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# Several comments about the combinatorics of $\tau$-covers

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**Abstract.** In a previous work with Mildenberger and Shelah, we showed that the combinatorics of the selection hypotheses involving $\tau$-covers is sensitive to the selection operator used. We introduce a natural generalization of Scheepers' selection operators, and show that:

(1) A slight change in the selection operator, which in classical cases makes no difference, leads to different properties when $\tau$-covers are involved.

(2) One of the newly introduced properties sheds some light on a problem of Scheepers concerning $\tau$-covers.

Improving an earlier result, we also show that no generalized Luzin set satisfies $\mathsf{U}_{\rm fin}(\Gamma, \mathrm{T})$.

**Keywords:** combinatorial cardinal characteristics of the continuum, $\gamma$-cover, $\omega$-cover, $\tau$-cover, selection principles, Borel covers, open covers

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## 1 Introduction

Topological properties defined by diagonalizations of open or Borel covers have a rich history in various areas of general topology and analysis, and they are closely related to infinite combinatorial notions, see [8, 12, 5, 13] for surveys on the topic and some of its applications and open problems.

Let $X$ be an infinite set. By *a cover of* $X$ we mean a family $\mathcal{U}$ with $X \notin \mathcal{U}$ and $X = \cup\mathcal{U}$. A cover $\mathcal{U}$ of $X$ is said to be

(1) a *large cover of* $X$ if: $(\forall x \in X)$ $\{ U \in \mathcal{U} : x \in U \}$ is infinite.

(2) an $\omega$*-cover of* $X$ if: $(\forall \text{ finite } F \subseteq X)(\exists U \in \mathcal{U})$ $F \subseteq U$.

(3) a $\tau$*-cover of* $X$ if: $\mathcal{U}$ is a large cover of $X$, and $(\forall x, y \in X)$ $\{ U \in \mathcal{U} : x \in U \text{ and } y \notin U \}$ is finite, or $\{ U \in \mathcal{U} : y \in U \text{ and } x \notin U \}$ is finite.

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(4) a $\gamma$-*cover of* $X$ if: $\mathcal{U}$ is infinite and $(\forall x \in X)\ \{ U \in \mathcal{U} : x \notin U \}$ is finite.

Let $X$ be an infinite, zero-dimensional, separable metrizable topological space (in other words, a set of reals). Let $\Omega$, $\mathrm{T}$ and $\Gamma$ denote the collections of all *open* $\omega$-covers, $\tau$-covers and $\gamma$-covers of $X$, respectively. Additionally, denote the collection of all open covers of $X$ by $\mathcal{O}$. Similarly, let $C_\Omega$, $C_\mathrm{T}$, $C_\Gamma$, and $C$ denote the corresponding collections of *clopen* covers. Our restrictions on $X$ imply that each member of any of the above classes contains a countable member of the same class [11]. We therefore confine attention in the sequel to *countable* covers, and restrict the above four classes to contain only their countable members. Having this in mind, we let $\mathcal{B}_\Omega$, $\mathcal{B}_\mathrm{T}$, $\mathcal{B}_\Gamma$, and $\mathcal{B}$ denote the corresponding collections of *countable Borel* covers.

Let $\mathscr{A}$ and $\mathscr{B}$ be any of the mentioned classes of covers (but of the same descriptive type, i.e., both open, or both clopen, or both Borel). Scheepers [7] introduced the following *selection hypotheses* that $X$ might satisfy:

- $\mathsf{S}_1(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of members of $\mathscr{A}$, there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{ U_n : n \in \mathbb{N} \} \in \mathscr{B}$.

- $\mathsf{S}_\mathrm{fin}(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of members of $\mathscr{A}$, there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathscr{B}$.

- $\mathsf{U}_\mathrm{fin}(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of members of $\mathscr{A}$ *which do not contain a finite subcover*, there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{ \cup\mathcal{F}_n : n \in \mathbb{N} \} \in \mathscr{B}$.

Some of the properties are never satisfied, and many equivalences hold among the meaningful ones. The surviving properties appear in Figure 1, where an arrow denotes implication [10]. It is not known whether any other implication can be added to this diagram – see [6] for a summary of the open problems concerning this diagram.

