Note di Matematica
Note Mat. 1(2008), 69-76
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v28n1p69
http://siba2.unile.it/notemat

# Remarks on digital deformation

**Laurence Boxer**[i]
*Department of Computer and Information Sciences, Niagara University, Niagara University, NY 14109, USA*
boxer@niagara.edu

Received: 24/10/2006; accepted: 09/01/2007.

**Abstract.** The paper [5] defines a notion of digital deformation and claims to prove that if $(X, p)$ is $k$-deformable into $(A, p)$, then these two pointed images have isomorphic fundamental groups. We present a simple counterexample to this claim.

**Keywords:** digital image, digitally continuous, deformation, homotopy, fundamental group, digital topology

**MSC 2000 classification:** primary 55Q99, 62HR35, 68U10

## 1 Introduction

In *digital topology*, we examine geometric properties of digital images via tools adapted from Euclidean topology. These tools include digital versions of continuous functions, *homotopy* (continuous deformation), homotopy type, and the fundamental group. A theme of several recent papers [3, 4, 5] is the relationship between the fundamental groups $\Pi_1^{k_0}(X, x)$ and $\Pi_1^{k_1}(f(X), f(x))$, where $f : (X, x) \to (f(X), f(x))$ is a $(k_0, k_1)$-continuous function.

Of interest in this paper is the case of $f$ being a pointed deformation. It is claimed in [5] that for pointed deformations $f : (X, p) \to (A, p)$, $\Pi_1^{k_0}(X, p)$ and $\Pi_1^{k_1}(A, p)$ are isomorphic. In this paper, we present a simple counterexample to this claim.

## 2 Preliminaries

### 2.1 General properties

Let $\mathbf{N}$ be the set of natural numbers and let $\mathbf{Z}$ denote the set of integers. Then $\mathbf{Z}^n$ is the set of lattice points in Euclidean $n$-dimensional space.

Adjacency relations frequently used for digital images include the following [8]. Two points $p$ and $q$ in $\mathbf{Z}^2$ are 8-adjacent if they are distinct and differ

[i]Department of Computer and Information Sciences, Niagara University, Niagara University, NY 14109, USA; and Department of Computer Science and Engineering, State University of New York at Buffalo.

by at most 1 in each coordinate; $p$ and $q$ in $\mathbf{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate. Two points $p$ and $q$ in $\mathbf{Z}^3$ are 26-*adjacent* if they are distinct and differ by at most 1 in each coordinate; they are 18-*adjacent* if they are 26-adjacent and differ in at most two coordinates; they are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate. For $k \in \{4, 8, 6, 18, 26\}$, a $k-$ *neighbor* of a lattice point $p$ is a point that is $k$-adjacent to $p$.

Let $\kappa$ be an adjacency relation defined on $\mathbf{Z}^n$. A digital image $X \subset \mathbf{Z}^n$ is $\kappa$-connected [6] if and only if for every pair of points $\{x, y\} \subset X$, $x \neq y$, there exists a set $\{x_0, x_1, \ldots, x_c\} \subset X$ such that $x = x_0$, $x_c = y$, and $x_i$ and $x_{i+1}$ are $\kappa$-neighbors, $i \in \{0, 1, \ldots, c-1\}$. A $\kappa$-*component* of $X$ is a maximal $\kappa$-connected subset of $X$.

**1 Definition** ([2])**.** Let $a, b \in \mathbf{Z}$, $a < b$. A digital interval is a set of the form

$$[a, b]_{\mathbf{Z}} \ = \ \{z \in \mathbf{Z} \mid a \leq z \leq b\}$$

in which 2-adjacency is assumed.

**2 Definition** ([3]; see also [11])**.** Let $X \subset \mathbf{Z}^{n_0}$, $Y \subset \mathbf{Z}^{n_1}$. Let $f : X \to Y$ be a function. Let $\kappa_i$ be an adjacency relation defined on $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$. We say $f$ is $(\kappa_0, \kappa_1)$-continuous if for every $\kappa_0$-connected subset $A$ of $X$, $f(A)$ is a $\kappa_1$-connected subset of $Y$.

We say a function satisfying Definition 2 is *digitally continuous.* This definition implies the following.

