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Linear natural liftings of forms to Weil bundles with Weil algebras $\mathbb{D}^r_k$

Jacek Dębecki
Instytut Matematyki, Uniwersytet Jagielloński
ul. Łojasiewicza 6, 30-348 Kraków, Poland
debecki@im.uj.edu.pl

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Abstract. We give an explicit description and calculate the dimension of the vector space of linear natural liftings of p-forms on n-dimensional manifolds M to q-forms on $T^{\mathbb{D}^r_k}M$, where $\mathbb{D}^r_k$ is the Weil algebra of r-jets at 0 of smooth functions $\mathbb{R}^k \longrightarrow \mathbb{R}$, for all non-negative integers n, p, q, r, k except the case $p = n$ and $q = 0$.

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Let A be a Weil algebra and T^A the Weil functor corresponding to A (see [2] or [4]). Let us denote by $\Omega^p M$ the vector space of p-forms on a manifold M. A linear natural lifting of p-forms to q-forms on T^A is a family of linear maps $L_M : \Omega^p M \longrightarrow \Omega^q(T^A M)$ indexed by n-dimensional manifolds and satisfying for all such manifolds M, N, every embedding $f : M \longrightarrow N$ and every $\omega \in \Omega^p N$ the condition $L_M(f^*\omega) = (T^A f)^*(L_N(\omega))$.

In [1] we have given a classification of liftings of this kind for all non-negative integers n, p and q except the case $p = n$ and $q = 0$. There we have established an isomorphism between the vector space of such liftings and the vector space in the table below for proper n, p and q.

	$0 \le p \le n-1$	$p = n$	$n+1 \le p$
$q = 0$	A_{p-q}		$\{0\}$
$1 \le q \le p$	$A_{p-q} \oplus A_{p-q+1}$	A_{p-q}	$\{0\}$
$q = p+1$	A_{p-q+1}	$\{0\}$	$\{0\}$
$p+2 \le q$	$\{0\}$	$\{0\}$	$\{0\}$

Here A_s for the Weil algebra A inducing T^A and a non-negative integer s is the vector space of skew-symmetric s-linear maps $F : A \times \cdots \times A \longrightarrow A^*$, where A^* denotes the vector space of linear functions $A \longrightarrow \mathbb{R}$, satisfying

$$F(a_1, \ldots, a_{t-1}, bc, a_{t+1}, \ldots, a_s)(d) = \\ F(a_1, \ldots, a_{t-1}, b, a_{t+1}, \ldots, a_s)(cd) + F(a_1, \ldots, a_{t-1}, c, a_{t+1}, \ldots, a_s)(bd) \quad (1)$$

for every $t \in \{1, \dots, s\}$ and all $a_1, \dots, a_{t-1}, a_{t+1}, \dots, a_s, b, c, d \in A$.

Unfortunately, the vector spaces A_s are a bit abstract and we cannot find out the dimension of A_s for every Weil algebra A and every non-negative integer s. This work is devoted to the study of a special case, namely if A is the algebra $\mathbb{D}_k^r$ of r-jets at 0 of smooth functions $\mathbb{R}^k \longrightarrow \mathbb{R}$. We will give an explicit description of $(\mathbb{D}_k^r)_s$ and calculate its dimension for all non-negative integers r, k, s. The importance of the special case we treat is that each Weil algebra A is a factor algebra of $\mathbb{D}_k^r$ for some r, k (see [3]), so A_s is a subspace of $(\mathbb{D}_k^r)_s$ for each s.

Fix non-negative integers r, k, s. We will denote by x^i for $i \in \{1, \dots, k\}$ the r-jet at 0 of the function $\mathbb{R}^k \ni u \longrightarrow u^i \in \mathbb{R}$ and we will write $x^\alpha = (x^1)^{\alpha^1} \cdots (x^k)^{\alpha^k}$ and $|\alpha| = \alpha^1 + \dots + \alpha^k$ for each $\alpha \in \mathbb{N}^k$, where $\mathbb{N}$ stands for the set of non-negative integers. It is obvious that x^ε for $\varepsilon \in \mathbb{N}^k$ such that $|\varepsilon| \le r$ form a basis of the vector space $\mathbb{D}_k^r$ and

$$x^\zeta x^\eta = \begin{cases} x^{\zeta+\eta} & \text{if } |\zeta + \eta| \le r, \\ 0 & \text{otherwise} \end{cases}$$

for all $\zeta, \eta \in \mathbb{N}^k$ such that $|\zeta| \le r$, $|\eta| \le r$.

