Note di Matematica
Note Mat. 1(2008), 195-202
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1p195
http://siba2.unile.it/notemat

Means via groups and some properties of autodistributive Steiner triple systems

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Received: 17/12/2001; accepted: 11/05/2007.

Abstract. The classical definition of arithmetical mean can be transferred to a group $(\mathbf{G}, +)$ with the property that for any $y \in \mathbf{G}$ there is a unique $x \in \mathbf{G}$ such that $x + x = y$ (uni-2-divisible group). Indeed we define in $\mathbf{G}$ a commutative and idempotent operation $\triangledown$ that recalls the classical means, even if $(\mathbf{G}, +)$ is not commutative. Afterwards in section 3 we show that, by means of the commutative and idempotent operation $\triangledown$ usually associated with an autodistributive Steiner triple system $(\mathbf{G}, \mathcal{L})$, we can endow any plane of $(\mathbf{G}, \mathcal{L})$ of a structure of affine desarguesian (Galois) plane.

Keywords: finite groups, group means, geometry, combinatorics

MSC 2000 classification: primary 14L35

1 Introduction

Let us consider groups $(\mathbf{G}, +)$ in which for any $y \in \mathbf{G}$ there is a unique $x \in \mathbf{G}$ such that $y = 2x$ (i.e. the function $\underline{2}$ mapping $x \in \mathbf{G}$ in $2x$ is bijective). We say that these groups are *uniquely 2-divisible* (briefly: *uni-2-divisible*).

Henceforth we will often represent a group $(\mathbf{G}, +)$ only with $\mathbf{G}$.

It is easy to see that whenever all the elements of a group $\mathbf{G}$ are of finite odd order, then the group is uni-2-divisible.

Clearly, the additive real group $\mathbb{R}$ is uni-2-divisible and several properties of the classical arithmetical mean depend on this fact.

Now let a, b, d belong to an arbitrary group $\mathbf{G}$, with $a + 2d = b$. If d' is the only element of $\mathbf{G}$ such that $d' + a = a + d$, then $b = a + 2d = 2d' + a$; therefore $a = b + 2(-d) = 2(-d') + b$. Thus it is natural to say that $a + d$ is a *midpoint* of a and b.

Let $\mathbf{G}$ be a uni-2-divisible group. Thus, for any $a, b \in \mathbf{G}$, in $\mathbf{G}$ there are a unique d and a unique d' such that $a + 2d = b = 2d' + a$; wiz. $d = (-a + b)/2$ and $d' = (b - a)/2$. Thus a and b have a unique midpoint that we indicate with $a \triangledown b$. Obviously, if $\mathbf{G}$ is commutative, then $a \triangledown b = (a + b)/2$ (cf. the classical case of real numbers); more generally we have:

$$a \bigtriangledown b = a + (-a+b)/2 = (b-a)/2 + a \\ = (a-b)/2 + b = b + (-b+a)/2 = b \bigtriangledown a. \quad (1)$$

Clearly $\bigtriangledown$ is a commutative and idempotent operation on $\mathbf{G}$ such that $+$ is distributive with respect to $\bigtriangledown$ (hence for any $g \in \mathbf{G}$, $\underline{g+}$ and $\underline{+g}$ are automorphisms of $(\mathbf{G}, \bigtriangledown)$). Moreover the translations $\underline{a\bigtriangledown}$ and $\underline{\bigtriangledown a}$ (but $\underline{a\bigtriangledown} = \underline{\bigtriangledown a}$ by commutativity of $\bigtriangledown$) are bijective; i.e.: $(\mathbf{G}, \bigtriangledown)$ is a quasigroup.

Let $\mathbf{G}$ be a uni-2-divisible group. Whenever the elements a and c of $\mathbf{G}$ commute, then the only midpoint of a and $a+c$ is $c/2+a = a+c/2$; hence a and $c/2$ commute too. Consequently, the center $\mathbf{C}$ of $\mathbf{G}$ is uni-2-divisible.