**3 Proposition** ([3]; see also [11])**.** *Let $X$ and $Y$ be digital images. Then the function $f : X \to Y$ is $(\kappa_0, \kappa_1)$-continuous if and only if for every $\{x_0, x_1\} \subset X$ such that $x_0$ and $x_1$ are $\kappa_0$-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1$-adjacent.*

For example, if $\kappa$ is an adjacency relation on a digital image $Y$, then $f : [a, b]_{\mathbf{Z}} \to Y$ is $(2, \kappa)$-continuous if and only if for every $\{c, c+1\} \subset [a, b]_{\mathbf{Z}}$, either $f(c) = f(c+1)$ or $f(c)$ and $f(c+1)$ are $\kappa$-adjacent.

## 2.2 Digital homotopy

Roughly, a homotopy between continuous functions is a continuous deformation of one of the functions into the other over a time period.

**4 Definition** ([3]; see also [7])**.** Let $X$ and $Y$ be digital images. Let $f, g : X \to Y$ be $(\kappa, \kappa')$-continuous functions. Suppose there is a positive integer $m$ and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;

- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by

$$F_x(t) \;=\; F(x,t) \text{ for all } t \in [0,m]_{\mathbf{Z}}$$

  is $(2, \kappa')$-continuous.

- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by

$$F_t(x) \;=\; F(x,t) \text{ for all } x \in X$$

  is $(\kappa, \kappa')$-continuous.

Then $F$ is a digital $(\kappa, \kappa')$-homotopy between $f$ and $g$, and $f$ and $g$ are digitally $(\kappa, \kappa')$-homotopic in $Y$. If for some $x \in X$ we have $F(x,t) = F(x,0)$ for all $t \in [0,m]_{\mathbf{Z}}$, we say $F$ holds $x$ fixed.

The notation

$$f \;\simeq_{\kappa,\kappa'}\; g$$

indicates that functions $f$ and $g$ are digitally $(\kappa, \kappa')$-homotopic in $Y$.

If $(X, \kappa)$ is a digital image and $x_0 \in X$, the triple $(X, x_0, \kappa)$ is a *pointed digital image.*

For $p \in Y$, we denote by $\overline{p}$ the constant function $\overline{p} : X \to Y$ defined by $\overline{p}(x) = p$ for all $x \in X$.

**5 Definition.** A digital image $(X, \kappa)$ is $\kappa$-contractible [7, 2] if its identity map is $(\kappa, \kappa)$-homotopic to a constant function $\overline{p}$ for some $p \in X$. If the homotopy of the contraction holds $p$ fixed, we say $(X, p, \kappa)$ is pointed $\kappa$-contractible.

**6 Example** ([2])**.** Every digital interval $[0, m]_{\mathbf{Z}}$ is pointed contractible.

## 2.3 Digital loops

**7 Definition** (See [7])**.** A digital $\kappa$-path in a digital image $X$ is a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbf{Z}} \to X$. If, further, $f(0) = f(m)$, we call $f$ a digital $\kappa$-loop, and the point $f(0)$ is the basepoint of the loop $f$. If $f$ is a constant function, it is called a trivial loop.

If $f$ and $g$ are digital $\kappa$-paths in $X$ such that $g$ starts where $f$ ends, the *product* (see [7]) of $f$ and $g$, written $f \cdot g$, is, intuitively, the $\kappa$-path obtained by following $f$ by $g$. Formally, if $f : [0, m_1]_{\mathbf{Z}} \to X$, $g : [0, m_2]_{\mathbf{Z}} \to X$, and $f(m_1) = g(0)$, then $(f \cdot g) : [0, m_1 + m_2]_{\mathbf{Z}} \to X$ is defined by

$$(f \cdot g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_{\mathbf{Z}}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_{\mathbf{Z}}. \end{cases}$$

Unlike its Euclidean model, a digital interval is a finite set, so were we to restrict homotopy classes of loops to loops defined on the same digital interval, we would limit the class of a given loop undesirably. The following notion of *trivial extension* permits a loop to "stretch" and remain in the same pointed homotopy class. Intuitively, $f'$ is a trivial extension of $f$ if $f'$ follows the same path as $f$, but more slowly, with pauses for rest (subintervals of the domain on which $f'$ is constant).