Of course, $A_0 = A^*$ for every Weil algebra A. Therefore we will be concerned only with the case $s > 0$. If $r = 0$ or $k = 0$ then $\mathbb{D}_k^r = \mathbb{R}$, so $(\mathbb{D}_k^r)_0 = \mathbb{R}^*$ and it is a simple matter to see that if $s > 0$ then $(\mathbb{D}_k^r)_s = \{0\}$. Therefore we will be concerned only with the case $r > 0$ and $k > 0$.

We can now formulate our main result.

1 Definition. Let Z denote the set of $(i_1, \dots, i_s, \alpha) \in \{1, \dots, k\}^s \times \mathbb{N}^k$ with the properties that $i_1 < \dots < i_s$ and either $|\alpha| < r$ or $|\alpha| = r$ and $i_s < \max\{\, l \in \{1, \dots, k\} \mid \alpha^l > 0 \,\}$.

2 Theorem. *The map $I : (\mathbb{D}_k^r)_s \longrightarrow \mathbb{R}^Z$ given by*

$$I(F)(i_1, \dots, i_s, \alpha) = F(x^{i_1}, \dots, x^{i_s})(x^\alpha)$$

for every $F \in (\mathbb{D}_k^r)_s$ and every $(i_1, \dots, i_s, \alpha) \in Z$ is an isomorphism of vector spaces.

Proof. The theorem will be proved by showing that for each $C \in \mathbb{R}^Z$ there is a unique $F \in (\mathbb{D}_k^r)_s$ such that

$$F(x^{i_1}, \dots, x^{i_s})(x^\alpha) = C(i_1, \dots, i_s, \alpha) \tag{2}$$

for every $(i_1, \dots, i_s, \alpha) \in Z$. Fix $C \in \mathbb{R}^Z$. Our construction of F will be divided into six steps.

Step 1. We define $F(x^{i_1}, \dots, x^{i_s})(x^\alpha)$ for $(i_1, \dots, i_s, \alpha) \in Z$ by (2).

Step 2. We define $F(x^{i_1},\dots,x^{i_s})(x^\alpha)$ for $(i_1,\dots,i_s,\alpha)\in\{1,\dots,k\}^s\times\mathbb{N}^k$ such that either $|\alpha|<r$ or $|\alpha|=r$ and $\max\{i_1,\dots,i_s\}<\max\{\,l\in\{1,\dots,k\}\mid \alpha^l>0\,\}$.

Since F should be skew-symmetric, we put either

$$F(x^{i_1},\dots,x^{i_s})(x^\alpha)=\operatorname{sgn}\sigma F(x^{i_{\sigma(1)}},\dots,x^{i_{\sigma(s)}})(x^\alpha)$$

if there is a permutation σ of $\{1,\dots,s\}$ such that $i_{\sigma(1)}<\dots<i_{\sigma(s)}$ (note that there is at most one σ with this property) or $F(x^{i_1},\dots,x^{i_s})(x^\alpha)=0$ otherwise.

Step 3. We define $F(x^{i_1},\dots,x^{i_s})(x^\alpha)$ for $(i_1,\dots,i_s,\alpha)\in\{1,\dots,k\}^s\times\mathbb{N}^k$ such that $i_1<\dots<i_s$ and $|\alpha|\le r$, but $(i_1,\dots,i_s,\alpha)\notin Z$.

If $G\in(\mathbb{D}^r_k)_s$, $t\in\{1,\dots,s\}$ and $\gamma_1,\dots,\gamma_s,\delta\in\mathbb{N}^k$ then, by induction on $|\gamma_t|$, (1) leads easily to

$$G(x^{\gamma_1},\dots,x^{\gamma_s})(x^\delta)=\\ \sum_{j\in\{\,l\in\{1,\dots,k\}\mid\gamma_t^l>0\,\}}\gamma_t^j G(x^{\gamma_1},\dots,x^{\gamma_{t-1}},x^j,x^{\gamma_{t+1}},\dots,x^{\gamma_s})(x^{\gamma_t-e_j+\delta}),\quad(3)$$

where $e_1,\dots,e_k$ stand for the standard basis of the module $\mathbb{Z}^k$.