Below each property $P$ in Figure 1 appears its *critical cardinality*, $\mathsf{non}(P)$, which is the minimal cardinality of a space $X$ not satisfying that property. The definitions of most of the cardinals appearing in this figure can be found in [2, 1], whereas $\mathfrak{od}$ is defined in [6], and the results were established in [4, 10, 9, 6].

A striking observation concerning Figure 1 is, that in the top plane of the figures, the critical cardinality of $\Pi(\Gamma,\mathscr{B})$ for $\Pi \in \{ \mathsf{S}_1, \mathsf{S}_\mathrm{fin}, \mathsf{U}_\mathrm{fin} \}$ is independent of $\Pi$ in all cases *except for that where* $\mathscr{B} = \mathrm{T}$. We demonstrate this anomaly further in Section 2, where we also give a partial answer to a problem of Scheepers. In Section 3 we show that no Luzin set satisfies $\mathsf{U}_\mathrm{fin}(\Gamma,\mathrm{T})$, improving a result from [10].

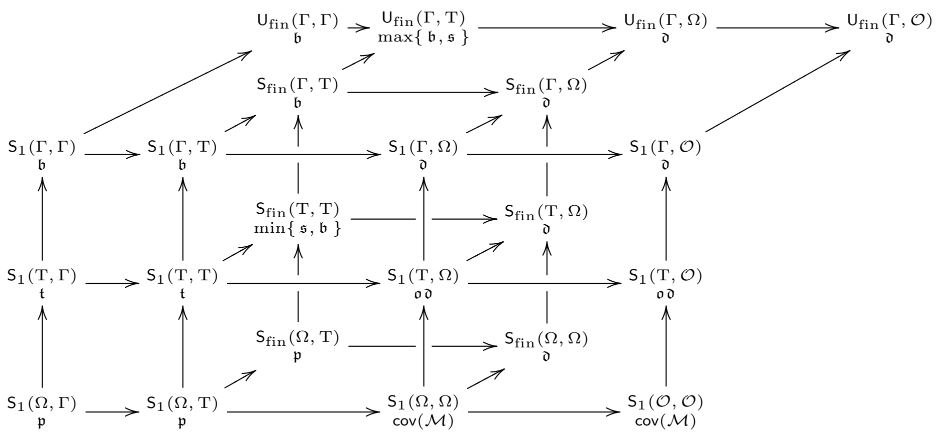

Figure 1. The surviving properties

## 2 Generalized selection hypotheses

**1 Definition.** Let $\kappa < \lambda$ be any (finite or infinite) cardinal numbers. Denote

- $\mathsf{S}_{[\kappa,\lambda)}(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of members of $\mathscr{A}$, there exist subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$ with $\kappa \le |\mathcal{F}_n| < \lambda$ for each $n \in \mathbb{N}$, and $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.

- $\mathsf{U}_{[\kappa,\lambda)}(\mathscr{A},\mathscr{B})$: For each sequence $\langle \mathcal{U}_n : n \in \mathbb{N}\rangle$ of members of $\mathscr{A}$ which do not contain subcovers of size less than $\lambda$, there exist subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$ with $\kappa \le |\mathcal{F}_n| < \lambda$ for each $n \in \mathbb{N}$, and $\{\, \cup\mathcal{F}_n : n \in \mathbb{N}\,\} \in \mathscr{B}$.

So that $\mathsf{S}_{[1,2)}(\mathscr{A},\mathscr{B})$ is $\mathsf{S}_1(\mathscr{A},\mathscr{B})$, $\mathsf{S}_{[0,\aleph_0)}(\mathscr{A},\mathscr{B})$ is $\mathsf{S}_{\rm fin}(\mathscr{A},\mathscr{B})$, and $\mathsf{U}_{[0,\aleph_0)}(\mathscr{A},\mathscr{B})$ is $\mathsf{U}_{\rm fin}(\mathscr{A},\mathscr{B})$.

**2 Definition.** Say that a family $\mathcal{A} \subseteq \{\,0,1\,\}^{\mathbb{N}\times\mathbb{N}}$ is *semi $\tau$-diagonalizable* if there exists a *partial* function $g : \mathbb{N} \to \mathbb{N}$ such that:

(1) For each $A \in \mathcal{A}$: $(\exists^\infty n \in \mathrm{dom}(g))\ A(n, g(n)) = 1$;

(2) For each $A, B \in \mathcal{A}$:
Either $(\forall^\infty n \in \mathrm{dom}(g))\ A(n, g(n)) \le B(n, g(n))$,
or $(\forall^\infty n \in \mathrm{dom}(g))\ B(n, g(n)) \le A(n, g(n))$.

