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More ubiquitous undetermined games and other results on uncountable length games in Boolean algebras

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Abstract. This paper surveys some of the known theory for countable length games related to distributive laws in Boolean algebras. The results can be naturally extended to uncountable length games, and detailed proofs are given. In particular, we show the following for uncountable length games related to distributive laws in Boolean algebras. When $|\kappa^{<\kappa}| = \kappa$, there is a Boolean algebra in which $\mathcal{G}_1^\kappa(2)$ is undetermined. $\mathcal{G}_1^\kappa(\infty)$ is equivalent to G_κ^{II}, the strategically closed forcing game. Under certain weak assumptions on cardinal arithmetic, Player II having a winning strategy for G_κ^{I} implies $\mathbb{B}$ has a dense subtree which is $< \kappa^+$-closed.

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1 Introduction

Distributive laws are a useful tool for classifying and characterising Boolean algebras as well as giving information about their forcing extensions. They provide a means of measuring how close to an algebra of sets a given Boolean algebra is: Every κ^+-algebra of sets is (κ, κ)-distributive, and a complete Boolean algebra is completely distributive if and only if it is isomorphic to a power set algebra [15]. Measure algebras (those Boolean algebras obtained by taking the σ-algebra of measurable sets of some probability measure space and modding out by the null sets) are weakly (ω, ω)-distributive but not $(\omega, 2)$-distributive. This provides one way of testing whether a Boolean algebra does not carry a strictly positive countably additive measure. By their forcing-equivalent state-

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ments, distributive laws give information about which new functions on ordinals a forcing extension by a Boolean algebra can have.

We are interested in the game analogues of general distributive laws in Boolean algebras. This has a heritage in the Banach-Mazur game played on the real numbers. Jech initiated the study of the analogue of the Banach-Mazur game in the context of Boolean algebras. This game has close ties with the (ω,∞)-distributive law. (See [7].) Later he expanded this study to countable length cut-and-choose games related to more general distributive laws. (See [8].) We built on Jech's work in [4], [2], and [3].

In this paper we survey some results of Foreman, Veličković, and Zapletal for countable length games and show how they easily generalise to uncountable length games. As the results for uncountable length games require some extra hypotheses, we provide proofs for the sake of availability in the literature. In section 3 we give an example of a Boolean algebra in which many cut-and-choose games are undetermined, using weaker hypothesis than in the previously existing examples. Section 4 provides a proof that the well-known strategically-closed forcing game is equivalent to a cut-and-choose game of the same length. In section 5 we give conditions when II having a winning strategy for G^{I}_κ in a Boolean algebra $\mathbb{B}$ implies that $\mathbb{B}$ contains a $<\kappa^+$-closed dense subset, extending some work of Foreman and Veličković. The paper concludes in section 6 with a generalisation of a theorem of Jech linking Axiom A with cut-and-choose games; this along with other results yield game-theoretic analyses of two uncountable height tree forcings.

2 Definitions, basic facts and notation

Throughout this paper, we restrict ourselves to the class of complete Boolean algebras. Let $\mathbb{B}$ denote an arbitrary complete Boolean algebra, and let $\mathbb{B}^+$ denote $\mathbb{B}\setminus\{\mathbf{0}\}$. Basic set-theoretic notation is used. We let $[\lambda]^\kappa=\{x\subseteq\lambda : |x|=\kappa\}$, $[\lambda]^{<\kappa}=\{x\subseteq\lambda : |x|<\kappa\}=\mathcal{P}_\kappa\lambda$, and $[\lambda]^{\le\kappa}=\{x\subseteq\lambda : |x|\le\kappa\}=\mathcal{P}_{\kappa^+}\lambda$. Both $X^{<\kappa}$ and $(X)^{<\kappa}$ are used to denote the set (or tree ordered by end extension) of sequences from ordinals $\alpha<\kappa$ into X. $\|\varphi\|$ denotes the Boolean value of φ in $\mathbb{B}$. We assume a knowledge of forcing and Boolean valued models.

The definition of distributive laws is presented here in the most general form encountered in this paper.

1 Definition ([15])**.** $\mathbb{B}$ satisfies the $(\kappa,\lambda,<\mu)$-*distributive law* ($(\kappa,\lambda,<\mu)$-d.l.) if for all families $\{b_{\alpha,\beta} : \alpha<\kappa,\ \beta<\lambda\}\subseteq\mathbb{B}$,

$$\bigwedge_{\alpha<\kappa}\bigvee_{\beta<\lambda} b_{\alpha,\beta} \;=\; \bigvee_{f:\kappa\to[\lambda]^{<\mu}}\ \bigwedge_{\alpha<\kappa}\ \bigvee_{\beta\in f(\alpha)} b_{\alpha,\beta}. \tag{1}$$

2 Notation. The $(\kappa, \lambda, <2)$-d.l. is usually referred to as the (κ, λ)-d.l., and the $(\kappa, \lambda, <\omega)$-d.l. is usually referred to as the weak (κ, λ)-d.l. We say that the $(<\kappa, \lambda)$-d.l. holds if the (ρ, λ)-d.l. holds for all $\rho < \kappa$. We say that the (κ, ∞)-d.l. holds if the (κ, λ)-d.l. holds for all cardinals λ.

3 Remark. For cardinals $\kappa_0 \leq \kappa_1$ and $2 \leq \mu_0 \leq \mu_1 \leq \lambda_0 \leq \lambda_1$, the $(\kappa_1, \lambda_1, <\mu_0)$-d.l. implies the $(\kappa_0, \lambda_0, <\mu_1)$-d.l.

4 Definition ([15])**.** A *partition of unity* (*of* a) is a collection $W \subseteq \mathbb{B}^+$ such that $\bigvee W = \mathbf{1}$ ($\bigvee W = a$) and for all $b, c \in W$ with $b \neq c$, $b \wedge c = \mathbf{0}$.

The following fact is well-known. A proof of (1) $\Longleftrightarrow$ (2) can be found in [15]. A proof of (1) $\Longleftrightarrow$ (3) for $\mu = 2$ can be found in [10], and a proof for the more general case for any $2 \leq \mu \leq \lambda$ follows easily.

5 Fact. The following are equivalent.

(1) $\mathbb{B}$ satisfies the $(\kappa, \lambda, <\mu)$-d.l.

(2) For any family W_α, $(\alpha < \kappa)$, of partitions of unity of $\mathbb{B}$ with each $|W_\alpha| \leq \lambda$, there exists a partition of unity W such that for each $b \in W$, for each $\alpha < \kappa$, $b \wedge c \neq \mathbf{0}$ for less than μ-many $c \in W_\alpha$.

(3) For each $\mathbb{B}$-name $\dot{g}$ for a function from $\check{\kappa}$ into $\check{\lambda}$ and any generic filter $G \subseteq \mathbb{B}^+$, there is a function $f : \kappa \to [\lambda]^{<\mu}$ in V such that $V[G] \models$"$\forall \alpha < \check{\kappa}$, $\dot{g}(\alpha) \in f(\alpha)$".

Recall the following game related to the $(\kappa, \lambda, <\mu)$-d.l., which we introduced in [4]. This generalises a game of Jech related to the weak (ω, λ)-d.l. in [8].

6 Definition ([4])**.** Let κ, λ be infinite cardinals and μ be a cardinal such that $2 \leq \mu \leq \lambda$. The game $\mathcal{G}^\kappa_{<\mu}(\lambda)$ is played between two players in a complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, Player I fixes some $a \in \mathbb{B}^+$. For $\alpha < \kappa$, the α-th round is played as follows: Player I chooses a partition W_α of a such that $|W_\alpha| \leq \lambda$; then Player II chooses some $E_\alpha \in [W_\alpha]^{<\mu}$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, E_0, W_1, E_1, \ldots, W_\alpha, E_\alpha, \ldots : \alpha < \kappa \rangle \tag{2}$$

called a *play* of the game. Player I *wins the play* (2) if and only if

$$\bigwedge_{\alpha < \kappa} \bigvee E_\alpha = \mathbf{0}. \tag{3}$$

$\mathcal{G}^\kappa_{<\mu}(\infty)$ is the game played like $\mathcal{G}^\kappa_{<\mu}(\lambda)$, except now Player I can choose partitions of any size.

7 Fact. If II has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$, then $\mathbb{B}$ satisfies the $(\kappa, \lambda, <\mu)$-d.l.

8 Notation. If $\mu = \nu^+$, we often write $\mathcal{G}^{\kappa}_{\nu}(\lambda)$ instead of $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$. In particular, we write $\mathcal{G}^{\kappa}_{1}(\lambda)$ for $\mathcal{G}^{\kappa}_{<2}(\lambda)$.

9 Remark. $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ can be played on a partial ordering $\mathbb{P}$. We say that II wins the play iff there is a $p \in \mathbb{P}$ such that $p \leq a$ and $\forall \alpha < \kappa$, E_α is pre-dense below p. If $\mathbb{P}$ is a separative partial ordering, then Player I (II) has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ in $\mathbb{P}$ iff Player I (II) has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ in $\mathrm{r.o.}(\mathbb{P})$.