The condition $(i_1,\dots,i_s,\alpha)\notin Z$ means that $|\alpha|=r$ and $i_s\ge\max\{\,l\in\{1,\dots,k\}\mid\alpha^l>0\,\}$. Taking $t=s$, $\gamma_1=e_{i_1},\dots,\gamma_{s-1}=e_{i_{s-1}}$, $\gamma_s=\alpha+e_{i_s}$ and $\delta=0$ in (3) we see that F should satisfy

$$0=(\alpha^{i_s}+1)F(x^{i_1},\dots,x^{i_s})(x^\alpha)+\\ \sum_{j\in\{\,l\in\{1,\dots,k\}\mid\alpha^l>0\,\}\setminus\{i_s\}}\alpha^j F(x^{i_1},\dots,x^{i_{s-1}},x^j)(x^{\alpha+e_{i_s}-e_j}),\quad(4)$$

since $x^{\alpha+e_{i_s}}=0$ and F should be s-linear. But $F(x^{i_1},\dots,x^{i_{s-1}},x^j)(x^{\alpha+e_{i_s}-e_j})$ for every $j\in\{\,l\in\{1,\dots,k\}\mid\alpha^l>0\,\}\setminus\{i_s\}$ has already been defined, as $\max\{i_1,\dots,i_{s-1},j\}<i_s=\max\{\,l\in\{1,\dots,k\}\mid(\alpha+e_{i_s}-e_j)^l>0\,\}$. Therefore we put

$$F(x^{i_1},\dots,x^{i_s})(x^\alpha)=\\ -\frac{1}{\alpha^{i_s}+1}\sum_{j\in\{\,l\in\{1,\dots,k\}\mid\alpha^l>0\,\}\setminus\{i_s\}}\alpha^j F(x^{i_1},\dots,x^{i_{s-1}},x^j)(x^{\alpha+e_{i_s}-e_j}).\quad(5)$$

Step 4. We define $F(x^{i_1},\dots,x^{i_s})(x^\alpha)$ for $(i_1,\dots,i_s,\alpha)\in\{1,\dots,k\}^s\times\mathbb{N}^k$ such that $|\alpha|\le r$.

This goes in the same way as step 2.

Step 5. We define $F(x^{\gamma_1},\dots,x^{\gamma_s})(x^\delta)$ for $\gamma_1,\dots,\gamma_s,\delta\in\mathbb{N}^k$ such that $|\gamma_1|\le r,\dots,|\gamma_s|\le r,|\delta|\le r$.

Since F should satisfy (3) and take linear values, we put either

$$F(x^{\gamma_1},\dots,x^{\gamma_s})(x^\delta) = \sum_{j_1\in\{\,l\in\{1,\dots,k\}\mid\gamma_1^l>0\,\}}\cdots\sum_{j_s\in\{\,l\in\{1,\dots,k\}\mid\gamma_s^l>0\,\}}\gamma_1^{j_1}\cdots\gamma_s^{j_s}\cdot F(x^{j_1},\dots,x^{j_s})(x^{\gamma_1-e_{j_1}+\dots+\gamma_s-e_{j_s}+\delta}) \tag{6}$$

if $|\gamma_1+\dots+\gamma_s+\delta|\le r+s$ or $F(x^{\gamma_1},\dots,x^{\gamma_s})(x^\delta)=0$ otherwise.

Step 6. We complete our construction easily, because x^ε for $\varepsilon\in\mathbb{N}^k$ such that $|\varepsilon|\le r$ form a basis of the vector space $\mathbb{D}_k^r$ and F should be s-linear with linear values.

Thus we have proved the uniqueness of F. By step 1, the map F we have constructed satisfies (2). By step 6, it is s-linear with linear values. By steps 4, 5, 6, it is easily seen to be skew-symmetric. What is left is to prove that it satisfies (1).

We begin by showing the crucial fact that for all $g_1,\dots,g_{s-1}\in\{1,\dots,k\}$ such that $g_1<\dots<g_{s-1}$ and every $\varepsilon\in\mathbb{N}^k$ such that $|\varepsilon|=r+1$

$$\sum_{h\in\{\,l\in\{1,\dots,k\}\mid\varepsilon^l>0\,\}}\varepsilon^h F(x^{g_1},\dots,x^{g_{s-1}},x^h)(x^{\varepsilon-e_h})=0. \tag{7}$$

If either $s=1$ or $s>1$ and $g_{s-1}<m$, where $m=\max\{\,l\in\{1,\dots,k\}\mid\varepsilon^l>0\,\}$ which implies $m\ge\max\{\,l\in\{1,\dots,k\}\mid(\varepsilon-e_m)^l>0\,\}$, then (7) is nothing but (4) with $i_1=g_1,\dots,i_{s-1}=g_{s-1}$, $i_s=m$ and $\alpha=\varepsilon-e_m$. So (7) holds, as (4) is equivalent to (5) which holds.