In the following theorem, note that $\min\{\,\mathfrak{s}, \mathfrak{b}, \mathfrak{od}\,\} \ge \min\{\,\mathfrak{s}, \mathfrak{b}, \mathsf{cov}(\mathcal{M})\,\} = \min\{\,\mathfrak{s}, \mathsf{add}(\mathcal{M})\,\}$.

**3 Theorem.**

*(1) X satisfies* $\mathsf{S}_{[0,2)}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\mathrm{T}})$ *if, and only if, for each Borel function* $\Psi : X \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$*: If* $\Psi[X]$ *is a* $\tau$*-family, then it is semi* $\tau$*-diagonalizable (Definition 2). The corresponding clopen case also holds.*

*(2) The minimal cardinality of a* $\tau$*-family that is not semi* $\tau$*-diagonalizable is at least* $\min\{\mathfrak{s},\mathfrak{b},\mathfrak{od}\}$*.*

*(3)* $\min\{\mathfrak{s},\mathfrak{b},\mathfrak{od}\} \leq \mathsf{non}(\mathsf{S}_{[0,2)}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\mathrm{T}})) = \mathsf{non}(\mathsf{S}_{[0,2)}(\mathrm{T},\mathrm{T})) = \mathsf{non}(\mathsf{S}_{[0,2)}(C_{\mathrm{T}},C_{\mathrm{T}}))$*.*

PROOF. (1) is proved as usual, (2) is shown in the proof of Theorem 4.15 of [6], and (3) follows from (1) and (2). *QED*

**4 Definition** ([9])**.** For functions $f,g,h \in \mathbb{N}^{\mathbb{N}}$, and binary relations $R,S$ on $\mathbb{N}$, define subsets $[f\,R\,g]$ and $[h\,R\,g\,S\,f]$ of $\mathbb{N}$ by:

$$[f\,R\,g] = \{\,n : f(n)Rg(n)\,\},\ [f\,R\,g\,S\,h] = [f\,R\,g] \cap [g\,S\,h].$$

For a subset $Y$ of $\mathbb{N}^{\mathbb{N}}$ and $g \in \mathbb{N}^{\mathbb{N}}$, we say that $g$ *avoids middles* in $Y$ with respect to $\langle R,S\rangle$ if:

(1) for each $f \in Y$, the set $[f\,R\,g]$ is infinite;

(2) for all $f,h \in Y$ at least one of the sets $[f\,R\,g\,S\,h]$ and $[h\,R\,g\,S\,f]$ is finite.

$Y$ satisfies the $\langle R,S\rangle$*-excluded middle property* if there exists $g \in \mathbb{N}^{\mathbb{N}}$ which avoids middles in $Y$ with respect to $\langle R,S\rangle$.

In [10] it is proved that $\mathsf{U}_{\mathrm{fin}}(\mathcal{B}_{\Gamma},\mathcal{B}_{\mathrm{T}})$ is equivalent to having all Borel images in $\mathbb{N}^{\mathbb{N}}$ satisfying the $\langle <,\leq\rangle$-excluded middle property (the statement in [10] is different but equivalent).

**5 Theorem.** *For a set of reals X, the following are equivalent:*

*(1) X satisfies* $\mathsf{U}_{[1,\aleph_0)}(\mathcal{B}_{\Gamma},\mathcal{B}_{\mathrm{T}})$*.*

*(2) Each Borel image of X in* $\mathbb{N}^{\mathbb{N}}$ *satisfies the* $\langle\leq,<\rangle$*-excluded middle property.*

*The corresponding assertion for* $\mathsf{U}_{[1,\aleph_0)}(C_{\Gamma},C_{\mathrm{T}})$ *holds when "Borel" is replaced by "continuous".*

PROOF. The proof is similar to the one given in [10] for $\mathsf{U}_{\mathrm{fin}}(\mathcal{B}_{\Gamma},\mathcal{B}_{\mathrm{T}})$, but is somewhat simpler.