**8 Definition** ([3])**.** Let $f$ and $f'$ be $\kappa$-paths in a digital image $X$. We say $f'$ is a trivial extension of $f$ if there are sets of $\kappa$-paths $\{f_1, f_2, \ldots, f_k\}$ and $\{F_1, F_2, \ldots, F_p\}$ in $X$ such that

(1) $k \leq p$;

(2) $f = f_1 \cdot f_2 \cdot \cdots \cdot f_k$;

(3) $f' = F_1 \cdot F_2 \cdot \cdots \cdot F_p$; and

(4) there are indices $1 \leq i_1 < i_2 < \cdots < i_k \leq p$ such that

- $F_{i_j} = f_j$, $1 \leq j \leq k$, and
- $i \notin \{i_1, i_2, \ldots, i_k\}$ implies $F_i$ is a trivial loop.

This notion lets us compare the digital homotopy properties of loops even if their domains have differing cardinality, since if $m_1 \leq m_2$, we obtain a trivial extension of a loop $f : [0, m_1]_{\mathbf{Z}} \to X$ to $f' : [0, m_2]_{\mathbf{Z}} \to X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_1; \\ f(m_1) & \text{if } m_1 \leq t \leq m_2. \end{cases}$$

The following notions are useful for defining the class of a pointed loop.

**9 Definition.** Let $f, g : [0, m]_{\mathbf{Z}} \to (X, x_0)$ be digital loops with basepoint $x_0$. If $H : [0, m]_{\mathbf{Z}} \times [0, M]_{\mathbf{Z}} \to X$ is a digital homotopy between $f$ and $g$ such that for all $t \in [0, M]_{\mathbf{Z}}$ we have

$$H(0, t) \;=\; H(m, t),$$

we say $H$ is loop-preserving. If, further, for all $t \in [0, M]_{\mathbf{Z}}$ we have

$$H(0, t) \;=\; H(m, t) \;=\; x_0,$$

we say $H$ holds the endpoints fixed.

The notion of $H$ holding the endpoints fixed was introduced in [4]. The term "loop-preserving" suggests that every (time) stage of the homotopy yields a digital loop.

Digital $\kappa$-loops $f$ and $g$ in X with the same basepoint $p$ *belong to the same $\kappa$-loop class in $X$* if there are trivial extensions $f'$ and $g'$ of $f$ and $g$, respectively, with domains of the same cardinality, and a loop-preserving homotopy between $f'$ and $g'$ that holds the endpoints fixed [3].

Membership in the same loop class in $(X, x_0)$ is an equivalence relation among digital $\kappa$-loops [3].

Let $[f]$ be the loop class of a loop $f$ in $X$. We have the following.

**10 Proposition** ([3, 7]). *Suppose $f_1, f_2, g_1, g_2$ are digital loops in a pointed digital image $(X, x_0)$, with $f_2 \in [f_1]$ and $g_2 \in [g_1]$. Then $f_2 \cdot g_2 \in [f_1 \cdot g_1]$.*

## 2.4 Digital fundamental group

The digital fundamental group is derived from the classical fundamental group of algebraic topology (see [10]).

Let $(X, p, \kappa)$ be a pointed digital image. Consider the set $\Pi_1^\kappa(X, p)$ of $\kappa$-loop classes $[f]$ in $X$ with basepoint $p$. By Proposition 10, the *product* operation

$$[f] * [g] \ = \ [f \cdot g]$$

is well-defined on $\Pi_1^\kappa(X, p)$.

The operation $*$ is associative on $\Pi_1^\kappa(X, p)$ [7].

**11 Lemma** ([3]). *Let $(X, p)$ be a pointed digital image. Let $\overline{p} : [0, m]_{\mathbf{Z}} \to X$ be the constant function $\overline{p}(t) = p$. Then $[\overline{p}]$ is an identity element for $\Pi_1^\kappa(X, p)$.*

**12 Lemma** ([3]). *If $f : [0, m]_{\mathbf{Z}} \to X$ represents an element of $\Pi_1(X, p)$, then the function $g : [0, m]_{\mathbf{Z}} \to X$ defined by*

$$g(t) = f(m - t) \text{ for } t \in [0, m]_{\mathbf{Z}}$$

*is an element of $[f]^{-1}$ in $\Pi_1^\kappa(X, p)$.*

**13 Theorem** ([3]). *$\Pi_1^\kappa(X, p)$ is a group under the $*$ product operation, the* $\kappa$-fundamental group of *$(X, p)$.*

We may interpret the following result to say that in a connected digital image $X$, the digital fundamental group is independent of the choice of basepoint.

