi^a(x,u)\frac{\partial}{\partial u^a} \tag{1}$$

where $x := (x_1,\ldots,x_p)$ are the independent variables, $u := (u_1,\ldots,u_q)$ the dependent ones, and $\xi_i\,,\varphi^a$ are given smooth functions. As well known [1,10,14, 19,21,26,28], a differential equation (or a system thereof)

$$\Delta := \Delta_\alpha(x,u^{(m)}) \,=\, 0 \quad ; \quad \alpha = 1,\ldots,\nu \tag{2}$$

(where $u^{(m)}$ denotes the functions u^a together with their derivatives with respect to x_i up to the order m), admits X as an exact symmetry (or is symmetric under X) if the following condition is satisfied

$$X^*(\Delta)|_{\Delta=0} \,=\, 0 \tag{3}$$

with usual notations and standard assumptions (as stated, e.g. , in [19]); we have denoted by X^* the "appropriate" prolongation of X for the equations at hand (or – alternatively – its infinite prolongation, indeed only a finite number of terms will appear in calculations). A condition equivalent to (3) (at least under mild hypotheses, see [19]) is the existence of a matrix functions $G = G_{\alpha\beta}(x,u^{(m)})$ such that

$$\left(X^*(\Delta)\right)_\alpha \,=\, G_{\alpha\beta}\,\Delta_\beta. \tag{4}$$

The equation (or system of equations) $\Delta = 0$ is said to be invariant under X (or to admit X as a *strong* symmetry) if

$$X^*(\Delta) \,=\, 0 \tag{5}$$

i.e., if $G_{\alpha\beta} = 0$. It can be shown [1, 4, 6, 19, 23] that if any $\Delta = 0$ is symmetric under a vector field X, then, rewriting the equation $\Delta = 0$ in terms of canonical coordinates (see below), the problem can be replaced by an equivalent problem which is invariant under X.

Given a differential problem $\Delta = 0$ which admits a symmetry X, its solutions which are invariant under the symmetry X are determined, as well known, by the system of q conditions

$$X_Q\, u^a := \varphi^a \ - \xi_i \,\frac{\partial u^a}{\partial x_i} \ = \ 0 \tag{6}$$

where $X_Q = \varphi^a(\partial/\partial u^a) - \xi_i(\partial/\partial x_i)$ is the vector field X written in "evolutionary form" [19].

Let us assume from now on, and unless stated otherwise, that our problem is a single PDE for only one independent variable $u = u(x)$ (i.e. $q = 1$); then, from the solution of the first-order PDE $X_Q u = 0$, one determines a set of new coordinates s, z and w, called canonical or symmetry-adapted variables (see e.g. [1, 19] and also [6]), where z, w are $X-$invariant variables, and the vector field X, and its prolongation X^* as well, take the form

$$X \ = \ X^* \ = \ \frac{\partial}{\partial s} \tag{7}$$

Expressing the original equation in terms of the new variables z, w, one obtains a *reduced* differential equation, say

$$\widehat{\Delta}(z, w) = 0 \tag{8}$$

for the $X-$invariant function $w = w(z)$. This procedure is well known, and several relevant examples can be found in the literature, and we do not insist on this point. See [23, 30] for a careful discussion about this procedure. If the PDE involves a function of two dependent variables only, then the reduced equation is just a ODE.

2 Conditional and "weak" conditional symmetries

It is well known that the idea and the method presented in the above section work even if X is not an exact symmetry for the problem, but is a *conditional* symmetry [2, 3, 9, 15, 27]. According to a current definition, a vector field X is a conditional symmetry for $\Delta = 0$ if it is an exact symmetry for the system

$$\Delta \ = \ 0 \qquad\qquad Q := X_Q u := \varphi - \xi_i u_i \ = \ 0 \tag{9}$$

(clearly, $u_i = \partial u/\partial x_i$), or if

$$X^*(\Delta)|_\Sigma = 0 \tag{10}$$

where Σ is the set of the simultaneous solutions of the two equations in (9), plus (possibly) some differential consequences of the second one: indeed, the equation $Q = 0$, which is a first-order equation, must be used together with the differential equations $\Delta = 0$ and $X^*(\Delta) = 0$ which are in general of order larger than one; then also differential consequences of $Q = 0$ must be introduced. Therefore—in defining conditional symmetry in this sense—it is understood that the system (9) must include the suitable differential consequences of the "invariant-surface condition" $Q = 0$, and X is requested to be an exact symmetry of this enlarged system (see [19, 20, 22, 23, 29] for a precise and detailed discussion on this point and the related notion of degenerate systems of PDE).