10 Remark. Let (X, τ) be a topological space. $U \in \tau$ is *regular open* if $\mathrm{int}(\mathrm{cl}(U)) = U$. Let $\mathrm{ro}(\tau)$ denote the set of regular open members of τ. (This constitutes a complete Boolean algebra under the appropriate Boolean operations (see [15]).) For $b \in \mathrm{ro}(\tau)$ nonempty, let $\mathrm{r}\mathcal{O}_b = \{\, \mathcal{U} \subseteq \mathrm{ro}(\tau) : \mathrm{int}(\mathrm{cl}(\bigcup \mathcal{U})) = b \,\}$. In the standard notation of SPM, the game $\mathcal{G}^{\omega}_{1}(2)$ is the game $\mathrm{G}_1(\mathcal{A}_b, \mathcal{B}_b, b \in \mathrm{ro}(\tau) \setminus \{\emptyset\})$: Player I fixes some nonempty $b \in \mathrm{ro}(\tau)$ at the beginning of the game. Then the game $\mathrm{G}_1(\mathcal{A}_b, \mathcal{B}_b)$ is played, where $\mathcal{A}_b = \{\, \mathcal{U} \in \mathrm{r}\mathcal{O}_b : |\mathcal{U}| = 2$ and $\mathrm{int}(\mathrm{cl}(U_0 \cap U_1)) = \emptyset \,\}$, and $\mathcal{B}_b = \{\, \mathcal{S} \in [\mathrm{ro}(\tau)]^{\leq \omega} : \mathrm{int}(\mathrm{cl}(\bigcap \mathcal{S})) \neq \emptyset \,\}$. Similarly for the other games.

The following fact relates the various games.

11 Fact. Let $\mathbb{B}$ be a complete Boolean algebra, and let $\kappa_0 \leq \kappa_1$ and $2 \leq \mu_0 \leq \mu_1 \leq \lambda_0 \leq \lambda_1$. If Player II has a winning strategy for $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1)$, then II has a winning strategy for $\mathcal{G}^{\kappa_0}_{<\mu_1}(\lambda_0)$. If Player I has a winning strategy for $\mathcal{G}^{\kappa_0}_{<\mu_1}(\lambda_0)$, then I has a winning strategy for $\mathcal{G}^{\kappa_1}_{<\mu_0}(\lambda_1)$.

The investigation of relationships between games and distributive laws began with Jech's work in [7], where he characterised the (ω, ∞)-d.l. in terms of Player I not having a winning strategy in the descending sequence game of length ω. Then he developed the theory of cut-and-choose games of length ω and related distributive laws in [8]. One of these games yields a property strictly intermediate between Axiom A and properness, and another of these games is used in Gray's Conjecture on Von Neumann's Problem concerning measurable Boolean algebras (see [8]). (Gray's conjecture has recently been refuted by Talagrand's solution to the von Neumann and Control Measure Problems.) In [4] we extended some of Jech's work to more general distributive laws.

12 Theorem (Dobrinen [4])**.** *Let $\mathbb{B}$ be a complete Boolean algebra.*

(1) If the (κ, λ)-d.l. fails in $\mathbb{B}$, then I has a winning strategy for $\mathcal{G}^{\kappa}_{1}(\lambda)$ in $\mathbb{B}$. This, in turn, implies that both the $(|\lambda^{<\kappa}|, \lambda)$-d.l. and the $(\kappa, |\lambda^{<\kappa}|)$-d.l. fail in $\mathbb{B}$. It follows that the (κ, ∞)-d.l. holds iff I does not have a winning strategy for $\mathcal{G}^{\kappa}_{1}(\infty)$.

(2) *If the $(\kappa, \lambda, < \mu)$-d.l. fails in $\mathbb{B}$, then I has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$. This, in turn, implies that the $(|(\lambda^{<\mu})^{<\kappa}|, \lambda, < \mu)$-d.l. fails in $\mathbb{B}$.*

Under GCH, this gives a game-theoretic characterisation of the (κ, λ)-d.l. whenever $\lambda < \kappa$ or $\mathrm{cf}(\lambda) \geq \kappa$, and a characterisation of the $(\kappa, \lambda, < \mu)$-d.l. whenever $\lambda < \kappa$, or $\kappa = \lambda$ and is regular. (See also [3] for further cases of triples of cardinals.)

3 A Boolean algebra in which $\mathcal{G}^{\kappa}_1(2)$ is undetermined

When games are the subject of study, it is of interest whether or not there are Boolean algebras in which neither player has a winning strategy. If that is the case, we say that the game is *undetermined* in that Boolean algebra.

Jech inaugurated this investigation by showing that $\mathcal{G}^{\omega}_1(\infty)$ is undetermined in the regular open algebra of the forcing which shoots a club through a stationary/co-stationary subset of $\aleph_1$. To get the size of partitions for Player I smaller, he used $\diamondsuit$ to construct a Suslin algebra in which $\mathcal{G}^{\omega}_1(2)$ is undetermined [8]. Zapletal improved on this to show that in ZFC, $\mathcal{G}^{\omega}_1(2)$ is undetermined in the regular open algebra of the forcing which shoots a club through a stationary/co-stationary subset of $\aleph_1$. He also gave another example in ZFC of a proper Boolean algebra in which $\mathcal{G}^{\omega}_1(2)$ is undetermined [18].

We are particularly interested in finding a Boolean algebra in which $\mathcal{G}^{\kappa}_1(2)$ is undetermined in ZFC. For $\kappa \geq \aleph_1$, we showed in [4] that for κ regular, $\diamondsuit_{\kappa^+}(\{\, \alpha < \kappa^+ : \mathrm{cof}(\alpha) = \kappa \,\}) + |\kappa^{<\kappa}| = \kappa$ suffice to construct a κ^+-Suslin algebra in which $\mathcal{G}^{\kappa}_1(2)$ is undetermined. By work of Cummings and Dobrinen, it is consistent with ZFC that for all cardinals κ, λ, μ with $\lambda \geq \mu \geq 2$, there is a Boolean algebra in which $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is undetermined. In fact, this holds in L [2]. However, this work assumes lots of diamond and also square, and there are many models of ZFC in which these do not hold. In [3], we showed if $|\kappa^{<\kappa}| = \kappa$ and θ is the least cardinal $> \kappa$ such that $|\theta^{<\kappa}| = \theta$, then there is a Boolean algebra in which the games $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ for all $2 \leq \mu \leq \kappa^+$ and $\lambda \geq \theta$ are undetermined. However, this did not address the case when $\lambda < \kappa$.

The main Theorem 23 of this section is that $|\kappa^{<\kappa}| = \kappa$ is sufficient to obtain a Boolean algebra in which $\mathcal{G}^{\kappa}_1(\lambda)$ is undetermined for all $2 \leq \lambda \leq |2^{\kappa}|$. To do so, it will be useful to have Lemma 15, a straightforward generalisation of Zapletal's Lemma 1 for $\mathcal{G}^{\omega}_1(2)$ in [18]. We begin with some basic facts about perfect subsets of generalised trees. Given $\sigma \in 2^{\kappa}$, let $B_\sigma = \{\, x \in 2^{\kappa} : \sigma \subseteq x \,\}$. We work in the topology generated by the basis $\{\, B_\sigma : \sigma \in 2^{<\kappa} \,\}$.

13 Definition. $X \subseteq 2^{\kappa}$ is *perfect* if X is closed and every element of X is not isolated.

Every closed subset of 2^κ can be coded by a subtree of $2^{<\kappa}$: For $X \subseteq 2^\kappa$, let $\tilde{X} = \{\, s \in 2^{<\kappa} : \exists x \in X(x \supseteq s) \,\}$. For $T \subseteq 2^{<\kappa}$ let $[T] = \{\, y \in 2^\kappa : \forall \alpha < \kappa,\ \exists s \in 2^\alpha \cap T(s \subseteq y) \,\}$. Given a closed $X \subseteq 2^\kappa$, $X = [\tilde{X}]$. Hence, there are $|2^{(2^{<\kappa})}|$ many perfect subsets of 2^κ.

14 Lemma. *In 2^κ there is a set of size $|2^\kappa|$ which contains no perfect set, assuming $|2^\kappa| = |\mathscr{P}(2^{<\kappa})|$.*

Proof. List all subtrees $\langle T_\gamma : \gamma < |2^\kappa| \rangle$ of $2^{<\kappa}$. We use transfinite induction to construct the desired set. Choose one $x_0 \in 2^\kappa$ and one $y_0 \in [T_0] \setminus \{\, x_0 \,\}$. Let $X_1 = \{\, x_0 \,\}$, $Y_1 = \{\, y_0 \,\}$. At Stage $\alpha < |2^\kappa|$, let $X_\alpha = \{\, x_\beta : \beta < \alpha \,\}$, $Y_\alpha = \{\, y_\beta : \beta < \alpha \,\}$. Choose one $x_\alpha \in 2^\kappa \setminus (X_\alpha \cup Y_\alpha)$ and one $y_\alpha \in [T_\alpha] \setminus (X_{\alpha+1} \cup Y_\alpha)$. Let $X = \{\, x_\alpha : \alpha < |2^\kappa| \,\}$, $Y = \{\, y_\alpha : \alpha < |2^\kappa| \,\}$. Then $|X| = |2^\kappa|$ and $X \cap Y = \emptyset$. $\forall \alpha < |2^\kappa|$, $y_\alpha \in [T_\alpha]$ and $y_\alpha \notin X$; so $X \not\supseteq [T_\alpha]$. Hence, X contains no perfect set. $\boxed{QED}$

The following is basically Zapletal's proof of Lemma 1 in [18].

15 Lemma. *Assume $|2^\kappa| = |\mathscr{P}(2^{<\kappa})|$ and Player II has a winning strategy σ for $\mathcal{G}_1^\kappa(2)$ in $\mathbb{B}$. For each maximal antichain $A \subseteq \mathbb{B}$ with $|A| \leq |2^\kappa|$, each $p \in \mathbb{B}^+$, and each $t \in \mathbb{B}^{<\kappa}$, there is an $a \in A$ and an $s \in \mathbb{B}^{<\kappa}$ such that $t \subseteq s$ and $g(p,s) \leq a$, where*

$$g(p,s) = p \wedge \left(\bigwedge_{i \in \mathrm{dom}(s),\ \sigma(p, s\restriction(i+1))=1} s(i) \ - \bigvee_{i \in \mathrm{dom}(s),\ \sigma(p, s\restriction(i+1))=0} s(i) \right). \quad (4)$$

Proof. Fix σ, A, p, t as in the hypotheses of the Lemma. Note: Since σ is a winning strategy for II, $g(p,s) > \mathbf{0}$ for each $p \in \mathbb{B}^+$ and each $s \in \mathbb{B}^{<\kappa}$. $|A| \leq |2^\kappa|$ implies that we can fix a sequence $\langle r_a : a \in A \rangle$ of distinct elements of 2^κ such that $\{\, r_a : a \in A \,\}$ does not contain a perfect subset of 2^κ. Let $\dot{u} : a \to r_a$. Then $\dot{u}$ is a $\mathbb{B}$-name for an element of 2^κ.