Moreover, a subgroup of $\mathbf{G}$ is uni-2-divisible if and only if it is closed under the mapping $\underline{2}^{-1}$. Therefore the set of uni-2-divisible subgroups of $\mathbf{G}$ is a closure system; hence if $g \in \mathbf{G}$, we will represent with $<< g >>$ the uni-2-divisible subgroup generated by g.

Clearly, the above subgroup $<< g >>$ is the set union of the chain of cyclic subgroups of type $< g/2^h >$, with an obvious meaning of symbol $g/2^h$, where $h \in \mathbb{N}$ (the set of natural numbers). Thus $<< g >>$ is a commutative group, as well as each $< g/2^h >$.

If $(\mathbf{G}, +)$ is a group, then through an element $0' \in \mathbf{G}$ one can define a new group operation on $\mathbf{G}$ by setting $a +' b = a - 0' + b$. Therefore, the left and the right translations $\underline{0'+}$, $\underline{+0'}$ are isomorphisms from $(\mathbf{G}, +)$ onto $(\mathbf{G}, +')$. Hence $0' = 0' + 0$ is the "zero" of $(\mathbf{G}, +')$ and $0' - b + 0'$ is the opposite $-'b$ of b with respect to $+'$; thus $a -' b = a +' (-'b) = a - b + 0'$.

If $(\mathbf{G}, +)$ is uni-2-divisible, then also $(\mathbf{G}, +')$ is uni-2-divisible; thus, for any $g \in \mathbf{G}$ we get $[(g-0')/2+0'] +' [(g-0')/2+0'] = (g-0')/2+0'-0'+(g-0')/2+0' = g$. Hence it is easy to verify that $\bigtriangledown$ coincides with the analogous operation $\bigtriangledown'$ associated with $+'$.

Now let $(\mathbf{G}, \bigtriangledown)$ be an arbitrary commutative and idempotent quasigroup. Then it is useful consider another operation $\multimap$ on $\mathbf{G}$ by setting $x \multimap z$ equal to the unique element y such that $x \bigtriangledown y = z$; hence $x \bigtriangledown (x \multimap z) = z = x \multimap (x \bigtriangledown z)$. Consequently, $x \multimap z = \underline{x\bigtriangledown}^{-1} z$; moreover, by $x \bigtriangledown (x \multimap z) = (x \multimap z) \bigtriangledown x = z$, we have also:

$$(x \multimap z) \multimap z = x. \quad (2)$$

It is obvious that $x \multimap z = z$ if and only if $x = z$. Moreover, since $\bigtriangledown$ is commutative, the following property holds:

$$x \bigtriangledown y = z \quad \Leftrightarrow \quad y \multimap z = x \quad \Leftrightarrow \quad x \multimap z = y. \quad (3)$$

If $\multimap$ is right distributive with respect to itself (briefly, *r-autodistributive*), we say that $\triangledown$ is a *mean*. In this case $\multimap$ is right distributive (r-distributive) also with respect to $\triangledown$. We say also that the structure $(\mathbf{G}, \triangledown, \multimap)$ is a mean.

Several authors studied quasigroups with a r-autodistributive operation (see [3] and [4]).

If $\mathbf{H}$ is a closed subset of a mean $(\mathbf{G}, \triangledown, \multimap)$, we say that $\mathbf{H}$ is *a sub-mean* of $(\mathbf{G}, \triangledown, \multimap)$. Then if $\mathbf{K} \subseteq \mathbf{G}$, $((\mathbf{K}))$ denote the sub-mean of $(\mathbf{G}, \triangledown, \multimap)$ generated by $\mathbf{K}$; in particular, if $\mathbf{K} = \{a_1, \ldots, a_n\}$, we will write $((a_1, \ldots, a_n))$ instead of $((\{a_1, \ldots, a_n\}))$. In a mean $(\mathbf{G}, \triangledown, \multimap)$ we have the following properties. The second property is a consequence of the first one, which follows from (2).

$$x \multimap (y \multimap z) = [(x \multimap z) \multimap z] \multimap (y \multimap z) = [(x \multimap z) \multimap y] \multimap z; \tag{4}$$

$$x \multimap (y \multimap x) = (x \multimap y) \multimap x \qquad \text{(flexibility property)}. \tag{5}$$

If $\triangledown$ is associated with a uni-2-divisible group $\mathbf{G}$, it is easy to verify that $x \multimap z = z - x + z$ (cf [7], Sec. 5); hence $+$ is distributive also with respect to $\multimap$. This means that $\underline{x+}$ and $\underline{+x}$ are automorphisms of the structure $(\mathbf{G}, \triangledown, \multimap)$. Moreover, it is easy to verify that $\multimap$ is r-autodistributive.