**14 Theorem** ([3]). *Let $X$ be a digital image with adjacency relation $\kappa$. If $p$ and $q$ belong to the same $\kappa$-component of $X$, then $\Pi_1^\kappa(X, p)$ and $\Pi_1^\kappa(X, q)$ are isomorphic groups.*

**15 Proposition** ([4]). *Let $X$ be a pointed $\kappa$-contractible digital image and let $p \in X$. Then $\Pi_1^\kappa(X, p)$ is a trivial group.*

## 2.5 Deformation and deformation retraction

We have the following.

**16 Definition** ([5]). Let $(X, \kappa)$ be a digital image and let $A$ be a nonempty subset of $X$. Then $X$ is $\kappa$-deformable into $A$ if there is a $\kappa$-homotopy $D : X \times [0, m]_{\mathbf{Z}} \to X$ such that $D(x, 0) = x$ and $D(x, m) \in A$ for all $x \in X$. $D$ is called a $\kappa$-deformation. If for some $x_0 \in A$ we have $D(x_0, t) = x_0$ for all $t \in [0, m]_{\mathbf{Z}}$, we say $X$ is pointed $\kappa$-deformable into $A$, and $D$ is a pointed $\kappa$-deformation.

Classical notions of topology [1] yielded the concepts of digital retraction and deformation retraction in [2]. Let $(X, \kappa)$ be a digital image and let $A$ be a nonempty subset of $X$. A *retraction* of $X$ onto $A$ is a $(\kappa, \kappa)$-continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. A *$\kappa$-deformation retraction* of $X$ to $A$ is a $\kappa$-homotopy $H : X \times [0, m]_{\mathbf{Z}} \to X$ such that the induced map $H(\_, 0)$ is the identity map $1_X$, and the induced map $H(\_, m)$ is a retraction of $X$ onto $A$.

## 3 Deformations, deformation retractions, and fundamental groups

Notice that a deformation retraction is a pointed deformation. We have the following.

**17 Theorem** ([4]). *Let $A$ be a nonempty subset of a digital image $X$ and let $H : X \times [0, m]_{\mathbf{Z}} \to X$ be a $\kappa$-deformation retraction of $X$ onto $A$. For $a \in A$, the inclusion map $i : (A, a) \hookrightarrow (X, a)$ induces an isomorphism of $\Pi_1^{\kappa}(A, a)$ and $\Pi_1^{\kappa}(X, a)$.*

However, a pointed deformation from a digital image $X$ into its nonempty subset $A$ need not yield an isomorphism of the fundamental groups of $X$ and $A$, despite the claim of Han (presented as Theorem 3 of [5]) to the contrary. Indeed, Han's claim is false even if the deformation is required to be onto $A$ at the end of the homotopy. Consider the pair $(X, A)$ defined as follows. $X = ([0, 2]_{\mathbf{Z}} \times [0, 2]_{\mathbf{Z}}) \cup \{(j, 0)\}_{j=3}^{7}$. Let $A \subset X$ be the set $A = ([0, 2]_{\mathbf{Z}} \times [0, 2]_{\mathbf{Z}}) \setminus \{(1, 1)\}$, a simple closed 4-curve; hence $\Pi_1^4(A, a)$ is isomorphic to $\mathbf{Z}$ [3, 8, 9]. It is easily seen that $X$ is 4-contractible via the function $h : X \times [0, 9]_{\mathbf{Z}} \to X$ defined by

$$h(x, y, t) \;=\; \begin{cases} (x, \max\{0, y - t\}) & \text{for } 0 \le t \le 2; \\ (\max\{0, x + 2 - t\},\, 0) & \text{for } 3 \le t \le 9. \end{cases}$$

Hence $X$ has a trivial 4-fundamental group.