Claim. $\exists s_0 \in \mathbb{B}^{<\kappa}$ such that $t \subseteq s_0$, and $\exists r \in 2^\kappa$ such that $\forall s_1 \in \mathbb{B}^{<\kappa}$, $\forall i < \kappa$, if $s_0 \subseteq s_1$ and $g(p, s_1)$ decides $\|\dot{u}(i) = 1\|$, then $g(p, s_1) \Vdash \dot{u}(i) = \check{r}(i)$.

Proof. Assume not. Then for each $s_0 \in \mathbb{B}^{<\kappa}$ with $s_0 \supseteq t$ and for all $r \in 2^\kappa$, there is an $s_1 \in \mathbb{B}^{<\kappa}$ and an $i < \kappa$ such that $s_0 \subseteq s_1$ and $g(p, s_1)$ decides $\|\dot{u}(i) = 1\|$, but $g(p, s_1) \nVdash \dot{u}(i) = \check{r}(i)$. By induction on $\mathrm{lh}(\eta)$ for $\eta \in 2^{<\kappa}$, we build $s_\eta \in \mathbb{B}^{<\kappa}$ such that

(1) $s_{\langle\rangle} = t$, and $\eta_0 \subseteq \eta_1 \leftrightarrow s_{\eta_0} \subseteq s_{\eta_1}$;

(2) $\forall \eta \in 2^{<\kappa}$, $\exists j$ with $\mathrm{lh}(\eta) < j < \kappa$ such that both $g(p, s_{\eta^\frown 0})$ and $g(p, s_{\eta^\frown 1})$ decide all the $\dot{u}(i)$, $i \leq j$, and they decide $\dot{u}(j)$ differently.

Let $s_{\langle\rangle} = t$. Suppose we have s_η and cannot continue the induction. Then for each $j > \mathrm{lh}(\eta)$, and all $s', s'' \supseteq s_\eta$, if $g(p, s')$ and $g(p, s'')$ decide $\dot{u}(i)$ for all $i \leq j$, then they agree on $\dot{u}(j)$. For any s' and any j, we can find an $s'' \supseteq s'$ such that $g(p, s'')$ decides $\dot{u}(i)$ for all $i \leq j$. Since II wins $\mathcal{G}_1^\kappa(2)$, the game keeps going at lengths less than κ, so we can find such a $g(p, s'')$. This defines an $r \in 2^\kappa$ in V, namely $r(i) = 0$ iff $\exists s'' \supseteq s_\eta$ such that $g(p, s'') \Vdash \dot{u}(i) = 0$. But by our assumption that the Claim is false, there is an $s' \supseteq s_\eta$ and a $j < \kappa$ such that $g(p, s')$ decides $\|\dot{u}(j) = 1\|$ and $g(p, s') \nVdash \dot{u}(j) = \check{r}(j)$. We can extend s' to an s'' such that $g(p, s'')$ decides $\dot{u}(i)$ for all $i \leq j$. Then $g(p, s'') \Vdash \dot{u}(j) = \check{r}(j)$. But then $g(p, s') \Vdash \dot{u}(j) = \check{r}(j)$, since $g(p, s'') \leq g(p, s')$ and $g(p, s')$ decides $\dot{u}(j)$. Contradiction. Thus, the induction continues.

For $\alpha < \kappa$ a limit ordinal and $\eta \in 2^\alpha$, let $s_\eta = \bigcup_{\beta<\alpha} s_{\eta\restriction\beta}$. Then by the induction, we get s_η for $\eta \in 2^{<\kappa}$ satisfying (1) and (2). For all $x \in 2^\kappa$, let $r_x \in 2^\kappa$ be the unique element of 2^κ such that $g(p, \bigcup_{i<\kappa} s_{x\restriction i}) \Vdash \dot{u} = \check{r}_x$. This is well-defined by (2). Also by (2), $\{ r_x : x \in 2^\kappa \}$ is a perfect subset of 2^κ. Each $r_x = r_a$ for some $a \in A$, since when $\dot{u}$ is decided, the decision is in V. But $\{ r_a : a \in A \}$ contains no perfect set. Contradiction. Therefore, the Claim holds. $\boxed{QED}$

Let s_0, r be as in the Claim. Let $s = s_0{}^\frown\|\dot{u} = \check{r}\|$. We will show that $\sigma(p, s) = 1$. Assume $\sigma(p, s) = 0$. Then $g(p, s) \Vdash \dot{u} \neq \check{r}$. For each $j < \kappa$, let $s^j = s^\frown\langle \|\dot{u}(i) = 1\| : i \leq j\rangle$. Then for each $j < \kappa$, $g(p, s^j)$ decides $\|\dot{u}(j) = 1\|$. By choice of s_0 and r, we have $g(p, s^j) \Vdash \dot{u}(j) = \check{r}(j)$. Contradiction. Thus, $g(p, s) \leq \|\dot{u} = \check{r}\|$. $r \in V$ implies that there is an $a \in A$ such that $r = r_a$. So $g(p, s) \leq \|\dot{u} = \check{r}_a\|$. Therefore, $g(p, s) \leq a$. $\boxed{QED}$

It is well-known that in a complete Boolean algebra, the $(\kappa, 2)$-d.l. holds iff the $(\kappa, |2^\kappa|)$-d.l. holds (see [15]). The analogous situation holds for the games $\mathcal{G}_1^\kappa(2)$ and $\mathcal{G}_1^\kappa(|2^\kappa|)$, at least when $|2^\kappa| = |\mathscr{P}(2^{<\kappa})|$. The equivalence for Player I holds in ZFC: it follows from the just mentioned fact about distributive laws along with Theorem 12(1).

16 Theorem. *Suppose $|2^\kappa| = |\mathscr{P}(2^{<\kappa})|$. If II has a winning strategy for $\mathcal{G}_1^\kappa(2)$, then II has a winning strategy for $\mathcal{G}_1^\kappa(|2^\kappa|)$.*

Proof. Let σ be a winning strategy for II for $\mathcal{G}_1^\kappa(2)$ in $\mathbb{B}$. Play $\mathcal{G}_1^\kappa(|2^\kappa|)$: Let I fix $p \in \mathbb{B}^+$ and play A_0, a maximal antichain below p with $|A_0| \leq |2^\kappa|$. By Lemma 15, there exist $a_0 \in A_0$ and $s_0 \in \mathbb{B}^{<\kappa}$ such that $g(p, s_0) \leq a_0$. Make II play a_0. For $\alpha < \kappa$ suppose I plays $A_{\alpha+1}$ and s_α is already given. By Lemma 15 there exist $a_{\alpha+1} \in A_{\alpha+1}$ and $s_{\alpha+1} \in \mathbb{B}^{<\kappa}$ such that $s_{\alpha+1} \supseteq s_\alpha$ and $g(p, s_{\alpha+1}) \leq a_{\alpha+1}$. For $\alpha < \kappa$ a limit ordinal, suppose I plays A_α, and let $s'_\alpha = \bigcup_{\beta<\alpha} s_\beta$. Then by Lemma 15, there exist $a_\alpha \in A_\alpha$ and $s_\alpha \in \mathbb{B}^{<\kappa}$ such that $s_\alpha \supseteq s'_\alpha$ and $g(p, s_\alpha) \leq a_\alpha$. Make II play a_α. Let $s = \bigcup_{\alpha<\kappa} s_\alpha$. $g(p, s) > \mathbf{0}$, since this is a play of $\mathcal{G}_1^\kappa(2)$ according to σ. $g(p,s) = \bigwedge_{\alpha<\kappa} g(p,s_\alpha) \leq \bigwedge_{\alpha<\kappa} a_\alpha$, so II wins $\mathcal{G}_1^\kappa(|2^\kappa|)$ according to our strategy. $\boxed{QED}$

17 Corollary. *Suppose $|2^\kappa| = |\mathscr{P}(2^{<\kappa})|$. If II has a winning strategy for $\mathcal{G}_1^\kappa(2)$ and $\mathbb{B}$ is $|2^\kappa|^+$-c.c., then II has a winning strategy for $\mathcal{G}_1^\kappa(\infty)$.*

To prepare for Theorem 23 we now review κ-club and κ-stationary subsets of $\mathscr{P}_{\kappa^+}(\lambda)$.

18 Definition ([3])**.** Let κ be regular and $\lambda > \kappa$. A set $C \subseteq \mathscr{P}_{\kappa^+}(\lambda)$ is *κ-club* if C is cofinal in $\mathscr{P}_{\kappa^+}(\lambda)$ and is closed under $\subseteq$-increasing sequences of length exactly κ. $S \subseteq \mathscr{P}_{\kappa^+}(\lambda)$ is *κ-stationary* if $S \cap C \neq \emptyset$ for all κ-club $C \subseteq \mathscr{P}_{\kappa^+}(\lambda)$.

19 Remark. κ-club and κ-stationary sets enjoy the usual nice properties: The intersection of κ many κ-club sets is again κ-club; κ-clubness is preserved under diagonal intersections; and each κ-stationary set can be decomposed into λ many disjoint κ-stationary sets. (See [3] for more on the general theory.)