But at this point a careful distinction is necessary.

Indeed, another definition is that X is a conditional symmetry for $\Delta = 0$ if the system (9) admits some solutions.

While it is true that if $\widehat{u}$ is a solution to (9) then also the following

$$X^*(\Delta)|_{u=\widehat{u}} = 0 \tag{11}$$

is satisfied, the converse of the statement is – strictly speaking – not true. Consider for instance (this example is due to Olver and Rosenau [20]) the heat equation $u_t - u_{xx} = 0$ and the vector field

$$X \;=\; x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + 3x^3\frac{\partial}{\partial u} \tag{12}$$

It is easy to verify that $X^*(\Delta)|_\Sigma$ does not vanish identically; on the other hand, if one looks for solutions of the system (9), one finds from $Q = 0$

$$u \;=\; w(z) + x^3 \qquad \text{with} \qquad z \;=\; xt$$

and substituting into the heat equation gives

$$t^3 w_{zz} - z(w_z - 6) \;=\; 0 \tag{13}$$

which is not a "pure" ODE for $w(z)$, but an equation containing also the "non-invariant" variable t. One may actually obtain $X-$invariant solutions to equation (13) by imposing $w_{zz} = 0$ and $w_z = 6$, which produces some solutions $\widehat{u}$ which in fact satisfy (11). But this situation is clearly different from the "pure" conditional symmetry case, i.e. where (10) is identically satisfied. What happens in the above example is that (12) is a symmetry of a new system which is obtained enlarging (9) according to a different prescription, i.e. in such a way to include all compatibility conditions of the differential consequences of both equations in (9) (or the “integrability conditions”) [19,20,23]. The counterpart of this fact is that, while in the first case, i.e. according to the first definition of conditional symmetry, the reduced equation is a *single* equation for $w = w(z)$ (just as in the case of exact symmetry, see (8)), in the other case one obtains a “combination” of equations of the form, as in (13)

$$R_1(s,z,w)\widehat{\Delta}_1(z,w) + R_2(s,z,w)\widehat{\Delta}_2(z,w) + \ldots = 0 \tag{14}$$

where R_ℓ are some coefficients depending also on some non-invariant coordinate s. Clearly, if one looks for invariant solutions, one must solve the system of equations $\widehat{\Delta}_\ell = 0$ in (14); in particular, when only two independent variables are involved, in the first case the reduced equation is a single ODE, whereas in the second case, as in the example above, the reduced equation takes the form (14) of a combination of ODE's.

Let us now remark that, in the example above, appending to the system (9) the new equation $\Delta^{(1)} := X^*(\Delta) = 0$, i.e. $u_t + 2u_{xx} - 18x = 0$, the vector field X becomes an exact symmetry of this augmented system. (It should be clear that this new equation is not to be confused with the differential consequences of $\Delta = 0$.) This suggests the following definition, which allows us to introduce a precise “classification” which distinguishes different types of conditional symmetries.

1 Definition. Given a PDE $\Delta = 0$, a vector field X is a *weak* conditional symmetry of order $r > 1$ if the system

$$\Delta = 0\ ,\ \Delta^{(1)} := X^*(\Delta) = 0, \ldots, X^*(\Delta^{(r-1)}) = 0 \tag{15}$$

$$Q = 0 \tag{16}$$

together with the differential consequences of (16), admits X as an exact symmetry, or – equivalently – if X satisfies

$$Q = \Delta = \Delta^{(1)} = \ldots = \Delta^{(r)} = 0. \tag{17}$$

Clearly, the case $r = 1$ corresponds to the first definition of “standard” or “true” conditional symmetry. It should be remarked incidentally that systems of equations as above may have no solutions at all. It is well known indeed that, even in the case of exact symmetries, it can happen – although unlikely – that the system $\Delta = Q = 0$ does not admit solutions, i.e. that no invariant solutions exist. It is therefore understood that all the systems we are going to consider from now on, do admit some solutions.

