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Transitive Subgeometry Partitions

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Abstract. The subgeometry partitions of $PG(n-1, q^m)$ that admit point-transitive groups are completely determined as those partitions by subgeometries isomorphic to $PG(n/(n,m)-1, q^{(n,m)})$ arising from $AG(m, q^n)$.

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1 Introduction

Recently, new types of subgeometry partitions of $PG(n-1, q^m)$ by subgeometries isomorphic to $PG(n-1, q)$ are constructed in Johnson [1]. These partitions are called 'm^{th}-root subgeometry partitions', where $(m, n) = 1$ and certain other restrictions apply. Furthermore, these subgeometries admit a group acting transitively on the set of subgeometries. Actually, other than these examples, very little is known about general subgeometry partitions. Are there other examples?

In this note, we show that every affine space $AG(m, q^n)$ produces subgeometry partitions of $PG(n-1, q^m)$ by subgeometries isomorphic to $PG(n/(m,n)-1, q^{(m,n)})$. Indeed, these subgeometry partitions admit a collineation group that is transitive on the points of $PG(n-1, q^m)$. Probably the main point is that it is easy to construct subgeometry partitions but we also show, using a straightforward application of results of Kantor, that the converse theorem is valid and prove the following general theorem.

1 Theorem. *A subgeometry partition of a projective space admits a point transitive group if and only if it is the subgeometry arising from an affine space $AG(m, q^n)$, producing a partition of $PG(n-1, q^m)$ by subgeometries isomorphic to $PG(n/(n,m)-1, q^{(n,m)})$.*

2 Field Subgeometry Partitions

Consider $GF(q^{mn})^*$. Define "points" as elements of $GF(q^{mn})^*$ and "components" as the images of $GF(q^n)$ under the group $GF(q^{mn})^*$. Then there are $(q^{mn}-1)/(q^n-1)$ components, so we have a transitive spread of n-dimensional $GF(q)$-subspaces of the associated m-dimensional vector space over $GF(q^n)$. That is, suppose $GF(q^n)$ and $GF(q^n)a$ for $a \in GF(q^{mn})^*$ share a non-zero element b, so that $b = ca$, for $b, c \in GF(q^n)^*$, then $c \in GF(q^n)$. Consider the group $GF(q^m)^*$. This group acts on the spread and the subgroup that fixes the component $GF(q^n)$ is $GF(q^{(m,n)})$. Since the original group is cyclic and transitive, we see that $GF(q^{(m,n)})$ fixes each component, and is then considered the 'kernel' of the group. Take an orbit Γ of length $(q^m-1)/(q^{(m,n)}-1)$ under $GF(q^m)^*$. Form the projective space $PG(n-1, q^m)$. We claim that Γ becomes a subgeometry isomorphic to $PG(n/(m,n)-1, q^{(m,n)})$. Most of the proof of the previous is accomplished in Johnson [2], where it is shown that the point-line geometry is at least a 'quasi-subgeometry'. It remains to show that we actually obtain a subgeometry.

To see this we may take without loss of generality the orbit that contains $GF(q^n)^* = L$. We need to show that a 2-dimensional $GF(q^m)$-space generated from two linearly independent vectors in L intersects Γ in exactly $(q^{(m,n)}-1)$ $GF(q^m)$-1-spaces.

So, let u and v be on L, that is, let $u, v \in GF(q^n)^*$ such that $\{u, v\}$ is linearly independent over $GF(q^m)$. Suppose that w is in Γ and in $\langle u, v\rangle_{GF(q^m)} - \{\langle u\rangle_{GF(q^m)}, \langle v\rangle_{GF(q^m)}\}$. Then since we have an orbit under $GF(q^m)$, we may assume that $w \in L = GF(q^n)^*$. Therefore,

$$\alpha u + \beta v = w,$$

where $\alpha, \beta \in GF(q^m)$ and $\alpha\beta \neq 0$. Since $u, v, w \in GF(q^n)$, then $(\alpha u + \beta v)^{q^n} = w = (\alpha^{q^n} u + \beta^{q^n} v)$, which implies that

$$(\alpha - \alpha^{q^n})u + (\beta - \beta^{q^n})v = 0.$$

However, $\sigma \in GF(q^m)$, implies $\sigma^{q^n} \in GF(q^m)$, so since u and v are linearly independent over $GF(q^m)$, it follows that

$$(\alpha - \alpha^{q^n}) = (\beta - \beta^{q^n}) = 0.$$

Therefore, $\alpha^{q^n} = \alpha$, implies that $\alpha^{(q^n-1, q^m-1)} = 1 = \alpha^{(q^{(m,n)}-1)}$. So, α and $\beta \in GF(q^{(m,n)})$. Hence, we obtain a subgeometry isomorphic to $PG(n/(m,n)-1, q^{(m,n)})$. The 'points' are the $GF(q^m)$-subspaces, which are permuted transitively by the cyclic group $GF(q^{mn})^*$.