The next theorem gives a functional representation of κ-club sets. For a purely combinatorial proof (not using infinitary logic) see [3].

20 Theorem (Kueker [16])**.** *Suppose $|\kappa^{<\kappa}| = \kappa \leq \lambda$ and $C \subseteq [\lambda]^{\leq\kappa}$ is κ-club. Then there exists a function $h : [\lambda]^{<\kappa} \to \lambda$ such that $C_h \subseteq C$, where $C_h = \{ x \in \mathscr{P}_{\kappa^+}(\lambda) : \forall y \in [x]^{<\kappa},\ h(y) \in x \}$.*

The next argument is an adaptation of one from [9].

21 Proposition. *Suppose $|\kappa^{<\kappa}| = \kappa < \lambda$, $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive, and II wins $\mathcal{G}_\kappa^\kappa(\lambda)$ in $\mathbb{B}$. Then $\mathbb{B}$ preserves all κ-stationary subsets of $\mathscr{P}_{\kappa^+}(\lambda)$.*

Proof. First, note that $\mathbb{B}$ preserves all cardinals $\leq \lambda$: the $(<\kappa,\kappa)$-d.l. implies preservation of all cardinals $\leq \kappa$ and II having a winning strategy for $\mathcal{G}_\kappa^\kappa(\lambda)$ implies preservation of all cardinals ρ with $\kappa < \rho \leq \lambda$. Let σ be a winning strategy for II in $\mathcal{G}_\kappa^\kappa(\lambda)$. Let S be a κ-stationary subset of $(\mathscr{P}_{\kappa^+}(\lambda))^V$. By Theorem 20, it suffices to show that for each $p \in \mathbb{B}^+$ and each $\mathbb{B}$-name $\dot{f}$ for which $p \Vdash \dot{f} : [\lambda]^{<\kappa} \to \lambda$, there is an $x \in S$ and a $q \leq p$ such that $q \Vdash x$ is closed under $\dot{f}$. Without loss of generality, assume $p = \mathbf{1}$. For each $y \in (\mathscr{P}_\kappa(\lambda))^V$ and every $\beta < \lambda$, let $a(y,\beta) = \|\dot{f}(y) = \beta\|$. Let $W(y) = \{ a(y,\beta) : \beta < \lambda \}$. Then $W(y)$ is a partition of unity. Let $A = \{ x \in (\mathscr{P}_{\kappa^+}(\lambda))^V : \|x$ is closed under $\dot{f}\| > \mathbf{0} \}$. We will show that A contains a κ-club in V.

Define $g : ((\mathscr{P}_{\kappa^+}(\lambda))^{<\kappa})^V \to (\mathscr{P}_{\kappa^+}(\lambda))^V$ in V as follows:
For $s \in ((\mathscr{P}_{\kappa^+}(\lambda))^{<\kappa})^V$ with $\mathrm{lh}(s)$ a successor cardinal, say $s = \langle s_\beta : \beta \leq \alpha \rangle$, let $g(s) = \{ \beta < \lambda : a(s_\alpha, \beta) \in \sigma(\langle W(s_0), \ldots, W(s_\alpha) \rangle) \}$. If $\mathrm{lh}(s)$ is a limit ordinal, just let $g(s) = \emptyset$. Then $g \in V$. Let $C = \{ x \in (\mathscr{P}_{\kappa^+}(\lambda))^V : \forall s \in ([x]^{<\kappa})^{<\kappa},\ g(s) \subseteq x \}$. Then $C \in V$ and C is a κ-club set in V.

We will show $C \subseteq A$. Let $x \in C$ and let $\langle y_\alpha : \alpha < \kappa \rangle$ enumerate $[x]^{<\kappa}$ in V.

For each $\alpha < \kappa$, let $E_\alpha = \sigma(\langle W(y_0), \ldots, W(y_\alpha)\rangle)$. $g(\langle y_0, \ldots, y_\alpha\rangle) \subseteq x$ implies $\bigvee E_\alpha = \bigvee\{\, a(y_\alpha, \beta) : \beta \in g(\langle y_0, \ldots, y_\alpha\rangle)\,\} \leq \bigvee\{\, a(y_\alpha, \beta) : \beta \in x\,\} = \|\dot f(y_\alpha) \in x\|$. Since σ is winning for II, we have $\bigwedge_{\alpha<\kappa} \bigvee E_\alpha > \mathbf{0}$. Hence, $\bigwedge_{\alpha<\kappa} \|\dot f(y_\alpha) \in x\| > \mathbf{0}$. By the $(<\kappa, \kappa)$-d.l., $[x]^{<\kappa}$ is the same in V as in $V^{\mathbb{B}}$; so $\bigwedge_{\alpha<\kappa} \|\dot f(y_\alpha) \in x\| = \|x$ is closed under $\dot f\|$. Hence, $x \in A$.

Let $x \in C \cap S$ and let $q = \|x$ is closed under $\dot f\|$. Then $q > \mathbf{0}$ and $q \Vdash$ "$x \in S$ and x is closed under $\dot f$". Therefore, $V^{\mathbb{B}} \models S$ is κ-stationary. $\boxed{QED}$

The next theorem shows that whenever $|\kappa^{<\kappa}| = \kappa$, one can always shoot a κ-club through a given κ-stationary set. This generalises an example of Kamburelis in [11] for countable length games (since ω-stationary is the same as stationary).

22 Theorem (Dobrinen [3])**.** *Suppose $|\kappa^{<\kappa}| = \kappa < \lambda$ and $S \subseteq [\lambda]^{\leq\kappa}$ is κ-stationary. Let $\mathbb{P}_S$ denote the set of all one-to-one functions f such that* $\mathrm{dom}(f)$ *is an ordinal less than κ^+,* $\mathrm{ran}(f) \subseteq \lambda$*, and for each ordinal $\zeta \leq \mathrm{dom}(f)$ with cofinality κ, $f[\zeta] \in S$. Let $g \leq f \leftrightarrow g \supseteq f$. $\mathbb{P}_S$ is a separative, atomless, (κ, ∞)-distributive, λ^+-c.c. forcing which adds a new κ-club set through S.*

We finally come to the main theorem of this section.

23 Theorem. *If $|\kappa^{<\kappa}| = \kappa$, then there is a (κ, ∞)-distributive Boolean algebra which has a dense $<\kappa$-closed subset, and in which for all $\lambda \geq 2$, $\mathcal{G}_1^\kappa(\lambda)$ is undetermined, and for all $\lambda \geq \kappa^+$ and all $\mu \leq \kappa^+$, $\mathcal{G}_{<\mu}^\kappa(\lambda)$ is undetermined.*

Proof. Let $S \subseteq \mathscr{P}_{\kappa^+}(\kappa^+)$ be a κ-stationary set such that $\mathscr{P}_{\kappa^+}(\kappa^+) \setminus S$ is also κ-stationary. (For a proof that such sets exist, see [3].) Let $\mathbb{B} = \mathrm{r.o.}(\mathbb{P})$. By Theorem 22, $\mathbb{B}$ kills the κ-stationarity of $\mathscr{P}_{\kappa^+}(\kappa^+) \setminus S$. Hence, by Proposition 21, II does not have a winning strategy for $\mathcal{G}_\kappa^\kappa(\kappa^+)$. This implies that II cannot have a winning strategy for $\mathcal{G}_{<\mu}^\kappa(\lambda)$ for all $\lambda \geq \kappa^+$ and all $\mu \leq \kappa^+$. In particular, this implies that II cannot have a winning strategy for $\mathcal{G}_1^\kappa(\kappa^+)$. Then, by Theorem 16, II cannot have a winning strategy for $\mathcal{G}_1^\kappa(2)$. Thus, for all $\lambda \geq 2$, II does not have a winning strategy for $\mathcal{G}_1^\kappa(\lambda)$ in $\mathbb{B}$.

On the other hand, by Theorem 12(1), Player I does not have a winning strategy for $\mathcal{G}_1^\kappa(\infty)$, since $\mathbb{B}$ is (κ, ∞)-distributive. Hence, I does not have a winning strategy for $\mathcal{G}_{<\mu}^\kappa(\lambda)$ for any $\lambda \geq \mu \geq 2$. $\boxed{QED}$

4 $\mathcal{G}_1^\kappa(\infty)$ is equivalent to the strategically closed forcing game

In this section, we show the full equivalence between $\mathcal{G}_1^\kappa(\infty)$ and the strategically closed forcing game G_κ^{II}. By results of Jech [8] and Veličković [17], we know that the ω-length versions of these games are equivalent. Foreman invented the following uncountable length games, which generalise the descending sequence game $\mathcal{G}$ of Jech in [7] to sequences of uncountable length. At limit ordinals, there is the question of which player gets to choose first, so Foreman defined two games. We slightly change his notation to match that of this paper. Our G^{I}_κ denotes his $G^{\mathrm{I}}_{\kappa^+}$, and our G^{II}_κ denotes his $G^{\mathrm{II}}_{\kappa^+}$.