2.1 Partial symmetries

Let us remark that, if in the previous Definition one removes the invariance condition $Q = 0$, then one would obtain the definition of *partial symmetry* [5,7]; indeed, if the system (15) admits, with $r > 1$, the symmetry X, then the solutions to (15) form a *proper* subset of the solution manifold S_Δ of the initial equation $\Delta = 0$, and this subset is a *symmetric set of solutions*, i.e. a subset of solutions which are transformed into one another by the transformations generated by X. This set, globally invariant under X, may contain or not $X-$invariant solutions; accordingly, partial symmetries generalize the notion of conditional symmetry and weak conditional symmetry, and are in some sense intermediate between conditional symmetries and exact symmetries (which clearly correspond to $r = 1$ in the definition (15)).

2.2 Examples

Let us illustrate and compare the different notions of symmetries considered up to now by means of some examples. These examples look really extremely simple, however it can be remarked that in all of them the vector field involved is the generator of translations along one variable (denoted in the examples below by t), and this is precisely the form of *any* vector field once written in canonical coordinates, see (7); therefore, the following examples can be viewed in some sense as the prototypical examples of the different notions of symmetries introduced above (see also [6]).

1) Consider the PDE for $u = u(x,t)$

$$\Delta := u_t + t(u_{xx} - u) = 0$$

The vector field

$$X = \frac{\partial}{\partial t} \tag{18}$$

is a "true" conditional symmetry for this equation, indeed the condition $Q = 0$, i.e. $u_t = 0$, is (trivially) an exact symmetry for the system $\Delta = Q = 0$, and – accordingly – produces a reduced equation $u_{xx} = u$ which is a "pure" ODE, as in (8).

2) The same vector field (18) is instead a weak conditional symmetry for the equation

$$\Delta := u_t + u_{xx} - u + t(u_x - u) = 0$$

Indeed, it is an exact symmetry of the enlarged system (notice in particular that $X^* = \partial/\partial t$ and therefore conditions (15) greatly simplify, see [6])

$$Q = u_t = 0 \quad , \quad \Delta = 0 \quad , \quad \Delta^{(1)} = \frac{\partial \Delta}{\partial t} = u_x - u = 0$$

and the invariance condition $u_t = 0$ produces, as expected, a "combination" of two ODE's, as in (14).

3) In the case of the ODE

$$\Delta := u_t - u_{tt} + u_{xx} - u + t(u_x - u) = 0$$

the vector field (18) is both a weak conditional symmetry and a partial symmetry. Indeed, the invariance condition $u_t = 0$ gives as above a combination of two ODE's with invariant solution $u = c \exp x$. But if we relax the condition $Q = 0$, and consider the additional condition $\Delta^{(1)} = X^*(\Delta) = 0$ which is now $u_x - u = 0$, we see that $X^* = \partial/\partial t$ is an exact symmetry of the enlarged system $\Delta = \Delta^{(1)} = 0$, and we find the larger "symmetric set" of solutions $u = c \exp x + c_1 \exp(x + t)$.

For other examples of weak and partial symmetries see [5–7].

3 $\lambda-$symmetries

Another important generalization of the notion of exact symmetries, also related to the problem of finding invariant solutions, is provided by the notion of $\lambda-$symmetries [13, 17, 18, 24].

We only recall here the basic definitions, starting from the case of a single dependent variable $u = u(x)$ of the p independent variables x.