In this particular case we say that both $\triangledown$ and $(\mathbf{G}, \triangledown, \multimap)$ are *group−means* (briefly, *g − means*). Till different notice a group $\mathbf{G}$ shall be uni-2-divisible.

2 Some remarks on means

Henceforth $\triangledown$ and $\multimap$ shall represent the operations of a mean $(\mathbf{G}, \triangledown, \multimap)$.

We say that a sequence $(a_0, a_1, \ldots)$ of elements of $\mathbf{G}$ is a $\multimap$-*sequence* if $a_{i-1} \multimap a_i = a_{i+1}$ (equivalently: $a_{i+1} \multimap a_i = a_{i-1}$ or $a_{i-1} \triangledown a_{i+1} = a_i$) for any index $i \neq 0$. Therefore $(a_0, a_1, \ldots)$ is completely determined by a_0 and a_1; hence we set $(a_0, a_1)_{\multimap} := (a_0, a_1, \ldots)$.

1 Remark. By property (5) we immediately get $a_0 \multimap a_2 = a_4$. Indeed $a_0 \multimap a_2 = (a_2 \multimap a_1) \multimap a_2 = a_2 \multimap (a_1 \multimap a_2) = a_2 \multimap a_3 = a_4$.

If $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ are two sequences such that $a_0 = a_i \triangledown b_i = b_0$ for any index i, then we say that they are *co-symmetric*.

2 Lemma. *Let the $\multimap$-sequences $(a_0, a_1)_{\multimap}$ and $(b_0, b_1)_{\multimap}$ fulfil the property $a_0 = a_1 \triangledown b_1 = b_0$. Then they are co-symmetric.*

Proof. Let $a_i \multimap a_0 = b_i$ for each $i < n$. Then by r-autodistributivity of $\multimap$ we have $a_n \multimap a_0 = (a_{n-2} \multimap a_{n-1}) \multimap a_0 = (a_{n-2} \multimap a_0) \multimap (a_{n-1} \multimap a_0) = b_{n-2} \multimap b_{n-1} = b_n$. Whence the claim by induction. $\boxed{QED}$

3 Theorem. *For any $a, a', a'', a''', a'''' \in \mathbf{G}$, let $a \bigtriangledown a'''' = a''$, $a \bigtriangledown a'' = a'$ and $a'' \bigtriangledown a'''' = a'''$. Then $a' \bigtriangledown a''' = a''$.*

Proof. Let us consider the sequence $(a_0, a_1, \ldots) = \multimap (a, a')$. Thus $a_0 = a$, $a_1 = a'$ and $a_2 = a''$; moreover, by Remark 1, $a_4 = a_0 \multimap a_2 = a \multimap a'' = a''''$. Furthermore, $a_3 = a_2 \bigtriangledown a_4 = a'' \bigtriangledown a'''' = a'''$. Whence the assertion by $a' \bigtriangledown a''' = a_1 \bigtriangledown a_3 = a_2 = a''$. $\boxed{QED}$

4 Remark. Let $\downarrow$ be the function mapping any sequence $(a_0, a_1, \ldots)$ into the sequence $(d_0, d_1, \ldots)$ obtained from $(a_0, a_1, \ldots)$ by inserting $a_i \bigtriangledown a_{i+1}$ between a_i and a_{i+1}. Hence $a_i = d_{2i}$ for any index i. If $(a_0, a_1, \ldots)$ is a $\multimap$-sequence, then — by Theorem 3 — also $\downarrow (a_0, a_1, \ldots)$ is a $\multimap$-sequence (and vice-versa, as an easy consequence of Remark 1).