24 Definition (Foreman [5]). The game G^{I}_κ is played in a Boolean algebra $\mathbb{B}$ as follows. On round 0, Player I chooses an $a_0 \in \mathbb{B}^+$; then Player II chooses some $b_0 \in \mathbb{B}^+$ such that $b_0 \le a_0$. In general, on round $\alpha < \kappa$, Player I chooses an $a_\alpha \in \mathbb{B}^+$ such that $a_\alpha \le b_\beta$ for all $\beta < \alpha$ (if possible). Then Player II chooses some $b_\alpha \in \mathbb{B}^+$ such that $b_\alpha \le a_\alpha$. Player I *wins the play* iff either for some $\alpha < \kappa$ I has no legal move at round α, or else the play is a sequence

$$a_0 \ge b_0 \ge a_1 \ge b_1 \ge \cdots \ge a_\alpha \ge b_\alpha \ge \cdots \tag{5}$$

of length κ and $\bigwedge_{\alpha<\kappa} b_\alpha = \mathbf{0}$. G^{II}_κ is played similarly: I starts the game, but II chooses first at limit ordinals $\alpha \ge \omega$; so the sequence looks like this:

$$a_0 \ge b_0 \ge a_1 \ge b_1 \ge \cdots \ge b_\omega \ge a_\omega \ge \cdots \ge b_\alpha \ge a_\alpha \ge \cdots \tag{6}$$

25 Note. $G^{\mathrm{I}}_\omega = G^{\mathrm{II}}_\omega =$ Jech's $\mathcal{G}$ in [7].

The games G^{I}_κ and G^{II}_κ can be played on a partial ordering $\mathbb{P}$.

26 Lemma (Foreman [5]). *Let $\mathbb{P}$ be a dense subset of $\mathbb{B}$. Then II (I) has a winning strategy for G^{I}_κ on $\mathbb{B}$ iff II (I) has a winning strategy for G^{I}_κ on $\mathbb{P}$. The same holds for G^{II}_κ.*

$\mathbb{P}$ is said to be *κ-strategically closed* if II has a winning strategy for G^{II}_κ. If $\mathbb{P}$ is $<\kappa^+$-closed, then II wins G^{I}_κ. It is known that G^{II}_κ is strictly easier for II to win than G^{I}_κ. (See [14] for a discussion.) The following example due to Gray applies. (His example can also be found in [5].)

27 Example (Gray [6]). There is a complete Boolean algebra in which I has a winning strategy for $G^{\mathrm{I}}_{\aleph_1}$ but in which II has a winning strategy for $G^{\mathrm{II}}_{\aleph_1}$.

On the other hand, it has been shown that the games G^{II}_κ and $\mathcal{G}^\kappa_1(\infty)$ are equivalent with regard to winning strategies for Player I.

28 Theorem ([4, 5]). *The following are equivalent.*

(1) I has a winning strategy for G^{II}_κ in $\mathbb{B}$;

(2) I has a winning strategy for $\mathcal{G}^\kappa_1(\infty)$ in $\mathbb{B}$;

(3) The (κ,∞)-d.l. fails in $\mathbb{B}$.

Foreman proved (1) iff (3) and we proved (2) iff (3). This extends work of Jech, showing that the games $\mathcal{G}$ and $\mathcal{G}^\omega_1(\infty)$ are equivalent for Player I [8].

Veličković showed the equivalence of the games $\mathcal{G}$ and $\mathcal{G}_1^{\omega}(\infty)$ for Player II [17]. This can be extended to G_κ^{II} and $\mathcal{G}_1^{\kappa}(\infty)$. The proof is a straightforward generalisation of the one given by Veličković, making the necessary cardinal changes and dealing with limit stages. Nevertheless, it is included here, since strategically closed forcings are quite useful in set theory and hence the equivalence is of interest.

29 Theorem. *Let κ be any infinite cardinal. II has a winning strategy for G_κ^{II} in $\mathbb{B}$ if and only if II has a winning strategy for $\mathcal{G}_1^{\kappa}(\infty)$ in $\mathbb{B}$.*

Proof. Let σ be a winning strategy for Player II for G_κ^{II} in $\mathbb{B}$. We construct a winning strategy τ for II for $\mathcal{G}_1^{\kappa}(\infty)$ in $\mathbb{B}$. In $\mathcal{G}_1^{\kappa}(\infty)$, suppose I fixes $a > \mathbf{0}$. In G_κ^{II}, let $a_0 = a$ and $b_0 = \sigma(\langle a_0 \rangle)$. In $\mathcal{G}_1^{\kappa}(\infty)$, I plays some partition W_0 of a. Have II choose some $u_0 \in W_0$ for which $u_0 \wedge b_0 > \mathbf{0}$ and define $\tau(\langle W_0 \rangle) = u_0$. In G_κ^{II}, let $a_1 = u_0 \wedge b_0$ and $b_1 = \sigma(\langle a_0, a_1 \rangle)$. In $\mathcal{G}_1^{\kappa}(\infty)$, I plays some partition W_1 of a. Have II choose some $u_1 \in W_1$ for which $u_1 \wedge b_1 > \mathbf{0}$ and define $\tau(\langle W_0, W_1 \rangle) = u_1$. Continue in this manner until round ω.

In G_κ^{II}, II plays first at round ω and beyond. For $\omega \leq \alpha < \kappa$, let $b_\alpha = \sigma(\langle a_\beta : \beta < \alpha \rangle)$. In $\mathcal{G}_1^{\kappa}(\infty)$, I plays some partition W_α of a. Have II choose some $u_\alpha \in W_\alpha$ for which $u_\alpha \wedge b_\alpha > \mathbf{0}$ and define $\tau(\langle W_0, \ldots, W_\alpha \rangle) = u_\alpha$. In G_κ^{II}, let $a_\alpha = u_\alpha \wedge b_\alpha$.

In this manner, we obtain a play

$$a_0 \geq b_0 \geq a_1 \geq b_1 \geq \cdots \geq b_\omega \geq a_\omega \geq \cdots \geq b_\alpha \geq a_\alpha, \ldots \tag{7}$$

of G_κ^{II} in which II plays according to the winning strategy σ. So $\bigwedge_{\alpha<\kappa} a_\alpha > \mathbf{0}$. Moreover, $\langle W_\alpha, u_\alpha : \alpha < \kappa \rangle$ is a play of $\mathcal{G}_1^{\kappa}(\infty)$ in which II plays according to τ, and

$$\bigwedge_{\alpha<\kappa} u_\alpha \geq \bigwedge_{\alpha<\kappa} (u_\alpha \wedge b_\alpha) = \bigwedge_{\alpha<\kappa} a_\alpha > \mathbf{0}. \tag{8}$$

Therefore, τ is a winning strategy for II for $\mathcal{G}_1^{\kappa}(\infty)$.

Now suppose τ is a winning strategy for II for $\mathcal{G}_1^{\kappa}(\infty)$ in $\mathbb{B}$. Note that the game $\mathcal{G}_1^{\kappa}(\infty)$ is equivalent to the game where at each stage $\alpha < \kappa$, I chooses a partition of the infimum of II's previous choices in $\mathcal{G}_1^{\kappa}(\infty)$ and II chooses one piece of that partition.

Claim. Suppose that $\alpha < \kappa$, $\langle a_\beta : \beta < \alpha \rangle$ is a decreasing sequence, and for each $\beta < \alpha$, $c_\beta = \bigwedge_{\gamma<\beta} a_\gamma$ and W_β is a partition of c_β. Let $c_\alpha = \bigwedge_{\beta<\alpha} a_\beta$. If $c_\alpha > \mathbf{0}$, then there exists some $\mathbf{0} < b_\alpha \leq c_\alpha$ with the property that for each $\mathbf{0} < u \leq b_\alpha$, there exists a partition W_α of c_α such that $u = \tau(\langle W_\beta : \beta \leq \alpha \rangle)$.

Suppose not. Then for each $\mathbf{0} < b \leq c_\alpha$, there exists some $\mathbf{0} < u \leq b$ such that for each partition W_α of c_α, $u \neq \tau(\langle W_\beta : \beta \leq \alpha \rangle)$. So the set of such u's is dense below c_α. Let W_α be a partition of c_α consisting of such u's. If I plays W_α

on round α, then $\tau(\langle W_\beta : \beta \leq \alpha\rangle)$ must be an element of W_α. But each $u \in W_\alpha$ is not equal to $\tau(\langle W_\beta : \beta \leq \alpha\rangle)$. Contradiction. Thus, the Claim holds.

We now construct a winning strategy σ for II for G^{II}_κ. In G^{II}_κ, let I choose $a_0 > \mathbf{0}$. Let II choose some $\mathbf{0} < b_0 \leq a_0$ satisfying the Claim for $c_0 = a_0$. Define $\sigma(\langle a_0\rangle) = b_0$. Next, I chooses some $\mathbf{0} < a_1 \leq b_0$. In $\mathcal{G}^\kappa_1(\infty)$, let $a = a_0$, and let W_0 be a partition of a_0 such that $a_1 = \tau(\langle W_0\rangle)$. Again, let II choose some $\mathbf{0} < b_1 \leq a_1$ satisfying the Claim for $c_1 = a_1$. Define $\sigma(\langle a_0, a_1\rangle) = b_1$. Let I choose some $\mathbf{0} < a_2 \leq b_1$. In $\mathcal{G}^\kappa_1(\infty)$, let W_1 be a partition of a_1 such that $a_2 = \tau(\langle W_0, W_1\rangle)$. Continue in this manner until round ω.