2 Definition. Given p smooth functions $\lambda_i = \lambda_i(x, u^{(1)})$, satisfying the compatibility conditions

$$D_i \lambda_j = D_j \lambda_i \tag{19}$$

where D_i denotes the total derivative with respect to x_i, the (infinite) λ-prolongation $X^*_{[\lambda]}$ of a vector field X

$$X \;=\; \xi_i(x, u) \frac{\partial}{\partial x_i} + \varphi(x, u) \frac{\partial}{\partial u}$$

is defined by (see [11, 12, 17, 18])

$$X^*_{[\lambda]} \;=\; X \;+\; \sum_J \Phi^{[\lambda]}_J \frac{\partial}{\partial u_J} \tag{20}$$

where $J = (j_1, \ldots, j_p)$ are multiindices and the coefficients $\Phi^{[\lambda]}_J$ are defined recursively, putting for simplicity $\Psi_J := \Phi^{[\lambda]}_J$, by

$$\Psi_{J,i} \;=\; (D_i + \lambda_i)\, \Psi_J \;-\; u_{J,k}\, (D_i + \lambda_i)\, \xi_k \tag{21}$$

with $\Psi_0 = \varphi$.

Clearly, standard prolongation [19] is recovered if $\lambda_i = 0$. For instance, the relationship between the coefficients Ψ_J of the second $\lambda-$prolongation and the coefficients Φ_J of the standard prolongation of the same vector field X is

$$\Psi_i \;=\; \Phi_i + \lambda_i\, Q \;\;;\; \Psi_{ij} \;=\; \Phi_{ij} + \lambda_i(D_jQ) + \lambda_j(D_iQ) + (D_i\lambda_j + \lambda_i\lambda_j)Q \quad (22)$$

where $Q = \varphi - \xi_i u_i$.

It is not difficult to see (cf. [11,12]) that the coefficients Ψ_J differ from the coefficients Φ_J of the standard prolongation by terms which are linear combinations of Q and of its differential consequences D_JQ. It follows that $\Psi_J = \Phi_J$ on the subspace $\mathcal{I}_X$ of X-invariant functions, identified by the vanishing of Q and D_JQ for all multiindices J. This also implies that the space $\mathcal{I}_X$ is invariant under $X^*_{[\lambda]}$. See [11,12] for details.

We now give the other basic definition related to this notion.

3 Definition. Given p smooth functions $\lambda_i = \lambda_i(x, u^{(1)})$, satisfying (19), a differential equation $\Delta = 0$ is said to be $\lambda-$symmetric under a vector field X if

$$X^*_{[\lambda]}\, \Delta|_{\Delta=0} \;=\; 0 \quad (23)$$

or, in a more refined geometrical language, a vector field X in the space M of independent and dependent variables is a λ-symmetry of the differential equation $\Delta = 0$ of order n, if the λ-prolonged vector field in the jet space $J^{(n)}M$ is tangent to the solution manifold $S_\Delta \subset J^{(n)}M$.

The notion of $\lambda-$symmetry admits a very interesting and deep geometrical interpretation, which however goes beyond the scope of this presentation [8,11, 12].

Notice that we have assumed here that $\lambda_i = \lambda_i(x, u^{(1)})$; this guarantees that if X is a Lie point-symmetry, then its λ-prolongation is a proper (rather than generalized) vector field in each jet space $J^{(k)}M$; the possibility of extending our discussion to more general functions λ_i, and therefore to generalized λ-symmetries, is certainly open and interesting, but we will not consider here this generalization.

To a clear examination and comparison of the peculiar properties of λ-symmetries it is convenient to distinguish their application to different cases. Although the notion of $\lambda-$symmetries has been introduced in the context of ODE's, where they play a very relevant role [17,18,24], it is more convenient for our purposes to postpone a (short) discussion of the case of ODE's after that of "scalar" PDE's.

1) "Scalar" PDE's

Let $\Delta = 0$ be a PDE for a single function $u = u(x)$ (i.e. $q = 1$) of the p variables x. It can be proved the following [8]:

4 Theorem. *Given p functions $\lambda_i(x, u^{(1)})$ satisfying (19) and a vector field X, the $\lambda-$prolongation $X^*_{[\lambda]}$ of X is proportional ("collinear") to the* standard *prolongation of another vector field $\widetilde{X}$. Precisely, there exists a nonvanishing function $\gamma = \gamma(x, u)$ satisfying*

$$\lambda_i \;=\; D_i\,\gamma/\gamma \;=\; D_i(\log|\gamma|) \tag{24}$$

such that

$$X^*_{[\lambda]} \;=\; \gamma^{-1}\,\widetilde{X}^* \qquad \textit{where} \qquad \widetilde{X} \;=\; \gamma\,X. \tag{25}$$

As a consequence, if X is a $\lambda-$symmetry for a PDE $\Delta = 0$, then there exists a vector field $\widetilde{X}$ defined as above which is a standard symmetry for this equation, and viceversa.