If $s>1$ and $g_{s-1}\ge m$, then $\max\{g_1,\dots,g_{s-1},h\}=g_{s-1}\ge\max\{\,l\in\{1,\dots,k\}\mid(\varepsilon-e_h)^l>0\,\}$ for every $h\in\{\,l\in\{1,\dots,k\}\mid\varepsilon^l>0\,\}$. Hence if $h\notin\{g_1,\dots,g_{s-1}\}$ then the skew-symmetry of F and (5) with $\{i_1,\dots,i_s\}=\{g_1,\dots,g_{s-1},h\}$ (which implies $i_s=g_{s-1}$) and $\alpha=\varepsilon-e_h$ give

$$F(x^{g_1},\dots,x^{g_{s-1}},x^h)(x^{\varepsilon-e_h}) = -\frac{1}{\varepsilon^{g_{s-1}}+1}\cdot \cdot\sum_{j\in\{\,l\in\{1,\dots,k\}\mid\varepsilon^l>0\,\}\setminus\{g_{s-1},h\}}\varepsilon^j F(x^{g_1},\dots,x^{g_{s-2}},x^j,x^h)(x^{\varepsilon-e_h+e_{g_{s-1}}-e_j}), \tag{8}$$

because if $\varepsilon^h>1$ then $F(x^{g_1},\dots,x^{g_{s-2}},x^h,x^h)(x^{\varepsilon-e_h+e_{g_{s-1}}-e_h})=0$, by the skew-symmetry of F. Substituting (8) into (7) and omitting the terms which vanish on account of the skew-symmetry of F we see that the left hand side of

(7) equals

$$-\frac{1}{\varepsilon^{g_{s-1}}+1}\sum_{h\in\{\,l\in\{1,\dots,k\}|\varepsilon^l>0\,\}\setminus\{g_1,\dots,g_{s-1}\}}\;\sum_{j\in\{\,l\in\{1,\dots,k\}|\varepsilon^l>0\,\}\setminus\{g_1,\dots,g_{s-1},h\}}\varepsilon^h\varepsilon^j\cdot$$
$$\cdot F(x^{g_1},\dots,x^{g_{s-2}},x^j,x^h)(x^{\varepsilon-e_h+e_{g_{s-1}}-e_j}).$$

But, using the skew-symmetry of F again, we have

$$\varepsilon^j\varepsilon^hF(x^{g_1},\dots,x^{g_{s-2}},x^h,x^j)(x^{\varepsilon-e_j+e_{g_{s-1}}-e_h})=$$
$$-\varepsilon^h\varepsilon^jF(x^{g_1},\dots,x^{g_{s-2}},x^j,x^h)(x^{\varepsilon-e_h+e_{g_{s-1}}-e_j}).$$

Therefore the left hand side of (7) equals 0. This establishes (7).

We recall that our aim is to show (1) for F we have constructed. Since F is skew-symmetric, it suffices to prove (1) only for $t=s$. This will be proved as soon as we can show that

$$F(x^{\alpha_1},\dots,x^{\alpha_{s-1}},x^{\beta+\gamma})(x^\delta)=$$
$$F(x^{\alpha_1},\dots,x^{\alpha_{s-1}},x^{\beta})(x^{\gamma+\delta})+F(x^{\alpha_1},\dots,x^{\alpha_{s-1}},x^{\gamma})(x^{\beta+\delta})\quad(9)$$

for $\alpha_1,\dots,\alpha_{s-1},\beta,\gamma,\delta\in\mathbb{N}^k$ such that $|\alpha_1|\le r,\dots,|\alpha_{s-1}|\le r$, $|\beta|\le r$, $|\gamma|\le r$, $|\delta|\le r$, because both the sides of (1) are $(s+2)$-linear with respect to $a_1,\dots,a_{s-1}$, b, c, d and x^ε for $\varepsilon\in\mathbb{N}^k$ such that $|\varepsilon|\le r$ form a basis of the vector space $\mathbb{D}^r_k$.

We now observe that (9) holds in four special cases.

Case 1. $\beta=0$ or $\gamma=0$. Then (9) is evident because, by steps 5 and 6, we have $F(x^{\alpha_1},\dots,x^{\alpha_{s-1}},1)(x^{\gamma+\delta})=0$ and $F(x^{\alpha_1},\dots,x^{\alpha_{s-1}},1)(x^{\beta+\delta})=0$.

Case 2. There is $i\in\{1,\dots,s-1\}$ such that $\alpha_i=0$. Then (9) is evident, as both the sides of (9) equal 0, by steps 5 and 6.

Case 3. $|\alpha_1+\dots+\alpha_{s-1}+\beta+\gamma+\delta|>r+s$. Then (9) is also evident, as both the sides of (9) equal 0, by steps 5 and 6.