For $\omega \leq \alpha < \kappa$, II chooses first at round α in G^{II}_κ. Let $c_\alpha = \bigwedge_{\beta<\alpha} a_\beta$. $c_\alpha > \mathbf{0}$ since each a_β is II's choice according to the winning strategy τ on $\langle W_0, \dots, W_\beta\rangle$. Since II gets to choose first at limit ordinals in G^{II}_κ, II can choose some $\mathbf{0} < b_\alpha \leq c_\alpha$ such that for each $\mathbf{0} < u \leq b_\alpha$, there is a partition W_α of c_α such that $u = \tau(\langle W_0, \dots, W_\alpha\rangle)$. Define $\sigma(\langle a_\beta : \beta < \alpha\rangle) = b_\alpha$. Let I choose some $\mathbf{0} < a_\alpha \leq b_\alpha$. Let W_α be a partition of c_α such that $a_\alpha = \tau(\langle W_0, \dots, W_\alpha\rangle)$.

$$\bigwedge_{\alpha<\kappa} a_\alpha = \bigwedge_{\alpha<\kappa} \tau(\langle W_0, \dots, W_\alpha\rangle) > \mathbf{0}, \tag{9}$$

since τ is a winning strategy for II for $\mathcal{G}^\kappa_1(\infty)$. $\boxed{QED}$

5 $<\kappa^+$-closed dense subsets of Boolean algebras and G^{I}_κ

Recall that if a partial ordering $\mathbb{P}$ has a $<\kappa^+$-closed dense subset, then Player II has a winning strategy for G^{I}_κ in $\mathbb{P}$. Conversely, Player II having a winning strategy for G^{I}_κ in $\mathbb{P}$ sometimes implies the existence of a $<\kappa^+$-closed dense subset. The main Theorem 38 of this section slightly improves on what was meant by a vague comment of Foreman in [5]. As no proof is given in that paper, we give one for sake of availability in the literature. The proof involves straightforward adaptations of Veličković's proofs of Theorem 30 and Lemma 36 for the countable length game $\mathcal{G}$. The proof is basically Veličković's. The analysis goes through dense subtrees. For a characterisation of when a partial ordering has a dense subtree, see [13].

30 Theorem (Veličković [17])**.** *Let T be a tree. If Player II has a winning strategy for $\mathcal{G}$ on T, then T has a $<\omega_1$-closed dense subset.*

This can be extended to uncountable length games, using G^{I}_κ in place of $\mathcal{G}$ and making some cardinal arithmetic assumptions.

31 Theorem. *Suppose T is a tree and Player II wins G^{I}_κ on T. Let λ be least such that T is nowhere (λ, ∞)-distributive, and suppose $\forall \kappa \leq \theta < \lambda$,*

$|\theta^{<\kappa}| < \lambda$. *Then T has a $< \kappa^+$-closed dense subset.*

PROOF. Without loss of generality, we can assume T is a normal tree. Let $\eta = \mathrm{ht}(T)$. For $\xi < \eta$, let T_ξ denote the ξ-th level of T. We first prove the theorem assuming that T is $(<\eta, \infty)$-distributive. Fix a winning strategy σ for II in G^{I}_κ on T. For $t \in T$ and $\vec{s} \in (T)^{<\kappa}$ a sequence of even or limit length, we say that $\vec{s}$ is a *partial play towards* t if $\vec{s}$ is played according to σ and $\inf \vec{s} \geq t$. Call $t \in T$ *good* if for every partial play $\vec{s}$ towards t and every $t' \in T$ with $\inf \vec{s} \geq t' > t$, there exists a partial play $\vec{s}\,'$ towards t extending $\vec{s}$ with $\inf \vec{s}\,' \leq t'$. ($\inf \vec{s}$ is well-defined if $\vec{s}$ has limit length, since T is normal.)

32 Claim. The set of good t's is $< \kappa^+$-closed.

PROOF. Let $\gamma \leq \kappa$ be a limit ordinal and $\langle t_i : i < \gamma \rangle$ be a strictly decreasing sequence of good conditions. (Note: the set of $t \in T$ for which there exists a partial play towards t is dense in T.) Let $\vec{s}_0$ be a partial play towards t_0. Then $\vec{s}_0$ is also a partial play towards t_1. Given $\alpha < \gamma$ and a partial play $\vec{s}_\alpha$ towards t_α, $\inf \vec{s}_\alpha \geq t_\alpha > t_{\alpha+1}$ implies there is a partial play $\vec{s}_{\alpha+1}$ towards $t_{\alpha+1}$ such that $\vec{s}_{\alpha+1} \supseteq \vec{s}_\alpha$ and $t_\alpha \geq \inf \vec{s}_{\alpha+1} \geq t_{\alpha+1}$. For a limit ordinal $\alpha < \gamma$, let $\vec{s}_\alpha = \bigcup_{\beta<\alpha} \vec{s}_\beta$. $\vec{s}_\alpha$ is a partial play of G^{I}_κ. Let $u_\alpha = \inf \vec{s}_\alpha$. Then $u_\alpha \geq t_\alpha$, since T is normal, so $\vec{s}_\alpha$ is a partial play towards $t_{\alpha+1}$. Let $\vec{s} = \bigcup_{\alpha<\gamma} \vec{s}_\alpha$. Let $u = \inf \vec{s}$. (u exists in T since $\vec{s}$ is a (possibly partial) play of G^{I}_κ according to σ.) u is good: Let $\vec{s}\,'$ be a partial play towards u and $t \in T$ such that $\inf \vec{s}\,' \geq t > u$. Then there is an $\alpha < \gamma$ such that $t > t_\alpha > u$. $\vec{s}\,'$ is also a partial play towards t_α. Since t_α is good, there is a partial play $\vec{s}\,''$ towards t_α, hence towards u, such that $\vec{s}\,'' \supseteq \vec{s}\,'$ and $\inf \vec{s}\,'' \leq t$. $\boxed{QED}$

We will say that $(t, t') \in T \times T$ is a *good pair* if $t' > t$ and for every partial play $\vec{s}$ towards t', there is a partial play $\vec{s}\,'$ towards t extending $\vec{s}$ such that $\inf \vec{s}\,' \leq t'$.

33 Claim. $\forall p \in T$, $\exists t < p$ such that (t, p) is a good pair.

PROOF. Let $p \in T$. For each partial play $\vec{s}$ towards p, let $D_{\vec{s}} = \{q \in T : q < p$ and $\exists \vec{s}\,'$ a partial play towards q such that $\vec{s}\,' \supseteq \vec{s}$ and $\inf \vec{s}\,' \leq p\}$. $D_{\vec{s}}$ is open dense below p: Let $t \leq p$. Then $\vec{s}$ is a partial play towards t. Let I choose some $s_0 \leq t$. Let $q = \sigma(\vec{s}^\frown s_0)$. Then $q \in D_{\vec{s}}$. To see that $D_{\vec{s}}$ is open, suppose $r \in D_{\vec{s}}$ and $t \leq r$. Then any $\vec{s}\,'$ which guarantees that $q \in D_{\vec{s}}$ also guarantees that $r \in D_{\vec{s}}$.

Let $\theta = \mathrm{ht}(p)$. If $\theta \geq \kappa$, then there are $\leq |\theta^{<\kappa}|$ many partial plays towards p. If $\theta < \kappa$, then there are $\leq |2^\theta|$ many partial plays towards p. By our hypothesis, these are both $< \eta$. Let $D_p = \bigcap \{ D_{\vec{s}} : \vec{s}$ is a partial play towards $p \}$. D_p is dense below p, since T is $(<\eta, \infty)$-distributive. Let $t \in D_p$. Then (t, p) is a good pair: Let $\vec{s}$ be a partial play towards p. $t \in D_{\vec{s}}$ implies $\exists \vec{s}\,' \supseteq \vec{s}$ such that $t \leq \inf \vec{s}\,'$ and $\inf \vec{s}\,' \leq p$. So (t, p) is a good pair. $\boxed{QED}$

34 Claim. The set of good t's is dense.

PROOF. Let $p_0 \in T$ be given. Let $\vec{s}_0$ be a partial play towards p_0. Let $p_1 < p_0$ be such that (p_1, p_0) is a good pair. Then there is a partial play $\vec{s}_1$ towards p_1 such that $\vec{s}_1 \supseteq \vec{s}_0$ and $\inf \vec{s}_1 \leq p_0$. Given $\vec{s}_\alpha$ and p_α, let $p_{\alpha+1} < p_\alpha$ be such that $(p_{\alpha+1}, p_\alpha)$ is a good pair. Since $\vec{s}_\alpha$ is a partial play towards p_α, there is a partial play $\vec{s}_{\alpha+1}$ towards $p_{\alpha+1}$ such that $\vec{s}_{\alpha+1} \supseteq \vec{s}_\alpha$ and $\inf \vec{s}_{\alpha+1} \leq p_\alpha$. For limit $\alpha < \kappa$, let $\vec{s}_\alpha = \bigcup_{\beta<\alpha} \vec{s}_\beta$. Then $\vec{s}_\alpha$ is a partial play of G^{I}_κ, so $\inf \vec{s}_\alpha$ exists in T. Let $p_\alpha = \inf \vec{s}_\alpha$. Then $\vec{s}_\alpha$ is a partial play towards p_α. Let $p = \inf_{\alpha<\kappa} p_\alpha$. p is good: Suppose $\vec{s}$ is a partial play towards p and $t' \in T$ satisfying $\inf \vec{s} \geq t' > p$. Then there is an $\alpha < \kappa$ such that $t' > p_\alpha$. $(p_{\alpha+1}, p_\alpha)$ is a good pair and $\vec{s}$ is a partial play towards p_α, so there is a partial play $\vec{s}\,'$ towards $p_{\alpha+1}$ such that $\vec{s}\,' \supseteq \vec{s}$ and $\inf \vec{s}\,' \leq p_\alpha$. Therefore, $\vec{s}\,'$ is a partial play towards p and $\inf \vec{s}\,' \leq t'$. $\boxed{QED}$

Hence, by Claims 32 and 34, the set of good $t \in T$ is dense and $< \kappa^+$-closed.

Now consider the general situation when T is not $(< \eta, \infty)$-distributive, where $\eta = \mathrm{ht}(T)$. To be precise, we can find a maximal antichain $M \subseteq T$ such that for each $b \in M$ there is a λ_b such that $T \restriction b$ is $(< \lambda_b, \infty)$-distributive and nowhere (λ_b, ∞)-distributive. Then we can do the proof on each piece $T \restriction b$, $b \in M$.