The construction of the function γ starting from equations (19) corresponds to finding a "potential" function $P(x, u)$ such that $\lambda_i = D_i(P)$; clearly, some difficulty may arise if the problem is posed in the large; then in general the result may hold only locally. For a full and detailed discussion on this point, see [8, 11, 12].

To illustrate the above theorem, let us write explicitly the coefficients $\widetilde{\Phi}_i$, $\widetilde{\Phi}_{ij}$ of the standard second prolongation of the vector field $\widetilde{X} = \gamma X$ and compare with the analogous coefficients Φ_i, Φ_{ij} of X and respectively $\Psi_i\,,\Psi_{ij}$ (see (22)) of its $\lambda-$prolongation: one finds, using also (24),

$$\begin{aligned}
\widetilde{\Phi}_i \;&=\; \gamma\,\Phi_i + (D_i\gamma)Q = \;\gamma\,\Psi_i\\
\widetilde{\Phi}_{ij} \;&=\; \gamma\,\Phi_{ij} + (D_i\gamma)(D_jQ) + (D_j\gamma)(D_iQ) + (D_{ij}\gamma)\,Q = \;\gamma\,\Psi_{ij}
\end{aligned}$$

Let us now remark that if a vector field X_0 is an exact symmetry of a given equation $\Delta = 0$, then $\beta\,X_0$, for any function $\beta(x, u)$, is in general not an exact symmetry, but only a (standard) conditional symmetry for that equation. We can then characterize $\lambda-$symmetries in this way:

5 Corollary. *A vector field X is a $\lambda-$symmetry of $\Delta = 0$ if and only if it is a conditional symmetry of the special form $X = \beta\widetilde{X}$ where $\widetilde{X}$ is an exact symmetry of $\Delta = 0$.*

It is also clear that, for what concerns the problem of finding invariant solutions, the procedure in the case of $\lambda-$symmetries is exactly the same as for standard symmetries, and that the invariant solutions under $\lambda-$symmetries are exactly the same as the invariant solutions under the "collinear" exact symmetry $\widetilde{X} = \gamma\,X$ defined above.

2) ODE's

The notion of $\lambda-$symmetry is particularly useful in the context of ODE's, where indeed they are able to provide a reduction of the order of the equation,

precisely as in the case of exact symmetries [17, 18, 24]. Here, the situation is completely different with respect to the case of PDEs, at least for two basic reasons:
a) First of all, the introduction of the collinear "exact" symmetry $\widetilde{X}$ is here not always possible. There are indeed several possibilities: in the cases where a function $\gamma(x,u)$ satisfying $\lambda = D_x(\gamma)/\gamma$, as in (24), can be found, then the λ-symmetry is equivalent to the standard symmetry $\widetilde{X}$, according to Theorem 1. But if e.g. $\lambda = \lambda(u)$, then γ takes the form

$$\gamma(x,u) \;=\; \exp\Big[\int^x \lambda(u(x'))\mathrm{d}x'\Big]$$

and the standard symmetry $\widetilde{X}$ would become a *nonlocal* symmetry (see [17, 18, 24], and also [8] for extending the notion of exponential nonlocal symmetries). But clearly it can also happen that no function γ is admitted (e.g., if $\lambda = u_x^2$).
b) Secondly, $\lambda-$symmetries in the case of ODE's not only lead to a reduction of the order of the equation, but are also able to give the most general solution of the equation; in the case of PDE's, the reduction is drastically different being obtained via the *restriction* to invariant solutions, i.e. the solutions satisfying the condition $Q = 0$.

We do not insist on this point, which is well known and admits many interesting applications, and relevant generalizations as well (e.g. the notion of *telescopic vector field* [24]).