1 ⇒ 2: Assume that $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a Borel image of $X$. Then $Y$ satisfies $\mathsf{U}_{[1,\aleph_0)}(\mathcal{B}_{\Gamma},\mathcal{B}_{\mathrm{T}})$. For each $n$, the collection $\mathcal{U}_n = \{\,U^n_m : m \in \mathbb{N}\,\}$, where $U^n_m = \{\,f \in \mathbb{N}^{\mathbb{N}} : f(n) \leq m\,\}$, is a clopen $\gamma$-cover of $\mathbb{N}^{\mathbb{N}}$. By standard arguments (see (1 ⇒ 2) in the proof of Theorem 2.3 of [6]) we may assume that no $\mathcal{U}_n$

contains a finite cover. For all $n$, the sequence $\{ U^n_m : m \in \mathbb{N} \}$ is monotonically increasing with respect to $\subseteq$, therefore—as large subcovers of $\tau$-covers are also $\tau$-covers—we may use $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$ instead of $\mathsf{U}_{[1,\aleph_0)}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$ to get a $\tau$-cover $\mathcal{U} = \{ \Psi^{-1}[U^n_{m_n}] : n \in \mathbb{N} \}$ for $X$. Let $g \in \mathbb{N}^\mathbb{N}$ be such that $g(n) = m_n$ for all $n$. Then $g$ avoids middles in $Y$ with respect to $\langle \leq, < \rangle$.

$2 \Rightarrow 1$: Assume that $\mathcal{U}_n = \{ U^n_m : m \in \mathbb{N} \}$, $n \in \mathbb{N}$, are Borel covers of $X$ which do not contain a finite subcover. Replacing each $U^n_m$ with the Borel set $\bigcup_{k \leq m} U^n_k$ we may assume that the sets $U^n_m$ are monotonically increasing with $m$. Define $\Psi : X \to \mathbb{N}^\mathbb{N}$ by: $\Psi(x)(n) = \min\{ m : x \in U^n_m \}$. Then $\Psi$ is a Borel map, and so $\Psi[X]$ satisfies the $\langle \leq, < \rangle$-excluded middle property. Let $g \in \mathbb{N}^\mathbb{N}$ be a witness for that. Then $\mathcal{U} = \{ U^n_{g(n)} : n \in \mathbb{N} \}$ is a $\tau$-cover of $X$.

The proof in the clopen case is similar. *QED*

**6 Corollary.** *The critical cardinalities of* $\mathsf{U}_{[1,\aleph_0)}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$, $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$, *and* $\mathsf{U}_{[1,\aleph_0)}(C_\Gamma, C_\mathrm{T})$, *are all equal to* $\mathfrak{b}$.

Proof. This follows from Theorem 5 and the corresponding combinatorial assertion, which was proved in [9]. *QED*

Recall from Figure 1 that the critical cardinality of $\mathsf{U}_{\mathrm{fin}}(\Gamma, \mathrm{T}) = \mathsf{U}_{[0,\aleph_0)}(\Gamma, \mathrm{T})$ is $\max\{ \mathfrak{s}, \mathfrak{b} \}$. Contrast this with Corollary 6.

According to Scheepers [12, Problem 9.5], one of the more interesting problems concerning Figure 1 is whether $\mathsf{S}_1(\Omega, \mathrm{T})$ implies $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$. If $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$ is preserved under taking finite unions, then we get a positive solution to Scheepers' Problem. (Note that $\mathsf{S}_1(\Omega, \mathrm{T})$ implies $\mathsf{S}_1(\Gamma, \mathrm{T})$.)

**7 Corollary.** *If* $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$ *is preserved under taking finite unions, then it is equivalent to* $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$ *and* $\mathsf{S}_1(\Gamma, \mathrm{T})$ *implies* $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$.

Proof. The last assertion of the theorem follows from the first since $\mathsf{S}_1(\Gamma, \mathrm{T})$ implies $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$.