Hence, without loss of generality, let λ be such that T is $(< \lambda, \infty)$-distributive and nowhere (λ, ∞)-distributive. Then there is a decreasing sequence $\langle D_\xi : \xi < \lambda \rangle$ of open dense subsets of T such that $\bigcap_{\xi<\lambda} D_\xi = \emptyset$. For each $\xi < \lambda$, pick by transfinite recursion a maximal antichain $A_\xi \subseteq D_\xi$ such that $\forall \xi' < \xi$, A_ξ refines $A_{\xi'}$. Let $R = \bigcup_{\xi<\lambda} A_\xi$. R with the induced ordering is a tree. $\mathrm{ht}(R) = \lambda$, so R is nowhere (λ, ∞)-distributive. R is dense in T: Let $t \in T$. Then there is a $\xi < \lambda$ such that $t \notin D_\xi$. If $s \in A_\xi$ and s and t are compatible, then actually $s < t$ (if $s \geq t$, then $s \notin D_\xi$, since D_ξ is open and $t \notin D_\xi$). Hence, R is $(< \lambda, \infty)$-distributive. Doing the first part of the proof on R yields a $< \kappa^+$-closed dense subset of R, and hence of T. $\boxed{QED}$

35 Corollary. *Suppose $|\kappa^{<\kappa}| = \kappa$, T is a tree with $\mathrm{ht}(T) = \kappa^+$, and II wins G^{I}_κ on T. Then T has a $< \kappa^+$-closed dense subset.*

Now we give game-theoretic conditions under which a Boolean algebra has a dense subset which is a tree. Veličković proved the following for the countable length game $\mathcal{G}$.

36 Lemma (Veličković [17])**.** *Let $\mathbb{B}$ be a complete Boolean algebra which has a dense set of size $\leq |2^{\aleph_0}|$. Assume II has a winning strategy for $\mathcal{G}$. Then $\mathbb{B}$ has a dense subset which is a tree under the induced ordering.*

Veličković's proof can be easily extended to uncountable length games, with a bit of cardinal arithmetic.

37 Lemma. *Suppose $\mathbb{B}$ has a dense subset of size $\leq |2^\kappa|$, and II has a winning strategy for G^{I}_κ in $\mathbb{B}$. Let λ be least such that $\mathbb{B}$ is not (λ, ∞)-distributive, and assume $|2^{<\kappa}| < \lambda$. Then $\mathbb{B}$ has a dense subset which is a tree under the induced ordering and on which II wins G^{I}_κ.*

Proof. First, note that if $\mathbb{B}$ has a dense subset of size $< |2^\kappa|$, then $\mathbb{B}$ must be atomic, since II has a winning strategy for G^{I}_κ. In that case, the atoms form a trivial tree on which II wins G^{I}_κ. So without loss of generality, let us assume $\mathbb{B}$ is atomless. Then $\mathbb{B}$ has a dense subset of size exactly $|2^\kappa|$.

Let λ be least such that $\mathbb{B}$ is not (λ, ∞)-distributive. There is a partition of unity $M \subseteq \mathbb{B}$ such that for each $b \in M$, there is a λ_b such that $\mathbb{B} \restriction b$ is nowhere (λ_b, ∞)-distributive but is $(< \lambda_b, \infty)$-distributive. Note that each $\lambda_b \geq \lambda$. We can then do the remainder of the proof on each $\mathbb{B} \restriction b$, $b \in M$.

So, without loss of generality, assume $\mathbb{B}$ is nowhere (λ, ∞)-distributive but is $(< \lambda, \infty)$-distributive. Note then that λ is regular. Let $\mathbb{P}$ be a dense subset of $\mathbb{B}$ with $|\mathbb{P}| = |2^\kappa|$. As in the previous proof, there are maximal antichains $A_\xi \subseteq \mathbb{P}$, $\xi < \lambda$, such that A_ξ refines A_η for all $\eta < \xi < \lambda$, and $\bigwedge_{\xi<\lambda} f(\xi) = \mathbf{0}$ for all $f \in \Pi_{\xi<\lambda} A_\xi$.

Claim. For each $p \in \mathbb{P}$, there is a $\xi < \lambda$ such that $A_\xi(p) := \{\, q \in A_\xi : p \wedge q > \mathbf{0} \,\}$ has size $|2^\kappa|$.

Proof. Let σ be a winning strategy for II for G^{I}_κ on $\mathbb{P}$. Construct by induction two Cantor trees of elements of $\mathbb{P}$, $\langle p_s : s \in 2^{<\kappa} \rangle$, $\langle q_s : s \in 2^{<\kappa} \rangle$. Set $q_{\langle\,\rangle} = p_{\langle\,\rangle} = p$. Given p_s, let $q_{s^\frown 0}, q_{s^\frown 1}$ be incompatible extensions of p_s in $\mathbb{P}$ such that there is a $\xi_s < \lambda$ for which $q_{s^\frown 0}, q_{s^\frown 1}$ are below different elements of A_{ξ_s}. Let $p_{s^\frown i} = \sigma(\langle q_{\langle\,\rangle}, p_{\langle\,\rangle}, q_{\langle s(0) \rangle}, p_{\langle s(0) \rangle}, \ldots, p_s, q_{s^\frown i} \rangle)$ for $i \leq 1$. If $\alpha < \kappa$ is a limit ordinal and $s \in 2^\alpha$, let $q_s \in \mathbb{P}$ such that $q_s \leq q_{s\restriction i}$ for all $i < \alpha$. (Such a q_s exists since σ is a winning strategy for II for G^{I}_κ.) Let $p_s = \sigma(\langle q_{s\restriction i}, p_{s\restriction i} : i < \alpha \rangle^\frown q_s)$.

Since σ is a winning strategy for II, for each $f : \kappa \to 2$ there is a p_f such that for each $\alpha < \kappa$, $p_f \leq p_{f\restriction\alpha}$. Let $\xi = \sup\{\, \xi_s : s \in 2^{<\kappa} \,\}$. $\xi < \lambda$, since $|2^{<\kappa}| < \lambda$ and λ is regular. Let $f \neq g$ and $\alpha < \kappa$ be least such that $f(\alpha) \neq g(\alpha)$. There are $a_f \neq a_g \in A_{\xi_{f\restriction\alpha}}$ such that $p_{f\restriction(\alpha+1)} \leq a_f$ and $p_{g\restriction(\alpha+1)} \leq a_g$. Therefore, $|\{\, q \in A_\xi : p \wedge q > \mathbf{0} \,\}| \geq |2^\kappa|$. $\boxed{QED}$

Now for each $\xi < \lambda$, let $E_\xi = \{\, p \in \mathbb{P} : |A_\xi(p)| = |2^\kappa| \,\}$. By induction, there is a 1-1 function $\varphi_\xi : E_\xi \to A_\xi$ such that for each $p \in E_\xi$, $p \wedge \varphi_\xi(p) > \mathbf{0}$. Enumerate E_ξ as $\langle p_\alpha : \alpha < \theta \rangle$ for some $\theta \leq |2^\kappa|$. For $\alpha < \theta$, let $\varphi_\xi(p_\alpha) \in A_\xi(p_\alpha) \setminus \{\, \varphi_\xi(p_\beta) : \beta < \alpha \,\}$.

$\forall \xi < \lambda$, $\forall p \in E_\xi$, choose one $q_p \leq p \wedge \varphi_\xi(p)$. Extend $\{\, q_p : p \in E_\xi \,\}$ to a maximal antichain C_ξ refining A_ξ. Build maximal antichains T_ξ for $\xi < \lambda$ as follows: $T_{\xi+1}$ refines C_ξ and T_η for all $\eta \leq \xi$; for limit $\xi < \lambda$, $T_\xi = \{\, \bigwedge_{\eta<\xi} f(\eta) : f \in \Pi_{\eta<\xi} T_\eta$ and $\bigwedge_{\eta<\xi} f(\eta) \neq \emptyset \,\}$: Let $T_0 = C_0$. (The T_ξ's are not required to be subsets of $\mathbb{P}$.) Finally, let $T = \bigcup\{ T_\xi : \xi < \lambda \}$. T is a tree and is a dense subset of $\mathbb{B}$. By Lemma 26, II has a winning strategy for G^{I}_κ on T. $\boxed{QED}$

38 Theorem. *Suppose $\mathbb{B}$ has a dense subset of size $\le |2^\kappa|$, and II has a winning strategy for G^{I}_κ on $\mathbb{B}$. Let λ_0 be least such that $\mathbb{B}$ is not (λ_0, ∞)-distributive, and λ_1 be least such that $\mathbb{B}$ is nowhere (λ_1, ∞)-distributive. Assume $|2^{<\kappa}| < \lambda_0$ and $\forall \kappa \le \theta < \lambda_1$, $|\theta^{<\kappa}| < \lambda_1$. Then $\mathbb{B}$ has a dense subtree which is $< \kappa^+$-closed.*

39 Corollary. *Assume $|\kappa^{<\kappa}| = \kappa$ and $|2^\kappa| = \kappa^+$. Suppose II has a winning strategy for G^{I}_κ on $\mathbb{B}$, and $\mathbb{B}$ has a dense subset of size κ^+ and is nowhere (κ^+, ∞)-distributive. Then $\mathbb{B}$ has a dense subtree which is $< \kappa^+$-closed.*

6 κ-length fusion and uncountable height tree forcings

In this section, we present a generalisation of Baumgartner's Axiom A to fusion sequences of uncountable length. We then use this to extend a result of Jech to uncountable length games. This in turn will help us analyse the game-theoretic properties of the regular open algebras of uncountable height tree forcings in Example 44.