5 Theorem. *Let the sequence $(a_0, a_1, \ldots)$ be equal to $(a_0, a_1)_{\multimap}$. Then for any index i we have $a_0 \bigtriangledown a_{i+1} = a_1 \bigtriangledown a_i$.*

Proof. We put $(d_0, d_1, \ldots) := \downarrow (a_0, a_1, \ldots)$. Hence, by Remark 4, it is sufficient to verify that $d_0 \bigtriangledown d_{2(i+1)} = d_2 \bigtriangledown d_{2i}$.

In fact the sequences $(b'_0, b'_1, \ldots) = (d_{i+1}, d_i)_{\multimap}$ and $(a'_0, a'_1, \ldots) = s(d_{i+1}, d_{i+2})$ are symmetric by Lemma 2. Therefore $d_2 \bigtriangledown d_{2i} = b'_{i-1} \bigtriangledown a'_{i-1} = d_{i+1} = b'_{i+1} \bigtriangledown a'_{i+1} = d_0 \bigtriangledown d_{2(i+1)}$. $\boxed{QED}$

Let us put $a_{-i} = a_i \multimap a_0$. Thus if we consider the $\multimap$-sequence $(c_0, c_1, \ldots) = (a_j, a_{j+1})_{\multimap}$, where $j \in \mathbb{Z}$, we have $c_i = a_{i+j}$ for any $i \in \mathbb{N}$.

For any $i, j \in \mathbb{Z}$, from Theorem 5 we immediately get the following equalities:

$$a_i \bigtriangledown a_j = a_0 \bigtriangledown a_{i+j}; \tag{6}$$

$$a_j = a_j \bigtriangledown a_j = a_i \bigtriangledown a_{2j-i}, \text{ hence } a_i \multimap a_j = a_{2j-i}. \tag{7}$$

Now, given the $\multimap$-sequence $(a_0, a_1)_{\multimap}$, let $\mathbf{H}_0$ be the set of all the terms a_i, with $i \in \mathbb{Z}$. Analogously, let us consider the set $\mathbf{H}_1$ obtained from the sequence $\downarrow ((a_0, a_1)_{\multimap})$, the set $\mathbf{H}_2$ obtained from the sequence $\downarrow^2 ((a_0, a_1)_{\mapsto})$, and so on. It is clear that the set $\mathbf{H}$ which is union of all $\mathbf{H}_i$ coincides with the sub-mean $((a_0, a_1))$.

6 Theorem. *Let the $\multimap$-sequence $(a_0, a_1)_{\multimap}$ possess at least two equal terms, with $a_0 \neq a_1$; moreover, let h be the minimum index such that $a_h = a_k$ for some $k < h$. The following properties hold:*

- (i_1) *$k = 0$; moreover, h is an odd number. Furthermore, $a_{h+1} = a_1$; hence $(a_0, a_1)_{\multimap} = (a_0, a_1, \ldots, a_{h-1}, a_0, a_1, \ldots)$.*
- (i_2) *for any $i, j \in \mathbb{N}$, $a_i = a_j$ if and only if $i \equiv j \pmod h$.*
- (i_3) *If $(\{0, \ldots, h-1\}, +')$ is the group of integers under the addition modulo h, let f be the bijection mapping any $i \in \{0, \ldots, h-1\}$ into $a_i \in$*

$\{a_0, a_1, \ldots, a_{h-1}\}$ *and consider the binary operation* $+$ *on* $\{a_0, a_1, \ldots, a_{h-1}\}$ *such that* f *is an isomorphism. Thus, for any* $a_p, a_q \in \{a_0, a_1, \ldots, a_{h-1}\}$, $a_p \bigtriangledown a_q = a_{(p+'q)/2}\,(= (a_p + a_q)/2)$.

(i_4) $((a_0, a_1)) = \{a_0, a_1, \ldots, a_{h-1}\}$*; moreover* $((a_0, a_1))$ *is a g-mean.*

PROOF.

(i_1) If $k > 0$, then $a_{k+1} \bigtriangledown a_{h-1} = a_k \bigtriangledown a_h = a_k = a_{k+1} \bigtriangledown a_{k-1}$, hence $a_{h-1} = a_{k-1}$; this contradicts the hypothesis.