3) "Vector" PDE's

We consider finally the case of "vector" PDE's, i.e. PDE involving $q > 1$ functions $u^a = u^a(x)$. Now the definitions of λ–prolongation and of λ–symmetry are more involved. We have to introduce p square $(q\times q)$ matrices $\Lambda_i = \Lambda_i(x, u^{(1)})$, depending on x_i, u^a and on the first-order derivatives $\partial u^a/\partial x_i$ (at least if we want to consider only proper and not generalized symmetries, as already remarked), and satisfying the new compatibility conditions [11, 12]

$$D_i\Lambda_j - D_j\Lambda_i + [\Lambda_i,\Lambda_j] \;=\; 0 \tag{26}$$

Given the vector field X

$$X \;=\; \xi_i(x,u)\frac{\partial}{\partial x_i} + \varphi^a(x,u)\frac{\partial}{\partial u^a}$$

the coefficients $\big(\Phi^{[\lambda]}\big)^a_J$, which shall be denoted by Ψ^a_J (or by Ψ_J when no confusion is possible) of its $\lambda-$prolongation

$$X^*_{[\lambda]} \;=\; X + \sum_a\sum_J \Psi^a_J\frac{\partial}{\partial u^a_J}$$

are now defined recursively by

$$\Psi^a_{J,i} \;=\; [D_i\delta^{ab} + (\Lambda_i)^{ab}]\Psi^b_J \;-\; (u^a)_{J,k}D_i\xi_k - (\Lambda_i)^{ab}(u^b)_{J,k}\xi_k \tag{27}$$

with $\Psi^a_0 = \varphi^a$ and $u^a_i = \partial u^a/\partial x_i$. For instance, we get

$$\Psi^a_i \;=\; \Phi^a_i + (\Lambda_i Q)^a$$

where Φ^a_J are the coefficients of standard prolongation [19], and $Q^a = \varphi^a - \xi_i u^a_i$; similarly,

$$\Psi^a_{ij} \;=\; \Phi^a_{ij} + (\Lambda_i(D_jQ))^a + (\Lambda_j(D_iQ))^a + ((D_i\Lambda_j + \Lambda_i\Lambda_j)Q)^a$$

One can see in particular that the identity $\Psi_{ij} = \Psi_{ji}$ is guaranteed precisely by the compatibility condition (26).

The results of Theorem 1 can be essentially extended also to this case; to this end, it is convenient to write the vector field X in its evolutionary from, namely

$$X_Q := \; Q^a\frac{\partial}{\partial u^a} \;=\; (\varphi^a - \xi_i u^a_i)\frac{\partial}{\partial u^a} \tag{28}$$

Introducing the shorthand notation Y for the (infinite) λ-prolongation of X_Q, we then have the following:

6 Theorem. *Let Y be the λ-prolongation of the vector field X_Q with some given $q \times q$ matrices Λ_i satisfying (26). Then there exists an invertible matrix function $\gamma = \gamma(x,u)$ such that*

$$Y \;=\; \gamma^{-1}X^*_{\gamma Q} \tag{29}$$

where $X^*_{\gamma Q}$ is the standard *prolongation of the evolutionary vector field*

$$X_{\widetilde{Q}} := \gamma X_Q \;=\; X_{\gamma Q} \tag{30}$$

The matrix γ is related to the Λ_i by the equation

$$\Lambda_i \;=\; \gamma^{-1}(D_i\gamma) \tag{31}$$

As a consequence, if X_Q is a λ-symmetry for a system of equations $\Delta_\alpha = 0$, then there is a vector field $X_{\widetilde{Q}} = X_{\gamma Q}$ which is standard symmetry for $\Delta_\alpha = 0$.