40 Definition. Let $(\mathbb{P}, \le)$ be a partial ordering. We will say that $\mathbb{P}$ satisfies *Axiom A*(κ) if there is a family of partial orderings $\langle \le_\alpha : \alpha < \kappa \rangle$ on $\mathbb{P}$ such that

(1) $\le_0$ is $\le$;

(2) $\forall \beta < \alpha < \kappa$, $q \le_\alpha p \to q \le_\beta p$;

(3) Whenever $\langle p_\alpha : \alpha < \kappa \rangle$ is a sequence satisfying $\forall \beta < \alpha < \kappa$, $p_\beta \ge_\beta p_\alpha$, then there is a $p \in \mathbb{P}$ such that $p \le_\alpha p_\alpha$ for all $\alpha < \kappa$;

(4) If $\dot{\xi}$ is a $\mathbb{P}$-name for an ordinal, $\alpha < \kappa$, and $\langle p_\beta : \beta < \alpha \rangle$ satisfies $\forall \gamma < \beta < \alpha$, $p_\gamma \ge_\gamma p_\beta$, then there is a set B and a $p_\alpha \in \mathbb{P}$ such that $|B| \le \kappa$, $p_\alpha \Vdash \dot{\xi} \in B$, and $\forall \beta < \alpha$, $p_\beta \ge_\beta p_\alpha$.

We say that a sequence $\langle p_\alpha : \alpha < \kappa \rangle$ satisfying $\forall \beta < \alpha < \kappa$, $p_\beta \ge_\beta p_\alpha$ is a *fusion sequence*; for $\alpha < \kappa$, a sequence $\langle p_\beta : \beta < \alpha \rangle$ satisfying $\forall \gamma < \beta < \alpha$, $p_\gamma \ge_\gamma p_\beta$ is a *partial fusion sequence.*

Note that $< \kappa^+$-closed forcings and κ^+-c.c. forcings satisfy Axiom A(κ). A seemingly weaker property is the following.

41 Definition. We shall say that a partial order $(\mathbb{P}, \le)$ satisfies Axiom $A'(\kappa)$ if there is a family of partial orderings $\langle \le_\alpha : \alpha < \kappa \rangle$ on $\mathbb{P}$ such that (1), (2), and (4) of Axiom A(κ) hold and $(3')$ holds, where

(3′) Whenever $\langle p_\alpha : \alpha < \kappa\rangle$ is a fusion sequence, then there is a $p \in \mathbb{P}$ such that $p \leq p_\alpha$ for all $\alpha < \kappa$.

Suppose $(\mathbb{P}, \leq)$ with $\langle \leq_\alpha : \alpha < \kappa\rangle$ satisfy (1), (2), and (3′). Let (4′) = (4′a)+(4′b), where

(4′a) $\forall p \in \mathbb{P}$, $\forall \alpha < \kappa$, if W is predense below p, then $\exists q \leq_\alpha p$ and $\exists W' \subseteq W$ such that $|W'| \leq \kappa$ and W' is predense below q.

(4′b) Given $\alpha < \kappa$, if $\langle p_\beta : \beta < \alpha\rangle$ is a partial fusion sequence, then $\exists p_\alpha$ such that $p_\beta \geq_\beta p_\alpha$ for all $\beta < \alpha$ (i.e. every partial fusion sequence is extendable).

42 Fact. If $(\mathbb{P}, \leq)$ with $\langle \leq_\alpha : \alpha < \kappa\rangle$ satisfy (1) and (2), then property (4) is equivalent to property (4′).

Proof. That (4) implies (4′b) is clear. To see that (4) implies (4′a), let $p \in \mathbb{P}$, $\alpha < \kappa$, and W be a set which is predense below p. $\langle p : \beta \leq \alpha\rangle$ is a partial fusion sequence, since $p \leq_\beta p$ for all $\beta \leq \alpha$. Enumerate W as $\{ r_\delta : \delta < \lambda \}$ for some cardinal λ and define $\dot\xi = \{ \langle \check\delta, r_\delta\rangle : \delta < \lambda \}$. (4) implies there is a set B and a $p_{\alpha+1} \in \mathbb{P}$ such that $|B| \leq \kappa$, $p_{\alpha+1} \Vdash \dot\xi \in B$, and for all $\beta \leq \alpha$, $p_{\alpha+1} \leq_\beta p$. Let $W' = \{ r_\delta : \delta \in B \}$. Then $p_{\alpha+1} \leq_\alpha p$ and W' is predense below $p_{\alpha+1}$.

Now assume (4′) and let $\dot\xi$ be a $\mathbb{P}$-name for an ordinal and $\langle p_\beta : \beta < \alpha\rangle$ be a partial fusion sequence. (4′b) implies there is a q such that $p_\beta \geq_\beta q$ for all $\beta < \alpha$. Let W be a maximal incompatible family below q such that $\forall r \in W$, r decides $\dot\xi$. (4′a) implies $\exists p_\alpha \leq_\alpha q$ and $\exists W' \subseteq W$ such that $|W'| \leq \kappa$ and W' is predense below p_α. Let $B = \{ \gamma : \exists r \in W'(r \Vdash \dot\xi = \gamma) \}$. Then $|B| \leq \kappa$ and $p_\alpha \Vdash \dot\xi \in B$. $p_\alpha \leq_\alpha q \leq_\beta p_\beta$ for all $\beta < \alpha$, so $p_\alpha \leq_\beta p_\beta$ for all $\beta < \alpha$. $\boxed{QED}$

Hence, Axiom $A(\kappa)$ implies Axiom $A'(\kappa)$. We have not checked whether the converse holds.

Jech showed that whenever $\mathbb{P}$ satisfies Axiom A, then II has a winning strategy for $\mathcal{G}^\omega_\omega(\infty)$ in r.o.($\mathbb{P}$) [8]. His proof easily extends to give the next proposition.

43 Proposition. *If* $(\mathbb{P}, \leq)$ *satisfies Axiom* $A'(\kappa)$*, then II has a winning strategy for* $\mathcal{G}^\kappa_\kappa(\infty)$ *in* r.o.($\mathbb{P}$).

Proof. Let $\mathbb{B}$ = r.o.($\mathbb{P}$). Let Player I fix some $a \in \mathbb{B}^+$. We describe a winning strategy for Player II. On the 0-th round, suppose I plays $W_0 = \{ b_{0\beta} : \beta < \lambda_0 \}$, a partition of a. Let $\dot\xi_0 = \{ \langle \check\beta, b_{0\beta}\rangle : \beta < \lambda_0 \}$, and let $q \in \mathbb{P}$ such that $q \leq a$. By (4), there is a set B_0 and a $p_0 \in \mathbb{P}$ such that $|B_0| \leq \kappa$, $p_0 \leq_0 q$, and $p_0 \Vdash \dot\xi_0 \in B_0$. $p_0 \leq \bigvee_{\beta \in B_0} b_{0\beta}$, since $b_{0\beta} = \|\dot\xi_0 = \check\beta\| \wedge a$ for each $\beta < \lambda_0$.

In general, play the α-th round as follows. As we begin the α-th round, we have already constructed a partial fusion sequence $\langle p_\gamma : \gamma < \alpha\rangle$. Let $W_\alpha = \{ b_{\alpha\beta} : \beta < \lambda_\alpha \}$ be Player I's move. Let $\dot\xi_\alpha = \{ \langle \check\beta, b_{\alpha\beta}\rangle : \beta < \lambda_\alpha \}$. By (4), there is a B_α of size $\leq \kappa$ and a $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash \dot{\xi}_\alpha \in B_\alpha$ and for each $\beta < \alpha$, $p_\alpha \leq_\beta p_\beta$. $p_\alpha \leq \bigvee_{\beta \in B_\alpha} b_{\alpha\beta}$, since $b_{\alpha\beta} = \|\dot{\xi}_\alpha = \check{\beta}\| \wedge a$ for each $\beta < \lambda_\alpha$.

By (3′), there is a $p \in \mathbb{P}$ such that for all $\alpha < \kappa$, $p \leq p_\alpha$. Therefore, $\mathbf{0} < p \leq \bigwedge_{\alpha<\kappa} p_\alpha \leq \bigwedge_{\alpha<\kappa} \bigvee_{\beta \in B_\alpha} b_{\alpha\beta}$. Hence, II wins $\mathcal{G}^\kappa_\kappa(\infty)$. $\boxed{QED}$

44 Examples. Assume $|2^{<\kappa}| = \kappa$. Let $\mathbb{P}(\kappa)$ denote perfect tree forcing on 2^κ. Kanamori investigated this forcing for $\kappa > \omega$ [12]. When $\kappa = \omega$ this is just Sacks forcing. Let $\mathbb{S}(\kappa)$ denote superperfect tree forcing on κ^κ, as defined by Brown in [1] for $\kappa > \omega$. When $\kappa = \omega$, this reduces to Miller forcing with splitting sets in some filter. In r.o.($\mathbb{P}(\kappa)$) and r.o.($\mathbb{S}(\kappa)$), II has a winning strategy for G^{I}_ρ for all $\rho < \kappa$, since both forcings are $< \kappa$-closed. Since both $\mathbb{S}(\kappa)$ and $\mathbb{P}(\kappa)$ satisfy Axiom $A(\kappa)$, II has a winning strategy for $\mathcal{G}^\kappa_\kappa(\infty)$, by Proposition 43. Hence, if $\kappa^{<\kappa} = \kappa$, they preserve all κ-stationary subsets of $[\lambda]^{\leq\kappa}$ for all $\lambda \geq \kappa$, by Fact 21. In r.o.($\mathbb{P}(\kappa)$), I wins $\mathcal{G}^\kappa_1(2)$, since a new function from κ into 2 is added, by Theorem 12 (1). If κ is strongly inaccessible, then II also has a winning strategy for $\mathcal{G}^\kappa_{<\kappa}(\infty)$ in r.o.($\mathbb{P}(\kappa)$). In r.o.($\mathbb{S}(\kappa)$), I wins $\mathcal{G}^\kappa_{<\kappa}(\kappa)$, since the forcing adds a new unbounded function from κ into κ, by Theorem 12 (2).

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