Furthermore, if $h = 2m > 0$, then $a_0 = a_h \bigtriangledown a_0 = a_{h-m} \bigtriangledown a_m = a_m \bigtriangledown a_m = a_m$. This is absurd, because $0 \neq m < h$.

Moreover, $a_{h+1} = a_1$. In fact $a_{h+1} \bigtriangledown a_1 = a_h \bigtriangledown a_2 = a_0 \bigtriangledown a_2 = a_1 \bigtriangledown a_1 = a_1$.

(i_2) The assertion is an obvious consequence of above property (i_1).

(i_3) Let $t = (p +' q)/2$; hence $t +' t = p +' q \cong p + q (\mod h)$; hence, $a_{t+t} = a_{t+'t} = a_{p+q}$ (see (i_2) above). Therefore $a_t = a_0 \bigtriangledown a_{p+q} = a_p \bigtriangledown a_q$. Whence the claim.

(i_4) In fact $\{a_0, a_1, \ldots, a_{h-1}\} \subseteq ((a_0, a_1))$. Moreover $\{a_0, a_1, \ldots, a_{h-1}\}$ is closed under $\bigtriangledown$ and $\multimap$. Whence the assertion by (i_3).

$\boxed{QED}$

Now let 0 and a be different elements of $\mathbf{G}$. We have the following

7 Theorem. *The sub-mean* $((0, a))$ *is a g-mean by means of a commutative binary operation* $+$ *on* $((0, a))$*, having* 0 *as the zero element.*

PROOF. If $(0, a)_{\multimap}$ possesses at least two equal terms, the assertion is true by (i_4) of Theorem 6, with $a_0 = 0$ and $a_1 = a$. Therefore we assume that the terms of $(0, a)_{\multimap}$ are pairwise distinct. Afterwards we define a commutative group operation $+_0$ on the set $\mathbf{H}_0$ above by putting $a_p +_0 a_q := a_{p+q}$, for any $p, q \in \mathbb{Z}$. Then we consider the analogous group $(\mathbf{H}_1, +_1)$; and so on with respect to any set $\mathbf{H}_i$.

Obviously, $(\mathbf{H}_i, +_i)$ is a subgroup of $(\mathbf{H}_{i+1}, +_{i+1})$. Therefore, considered the set union $\mathbf{H}$ of all $\mathbf{H}_i$, there is a unique (commutative) group operation $+$ on $\mathbf{H}$ such that each $(\mathbf{H}_i, +_i)$ is a subgroup of $(\mathbf{H}, +)$.

Thus, since $\mathbf{H} = ((0, a))$, it is easy to verify that on $((0, a))$ the mean $\bigtriangledown$ is determined by $(\mathbf{H}, +)$. In fact, if $b, c \in \mathbf{H}$, then there exists a set $\mathbf{H}_i$ such that $b, c \in \mathbf{H}_i$. Therefore $b = t_{2p}$ and $c = t_{2q}$, where t_{2p} and t_{2q} are suitable terms of $\mathbf{H}_{i+1}$. Hence $b \bigtriangledown c = t_{2p} \bigtriangledown t_{2q} = t_{p+q} = (t_{2p} + t_{2q})/2$. $\boxed{QED}$

Thus, for any $x \in \mathbf{G}$, we put $-x = x \multimap 0$; hence $-(-x) = x$. Moreover, we put $2x = 0 \multimap x$ and $x/2 = x \bigtriangledown 0$.

Then we have the following

8 Theorem. *For any $x, y \in \mathbf{G}$, $-(x \multimap y) = (-x) \multimap (-y)$ and $-(x \bigtriangledown y) = (-x) \bigtriangledown (-y)$.*

Proof. In fact $-(x \multimap y) = (x \multimap y) \multimap 0 = (x \multimap 0) \multimap (y \multimap 0) = (-x) \multimap (-y)$. The second equality is an obvious consequence of the first one. $\boxed{QED}$

3 The case of autodistributive Steiner triple systems

We recall that if $\mathbf{G}$ and $\mathcal{L}$ are sets such that the elements of $\mathcal{L}$ are subsets of $\mathbf{G}$, then $(\mathbf{G}, \mathcal{L})$ is said a *line space* — hence the elements of $\mathbf{G}$ and of $\mathcal{L}$ are said respectively *points* and *lines* — whenever distinct lines intersect at most in one point and for any distinct points a and b there exists a (unique) line containing them.