2 Theorem. *From any field $GF(q^{mn})$, there exists a subgeometry partition of $PG(n-1, q^m)$ by subgeometries isomorphic to $PG(n/(m,n)-1, q^{(m,n)})$.*

This particular subgeometry admits a collineation group that acts transitive on subgeometries and transitive on the points.

PROOF. If a subgeometry admits a collineation group that acts transitively on the points then we have a group transitive on the $GF(q^m)^*$-orbits, of an associated vector space. But, this implies that we have a group transitive on the non-zero vectors. Therefore, by adjoining the translation group, we have a doubly transitive group on points acting on the associated design.

However, Kantor [3] has determined the doubly transitive designs, which are either

(a) $PG(a, h)$,

(b) $AG(d, h)$,

(c) the points and secant lines of a unital, where $q+1 = h^2+1$,

(d) the affine Hall plane of order 9 **,**

(e) the affine Hering plane of order 27 **, or**

(f) one of two designs of Hering of type $2-(3^6, 3^2, 1)$.

So, in this situation, since we have an affine design, first assume that $m > 2$. Then, only the $AG(d, h)$-case applies. Hence, we have that the lines are translates of n-dimensional $GF(q)$-subspaces and these become 1-dimensional $GF(q^n)$-spaces in $AG(d, h)$. Hence, $d = m$ and we obtain the affine space $AG(m, q^n)$.

Now assume that $m = 2$. This means we have an affine translation plane of order q^n and we have a group $GF(q^2)^*$ acting as a collineation group of the affine plane with orbits of length $(q^m-1)/(q^{(m,n)}-1)$. For affine translation planes of order 9 then $GF(q^2)^*$ has component orbits of length 1, since m=2=n and q=3 which implies that the plane is Desarguesian. For the Hering plane of order 3^3, then $GF(q^2)^*$ has component orbits of length $q+1=4$. However, the Hering plane admits $SL(2,13)$ as a collineation group and if $GF(3^2)^*$ is normalized, an element of order 13 must fix each of the seven orbits under $GF(3^2)^*$ and then fix each of the orbits pointwise, a contradiction. $\boxed{QED}$

3 The Main Theorem

PROOF. We now complete the proof of Theorem 1.

It is assumed that the point transitive group permutes the set of subgeometries. Hence, they must all be isomorphic to a given projective space $PG(z-1, p^e)$, for p a prime. Therefore, the projective space itself is isomorphic to $PG(n-1, q^m)$, for $q = p^r$. By the main results of Johnson [2], there is an associated spread of dimension nm over $GF(q)$ admitting $GF(q^m)^*$ as a collineation group. The subgeometries isomorphic to $PG(z-1, p^e)$, unwrap into fans Γ, which are orbits of n-dimensional $GF(q)$-subspaces under the group $GF(q^m)^*$. Assume that the stabilizer of a given component L has order j dividing $q^m - 1$. Then there are $(q^m-1)/j$ components in Γ, which means that there are exactly $(q^n-1)/j$ points when the fan is folded, thus producing a subgeometry $PG(z-1, p^e)$. Since $PG(z-1, p^e)$ contains $(p^{ez}-1)/(p^e-1)$ points, then $q^n = p^{ez}$, so that $j = p^e - 1$.

So, we have a spread of n-dimensional $GF(q)$-subspaces of a nm-dimensional $GF(q)$-vector space. The associated design is an affine space $AG(m, q^n)$ by the previous argument. Therefore, the n-dimensional $GF(q)$-subspaces L are 1-dimensional $GF(q^n)$-subspaces and $GF(p^e)^*$ is a subgroup of $GF(q^m)^*$ that acts on L and is therefore a cyclic subgroup of $\Gamma L(1, q^n)$, and corresponds to a subfield $GF(p^e)$ of $GF(q^m)$. We also know that there must be a subgroup of $GF(q^m)^*$ of order at least $q-1$ which is a scalar group, so that $q \leq p^e \leq q^{(n,m)}$. Now we know that there is a subgeometry partition of $PG(n-1, q^m)$ by subgeometries isomorphic to $PG(n/(n,m)-1, q^{(n,m)})$. Also, there are two groups K_1 and K_2 both isomorphic to $GF(q^m)^*$ and two groups G_i, $i = 1, 2$, acting transitively on the components of the spread, where $K_i \subseteq G_i$, $i = 1, 2$. The stabilizer of L in G_i is in $\Gamma L(1, q^n)$ and contains K_{i_L}. It follows immediately that $K_{1_L} = K_{2_L}$. One of these groups may be taken as $GF(q^{(n,m)})^*$. Hence, $p^e = q^{(n,m)}$. This completes the proof of the theorem. $\boxed{QED}$

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