Assume that $X$ does not satisfy $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$. Then, by Hurewicz' Theorem [3], there exists an unbounded continuous image $Y$ of $X$ in $\mathbb{N}^\mathbb{N}$. For each $f \in Y$, define $f_0, f_1 \in \mathbb{N}^\mathbb{N}$ by $f_i(2n + i) = f(n)$ and $f_i(2n + (1 - i)) = 0$. For each $i \in \{ 0, 1 \}$, $Y_i = \{ f_i : f \in Y \}$ is a continuous image of $Y$. It is not difficult to see that $Y_0 \cup Y_1$ does not satisfy the $\langle \leq, < \rangle$-excluded middle property [9]. By Theorem 5, $Y_0 \cup Y_1$ does not satisfy $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$, thus, by the theorem's hypothesis, one of the sets $Y_i$ does not satisfy that property. Therefore $Y$ (and therefore $X$) does not satisfy $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$ either. *QED*

We do not know whether $\mathsf{U}_{[1,\aleph_0)}(\Gamma, \mathrm{T})$ is preserved under taking finite unions. We also do not know the situation for $\mathsf{U}_{\mathrm{fin}}(\Gamma, \mathrm{T})$. The following theorem is only interesting when $\mathfrak{s} < \mathfrak{b}$.

**8 Theorem.** *If there exists a set of reals $X$ satisfying $\mathsf{U}_{\text{fin}}(\Gamma, \mathrm{T})$ but not $\mathsf{U}_{\text{fin}}(\Gamma, \Gamma)$, then $\mathsf{U}_{\text{fin}}(\Gamma, \mathrm{T})$ is not preserved under taking unions of $\mathfrak{s}$ many elements.*

Proof. The proof is similar to the last one, except that here we define $\mathfrak{s}$ many continuous images of $Y$ as we did in [9] to prove that the critical cardinality of $\mathsf{U}_{\text{fin}}(\Gamma, \mathrm{T})$ is $\max\{\mathfrak{s}, \mathfrak{b}\}$. QED

# 3 Luzin sets

A set of reals $L$ is a *generalized Luzin set* if for each meager set $M$, $|L \cap M| < |L|$. In [10] we constructed (assuming a portion of the Continuum Hypothesis) a generalized Luzin set which satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but not $\mathsf{U}_{\text{fin}}(\Gamma, \mathrm{T})$. We now show that the last assertion always holds.

**9 Theorem.** *Assume that $L \subseteq \mathbb{N}^\mathbb{N}$ is a generalized Luzin set. Then $L$ does not satisfy the $\langle <, \leq \rangle$-excluded middle property. In particular, $L$ does not satisfy $\mathsf{U}_{\text{fin}}(C_\Gamma, C_\mathrm{T})$.*

Proof. We use the following easy observation.

**10 Lemma** ([10])**.** *Assume that $A$ is an infinite set of natural numbers, and $f \in \mathbb{N}^\mathbb{N}$. Then the sets*

$$\begin{aligned} M_{f,A} &= \{ g \in \mathbb{N}^\mathbb{N} : [g \leq f] \cap A \textit{ is finite} \} \\ \tilde{M}_{f,A} &= \{ g \in \mathbb{N}^\mathbb{N} : [f < g] \cap A \textit{ is finite} \} \end{aligned}$$

*are meager subsets of* $\mathbb{N}^\mathbb{N}$. QED

Fix any $f \in \mathbb{N}^\mathbb{N}$. We will show that $f$ does not avoid middles in $Y$ with respect to $\langle <, \leq \rangle$. The sets $M_{f,\mathbb{N}} = \{ g \in \mathbb{N}^\mathbb{N} : [g \leq f] \text{ is finite} \}$ and $\tilde{M}_{f,\mathbb{N}} = \{ g \in \mathbb{N}^\mathbb{N} : [f < g] \text{ is finite} \}$ are meager, thus there exists $g_0 \in L \setminus (M_{f,\mathbb{N}} \cup \tilde{M}_{f,\mathbb{N}})$. Now consider the meager sets $M_{f,[f<g_0]} = \{ g \in \mathbb{N}^\mathbb{N} : [g \leq f < g_0] \text{ is finite} \}$ and $\tilde{M}_{f,[g_0 \leq f]} = \{ g \in \mathbb{N}^\mathbb{N} : [g_0 \leq f < g] \text{ is finite} \}$, and choose $g_1 \in L \setminus (M_{f,[f<g_0]} \cup \tilde{M}_{f,[g_0 \leq f]})$. Then both sets $[g_0 < f \leq g_1]$ and $[g_1 < f \leq g_0]$ are infinite. QED

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