A line space is said a *Steiner triple system* if the lines possess exactly three points. In this latter particular case one define a commutative and idempotent quasigroup operation $\bigtriangledown$ by setting, for any distinct $x, y \in \mathbf{G}$, $x \bigtriangledown x = x$ and $x \bigtriangledown y$ equal to the unique point z such that $\{x, y, z\} \in \mathcal{L}$. Obviously, for any $x, y \in \mathbf{G}$, one has $x \bigtriangledown (x \bigtriangledown y) = y$.

Conversely, whenever $\bigtriangledown$ is a commutative and idempotent quasigroup operation on $\mathbf{G}$ such that $x \bigtriangledown (x \bigtriangledown y) = y$ for any $x, y \in \mathbf{G}$ — hence we will say that $\bigtriangledown$ is an *Steiner triple operation* — it is obvious that the set of triples of elements of $\mathbf{G}$ of the type $\{x, y, x \bigtriangledown y\}$, with $x \neq y$, endows $\mathbf{G}$ of a structure of Steiner triple system. Moreover, it is clear that the associated operation $\multimap$ coincides with $\bigtriangledown$.

Whenever a Steiner triple operation $\bigtriangledown$ is autodistributive, we say that the corresponding Steiner triple system is autodistributive.

Henceforth we will limit ourselves to the case of autodistributive Steiner triple operations. Obviously, in this case property (3) of section 1 becomes:

$$x \bigtriangledown y = z \quad \Leftrightarrow \quad y \bigtriangledown z = x \quad \Leftrightarrow \quad z \bigtriangledown x = y.$$

Now we fix an element $0 \in \mathbf{G}$. Hence for any $x \in \mathbf{G}$ (cf. section 2) $x/2 = 0 \bigtriangledown x = 0 \multimap x = 2x = x \multimap 0 = -x$.

Then we define on $\mathbf{G}$ a commutative binary operation $\oplus$ by putting $x \oplus y := 0 \bigtriangledown (x \bigtriangledown y)$ $[= 2(x \bigtriangledown y) = (0 \bigtriangledown x) \bigtriangledown (0 \bigtriangledown y)]$.

It is easy to verify that on the set $\{0, x, 2x\}$ $\oplus$ coincides with the group operation $+$ defined in section 2. The following equalities hold:

$$x \bigtriangledown y = 0 \bigtriangledown (x \oplus y) = 0 \bigtriangledown [(0 \bigtriangledown x) \bigtriangledown (0 \bigtriangledown y)] = 0 \bigtriangledown (2x \bigtriangledown 2y) = 2x \bigtriangledown 2y; \quad (8)$$

consequently, we get:

$$(x \oplus y) \oplus (x \oplus y) = 0 \bigtriangledown (x \oplus y) = 0 \bigtriangledown [0 \bigtriangledown (x \bigtriangledown y)] = x \bigtriangledown y = 2x \oplus 2y. \quad (9)$$

In the sequel often we will write $x - y$ instead of $x \oplus (-y)$.

If $\oplus$ is associative, it is clear that $(\mathbf{G}, \oplus)$ is a commutative and b-2-divisible group. Moreover, it is easy to verify that $\bigtriangledown$ is the g-mean associated with $\oplus$.

9 Remark. If $(\mathbf{G}, +)$ is a group of exponent 3 (wiz. $3x = 0$, for any $x \in \mathbf{G}$), $\bigtriangledown = \multimap$. Indeed $a \bigtriangledown b = a + (-a + b)/2 = a + 2(-a + b) = b - a + b = a \multimap b$.

Conversely, if $(\mathbf{G}, \bigtriangledown, \multimap)$ is the g-mean associated with a group $(\mathbf{G}, +)$ and $\bigtriangledown = \multimap$, then it is easy to verify that $(\mathbf{G}, +)$ has exponent 3. Therefore $x \oplus y = y - x + y + y - x + y$. Thus, whenever the group is commutative, $\oplus = +$.