We show there is a 4-deformation of $X$ onto $A$ as follows. Consider the function $H : X \times [0,8]_{\mathbf{Z}} \to X$ given as follows.

$$H(x,y,t) = \begin{cases} (\max\{0, x-t\},\, y) & \text{if } t \in [0,2]_{\mathbf{Z}}; \\ (\max\{2, x-t\},\, 0) & \text{if } (x,y) \in \{(j,0)\}_{j=5}^{7},\ 3 \le t \le 5; \\ H(x,y,2) & \text{if } (x,y) \notin \{(j,0)\}_{j=5}^{7},\ 3 \le t \le 5; \\ (2,1) & \text{if } (x,y) \in \{(j,0)\}_{j=5}^{7},\ t = 6; \\ H(x,y,5) & \text{if } (x,y) \notin \{(j,0)\}_{j=5}^{7},\ t = 6; \\ (2,2) & \text{if } (x,y) \in \{(6,0),\, (7,0)\},\ t = 7; \\ H(x,y,6) & \text{if } (x,y) \notin \{(6,0),\, (7,0)\},\ t = 7; \\ (1,2) & \text{if } (x,y) = (7,0),\ t = 8; \\ H(x,y,7) & \text{if } (x,y) \ne (7,0),\ t = 8. \end{cases}$$

It is easily seen that this function is a 4-homotopy between $1_X$ and the function $H_8 : X \to X$, defined by $H_8(x,y) = H(x,y,8)$, that is onto $A$.

What is valid from Han's paper is the following.

**18 Theorem.** *Let $X$ be a digital image and let $A$ be a non-empty subset of $X$. Let $D : X \times [0,m]_{\mathbf{Z}} \to X$ be a pointed $\kappa$-deformation of $X$ into $A$, with $D(p,t) = p$ for some $p \in A$ and all $t \in [0,m]_{\mathcal{Z}}$. Let $r : X \to A$ be the map defined by $r(x) = D(x,m)$ for all $x \in X$. Then the induced homomorphism $r_* : \Pi_1^\kappa(X,p) \to \Pi_1^\kappa(A,p)$ is one to one.*

PROOF. [5] Let $i : A \to X$ be the inclusion map. Then $D$ is a homotopy between $1_X$ and $i \circ r$. Therefore, $1_{\Pi_1^\kappa(X,p)} = i_* \circ r_*$. The assertion follows. QED

## 4 Summary

We have shown that the claim of [5], that a pointed digital deformation induces an isomorphism between fundamental groups, is false.

**Acknowledgements.** We gratefully acknowledge a suggestion of an anonymous referee.

## References

[1] K. Borsuk: Theory of Retracts, Polish Scientific Publishers, Warsaw 1967.

[2] L. Boxer: *Digitally continuous functions*, Pattern Recognition Letters, **15** (1994), 833–839.

[3] L. Boxer: *A classical construction for the digital fundamental group*, Journal of Mathematical Imaging and Vision, **10** (1999), 51–62.

[4] L. Boxer: *Properties of digital homotopy*, Journal of Mathematical Imaging and Vision, **22** (2005), 19–26.

[5] S.-E. Han: *Generalized digital* $(k_0, k_1)$*-homeomorphism*, Note di Matematica, **22** (2003), 157–166.

[6] G. T. Herman: *Oriented surfaces in digital spaces*, CVGIP: Graphical Models and Image Processing, **55** (1993), 381–396.

[7] E. Khalimsky: *Motion, deformation, and homotopy in finite spaces*, in Proceedings IEEE Intl. Conf. on Systems, Man, and Cybernetics, (1987) 227–234.

[8] T. Y. Kong: *A digital fundamental group*, Computers and Graphics, **13** (1989), 159–166.

[9] T. Y. Kong, A. W. Roscoe, A. Rosenfeld: *Concepts of digital topology*, Topology and its Applications, **46** (1992), 219–262.

[10] W. S. Massey: Algebraic Topology: An Introduction, Harcourt, Brace, and World, New York 1967.

[11] A. Rosenfeld: *'Continuous' functions on digital pictures*, Pattern Recognition Letters, **4** (1986), 177–184.