Proof (a sketch). The proof requires some refined techniques from differential geometry [8,16,25]. The idea is the following: introducing the operator ∇_i

$$(\nabla_i)^{ab} := D_i\delta^{ab} + (\Lambda_i)^{ab} \tag{32}$$

we immediately deduce that the compatibility condition (26) becomes

$$[\nabla_i, \nabla_j] \;=\; 0 \tag{33}$$

which generalizes the scalar condition (19). It can be shown that this implies the existence (at least locally) of an invertible matrix function $\gamma(x,u)$ such that (29) is satisfied. Conversely, if Λ_i satisfy (29), then also (26) (or (33)) hold, indeed

$$\begin{aligned} D_i\Lambda_j \;&=\; D_i(\gamma^{-1})D_j(\gamma) + \gamma^{-1}D_{ij}(\gamma) = \gamma^{-1}D_i(\gamma)\gamma^{-1}D_j(\gamma) + \gamma^{-1}D_{ij}(\gamma) \\ &=\; \Lambda_i\Lambda_j + \gamma^{-1}D_{ij}(\gamma) \end{aligned}$$

It now remains to show that for the coefficients Ψ^a_J of Y one has $\Psi^a_J = (\gamma^{-1})^{ab}\widetilde{\Phi}^b_J = (\gamma^{-1}\widetilde{\Phi}_J)^a$ where $\widetilde{\Phi}^a_J$ are the coefficients of the standard prolongation of $X_{\widetilde{Q}} = X_{\gamma Q}$. Let us consider only the first prolongation (higher-orders may be dealt with recursively):

$$\widetilde{\Phi}^a_i \;=\; D_i(\gamma Q)^a \;=\; (D_i\gamma^{ab})Q^b + \gamma^{ab}D_iQ^b = \gamma^{ab}\Big((\Lambda Q)^b + D_iQ^b\Big) = (\gamma\Psi_i)^a$$

thanks to (29). $\boxed{QED}$

3.1 Partial and weak conditional λ-symmetries

Clearly, also *partial* λ-symmetries (see Section 2.1) for PDEs can be introduced [8]. Indeed, given a PDE (or a system thereof) $\Delta = 0$ and a vector field X_Q with λ-prolongation Y, let us assume that

$$Y\,(\Delta)\,|_{\Delta=0} \;\neq 0 \tag{34}$$

Then, we can consider $\Delta^{(1)} := Y(\Delta) = 0$ as a new equation and check whether Y is a λ-symmetry for the enlarged system $\Delta = \Delta^{(1)} = 0$ (possibly iterating the procedure to some order $r > 1$), exactly as in (15). It is now clear that, if this is the case, the reduction procedure by means of the condition $Q = 0$ will transform the partial λ-symmetry into a weak conditional symmetry, giving rise to equations of the form (15). The final example in this paper will illustrate this case.

3.2 Examples

We have seen that any λ-symmetry X can be replaced by an equivalent standard symmetry $\widetilde{X} = \gamma X$. As already remarked, this equivalence is granted in general only locally, being connected to the existence of a function (or a matrix function) γ which is obtained by "integration" of the differential form

$\mu := \lambda_i \mathrm{d}x_i$ (or $\mu := \Lambda_i \mathrm{d}x_i$) via the compatibility conditions (19) (or (26)). For a full discussion of this point and of various related differential geometric aspects we refer to [8,11,12]. Let us also point out that actually the compatibility conditions (19) or (26) need not hold necessarily in all points of the space $J^{(n)}M$, but only along the solution manifold S_Δ of the differential problem $\Delta = 0$; pursuing this idea, one could show that in this case there is an equivalence of λ-symmetries with standard "nonlocal symmetries of exponential type", but we will not consider here this possibility (see [8]).

We will now give an example of a λ-symmetry for a PDE which is only locally equivalent to a standard symmetry: this fact is reflected by the presence of a multivalued function γ. The two other examples presented below concern systems of two PDEs; in both cases we will give the matrix γ, and obtain the symmetry-invariant solution imposing the invariance condition $Q^a = 0$, exactly as in the case of conditional symmetries (specifically: in Example 2 a standard conditional symmetry is involved, in the final Example a weak conditional symmetry of order $r = 2$).