Case 4. $|\alpha_1+\dots+\alpha_{s-1}+\beta+\gamma+\delta|\le r+s$, $|\beta+\gamma|\le r$, $|\beta+\delta|\le r$, $|\gamma+\delta|\le r$. Then (9) can be easily checked, because we may apply (6) to the left hand side of (9) as well as to each of two terms of its right hand side.

Assume that it is none of the above cases. Since it is not case 2, $|\alpha_1|\ge 1,\dots,|\alpha_{s-1}|\ge 1$. Since it is not case 3, $|\alpha_1+\dots+\alpha_{s-1}+\beta+\gamma+\delta|\le r+s$. Combining these yields $|\beta+\gamma+\delta|\le r+1$. If it were true that $|\beta+\gamma|\le r$, it would also be true that $|\beta+\delta|>r$ or $|\gamma+\delta|>r$, as it is not case 4, and so that $\gamma=0$ or $\beta=0$ respectively, contrary to the fact that it is not case 1. Therefore $|\beta+\gamma|=r+1$, and so $\delta=0$, $|\alpha_1|=1,\dots,|\alpha_{s-1}|=1$.

Summing up, it remains to prove (9) only if $|\alpha_1|=1,\dots,|\alpha_{s-1}|=1$, $|\beta|\le r$, $|\gamma|\le r$, $|\beta+\gamma|=r+1$ and $\delta=0$. Then there are $g_1,\dots,g_{s-1}\in\{1,\dots,k\}$

such that $\alpha_1 = e_{g_1}, \dots, \alpha_1 = e_{g_{s-1}}$. Since F is skew-symmetric, without loss of generality we can assume that $g_1 < \cdots < g_{s-1}$. Moreover, $x^{\beta+\gamma} = 0$ and F is s-linear, hence the left hand side of (9) equals 0. Using (6) we can rewrite (9) as

$$0 = \sum_{h \in \{\, l \in \{1,\dots,k\} \mid \beta^l > 0 \,\}} \beta^h F(x^{g_1}, \dots, x^{g_{s-1}}, x^h)(x^{\beta - e_h + \gamma}) + \\ \sum_{h \in \{\, l \in \{1,\dots,k\} \mid \gamma^l > 0 \,\}} \gamma^h F(x^{g_1}, \dots, x^{g_{s-1}}, x^h)(x^{\gamma - e_h + \beta}),$$

which is nothing but (7) with $\varepsilon = \beta + \gamma$. This completes the proof of the theorem. $\boxed{QED}$

3 Corollary. *The dimension of the vector space* $(\mathbb{D}^r_k)_s$ *equals*

$$\binom{r+s-1}{s}\binom{r+k}{r+s}$$

Proof. We will compute the number of elements of Z, which is equal to the dimension of $(\mathbb{D}^r_k)_s$, by the theorem.

For each $v \in \{0, \dots, r-1\}$ the number of $(i_1, \dots, i_s, \alpha) \in \{1, \dots, k\}^s \times \mathbb{N}^k$ such that $i_1 < \cdots < i_s$ and $|\alpha| = v$ equals

$$\binom{k}{s}\binom{v+k-1}{k-1}$$

Furthermore, we have

$$\binom{k}{s}\sum_{v=0}^{r-1}\binom{v+k-1}{k-1} = \binom{k}{s}\binom{r+k-1}{k} = \binom{r+s-1}{s}\binom{r+k-1}{r+s-1}$$

For each $m \in \{s+1, \dots, k\}$ the number of $(i_1, \dots, i_s, \alpha) \in \{1, \dots, k\}^s \times \mathbb{N}^k$ such that $i_1 < \cdots < i_s$, $|\alpha| = r$ and $i_s < m = \max\{\, l \in \{1, \dots, k\} \mid \alpha^l > 0 \,\}$ equals

$$\binom{m-1}{s}\binom{r+m-2}{m-1}.$$

Furthermore, we have

$$\sum_{m=s+1}^{k}\binom{m-1}{s}\binom{r+m-2}{m-1} = \\ \binom{r+s-1}{s}\sum_{m=s+1}^{k}\binom{r+m-2}{r+s-1} = \binom{r+s-1}{s}\binom{r+k-1}{r+s}.$$

Hence the number of elements of Z equals

$$\binom{r+s-1}{s}\binom{r+k-1}{r+s-1}+\binom{r+s-1}{s}\binom{r+k-1}{r+s}=\binom{r+s-1}{s}\binom{r+k}{r+s}.$$

This completes the proof of the corollary. $\boxed{QED}$

Note that the corollary is still true if $r=0$ or $k=0$ or $s=0$, as is easy to check.

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