10 Theorem. *For any $x, y \in \mathbf{G}$, the following equalities hold:*

(j_0) $(x \bigtriangledown y) \oplus z = (x \oplus z) \bigtriangledown (y \oplus z)$;

(j_1) $(x - y) \oplus y = x$;

(j_2) $(x \oplus y) \oplus y = x \oplus (-y) = x \oplus (y \oplus y)$;

(j_3) $[(-x) \oplus y] \oplus [(x \oplus y)] = -y$.

Proof.

(j_0) It is obvious, by definition of $\oplus$ and by autodistributivity of $\bigtriangledown$.

(j_1) In fact $(x-y) \oplus y = 0 \bigtriangledown [0 \bigtriangledown (x \bigtriangledown (0 \bigtriangledown y)) \bigtriangledown y] = (x \bigtriangledown (0 \bigtriangledown y)) \bigtriangledown (0 \bigtriangledown y) = x$.

(j_2) Indeed, by commutativity of $\oplus$ and by some properties of $\bigtriangledown$, we get $(x \oplus y) \oplus y = 0 \bigtriangledown [(0 \bigtriangledown (x \bigtriangledown y)) \bigtriangledown y] = 0 \bigtriangledown [(0 \bigtriangledown y) \bigtriangledown x] = 0 \bigtriangledown [(y \oplus y) \bigtriangledown x] = x \oplus (y \oplus y)$.

(j_3) In fact (see (j_0)) $[(-x) \oplus y] \oplus [x \oplus y] = 0 \bigtriangledown [((-x) \oplus y) \bigtriangledown (x \oplus y)] = 0 \bigtriangledown [((-x) \bigtriangledown x) \oplus y] = 0 \bigtriangledown [0 \oplus y] = 0 \bigtriangledown y = -y$.

$\boxed{QED}$

11 Theorem. *Let $a, b \in \mathbf{G}$, with $a \neq 0$ and $b \notin ((0, a))$. Then the structure $(\ ((0, a, b)), \oplus)$ is isomorphic to $(\ ((0, a)), +) \times (\ ((0, b)), +)$.*

Proof. In order to prove the theorem it is sufficient to verify that if h, k, h', k' belong to $\{-1, 0, 1\}$, then $(ha \oplus kb) \oplus (h'a \oplus k'b) = (h + h)'a \oplus (k + k')b$.

Since $\oplus$ is abelian, if at least one of the coefficients h, k, h', k' is 0, then this latter claim is true by property (j_2) and (j_1) in Theorem 10. Moreover the claim is trivial also whenever $h'a \oplus k'b = ha \oplus kb$ or $h'a \oplus k'b = (-h)a \oplus (-k)b$ [$= -(ha \oplus kb)$].

Hence it remains to considering a part of the case in which either $h' = -h$ or $k' = -k$. Then the claim is an easy consequence of (j_3) in Theorem 10. $\boxed{QED}$

12 Remark. We point out that as an immediate corollary of theorem above we get that $\oplus$ endows $((0, a, b))$ of the structure of affine desarguesian (Galois) plane of order 3.

We conclude by emphasizing that if $\mathbf{H}$ is a uni-2-divisible subgroup of a uni-2-divisible group $(\mathbf{G}, +)$, then $\mathbf{H}$ is a submean of $(\mathbf{G}, \bigtriangledown, \multimap)$. But there can be some submeans $\mathbf{H}$ of $(\mathbf{G}, \bigtriangledown, \multimap)$, with $0 \in \mathbf{H}$, which are not subgroups of $(\mathbf{G}, +)$. For instance, one can consider the non-commutative group $(\mathbf{G}, +)$ of order 27 and exponent 3 (see [8], p. 146, exercise 6). Thus whenever $a, b \in \mathbf{G}$, with $a + b \neq b + a$, the set $((0, a, b))$ is a submean of $(\mathbf{G}, \bigtriangledown, \multimap)$, but it is not a subgroup. Indeed $((0, a, b))$ has 9 elements; meantime the subgroup generated by a and b coincides with $\mathbf{G}$ (hence it has 27 elements).

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