1) This is an example in the punctured plane $\mathbf{R}^2 - \{0\}$. With $x_1 = x\,, x_2 = y$, let X be given by the standard rotation generator

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

Writing for ease of notation $r^2 = x^2 + y^2$, let $\lambda_1 = (-y/r^2)$ and $\lambda_2 = (x/r^2)$; this corresponds to

$$\gamma \ = \ \exp[\arctan(y/x)] \ := \ \exp(\theta) \ ,$$

and then to the vector field $\widetilde{X} = [\exp(\theta)]\partial_\theta$. Note that here γ is well defined only locally, as it is a multivalued function. One can check that

$$\zeta_1 \ := \ e^\theta u_\theta \quad \text{and} \quad \zeta_2 \ := \ e^{2\theta}(u_{\theta\theta} + u_\theta)$$

are invariant under the λ-prolongation of the above vector field X. In the x, y coordinates (but retaining the notation $\theta := \arctan(y/x)$) these read

$$\begin{aligned} \zeta_1 &= (\exp\theta)\ (xu_y - yu_x), \\ \zeta_2 &= (\exp(2\theta))\ \ (y^2u_{xx} + x^2u_{yy} - 2xyu_{xy} - xu_x - yu_y + xu_y - yu_x\ ). \end{aligned}$$

Let ζ_3 be any smooth nontrivial function $\zeta_3 = \zeta_3(r, u_r, u_{rr})$. Then any PDE of the form

$$\Delta \ := \ F(\zeta_1, \zeta_2, \zeta_3) \ = \ 0$$

is a second-order equation invariant under the λ-prolongation of X. Its symmetry reduction gives an ODE of the form

$$\widehat{\Delta} \ = \ \widehat{F}(r, w_r, w_{rr}) \ := \ F(0, 0, \zeta_3) \ = \ 0$$

for the function $u = w(r)$.

2) In this and the last example we will consider the case of two dependent variables (i.e. $q = 2$) and two independent variables; we shall write $u^1 = u(x, y)$ and $u^2 = v(x, y)$. It is not difficult to verify that any system of the form

$$\begin{aligned} u_x &= -v_x G(x) + v_x f(y, u, v, v_y, x v_x) \\ u_y &= x v_x \log(|v_x|) + g(y, u, v, v_y, x v_x) \end{aligned}$$

where G and f, g are arbitrary functions of the indicated arguments, admits the vector field

$$X \ = \ x \frac{\partial}{\partial x}$$

as $\lambda-$symmetry with Λ_i given by

$$\Lambda_1 = \begin{pmatrix} 0 & G_x(x) \\ 0 & 0 \end{pmatrix} \quad ; \quad \Lambda_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The matrix γ is given by

$$\gamma = \begin{pmatrix} 1 & G(x) + y \\ 0 & 1 \end{pmatrix}$$

The reduction, imposing $Q^a = 0$, i.e., $u_x = v_x = 0$, gives $u = u(y)$, $v = v(y)$; the first equation turns out to be identically satisfied, and the second one becomes an equation involving u, v with their derivatives with respect to y.

3) With the same notations as in the example above, now consider the system

$$\begin{aligned} u_x &= -v_x \log(|v_x|) + v \\ v_x &= 2v_y - y^2 + u_y + (v_x - v_y)^2 \end{aligned}$$

with the vector field

$$X \ = \ \frac{\partial}{\partial x} + v \frac{\partial}{\partial u}$$

and with

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

corresponding to a matrix function γ

$$\gamma = \begin{pmatrix} \exp x & 0 \\ 0 & \exp(x + y) \end{pmatrix}$$

Direct computation shows that X is a partial λ-symmetry of this system: indeed, according to Section 3.1, the first application of the λ-prolongation does not give zero but produces the new system

$$0 = 0 \qquad ; \qquad v_x = v_y$$

and one needs another application of the λ-prolongation of X. The reduction procedure, imposing $Q^a = 0$, i.e. $u_x = v$, $v_x = 0$ or $v = v(y)$, $u = w(y) + xv(y)$, gives

$$w_y + xv_y + v_y^2 + 2v_y - y^2 = 0$$

which has the form (15), as expected in the case of weak λ-symmetries (cf. Definition 1 and Sections 2.1 and 3.1); this forces $v_y = 0$ and leads to the solution $u = y^3/3 + cx$, $v = c =$ const.

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