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The conjugacy classes and commuting degree of the p-groups with a cyclic subgroup of index p^2

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Abstract. We compute the number of conjugacy classes in p-groups of order p^n and exponent p^{n-2} classified by Ninomiya in 1994. This enables us to obtain the commuting degrees of these groups.

Keywords: Commuting degree, conjugacy class, p-group

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1 Introduction

The commuting degree $d(G)$ of a finite group G is defined as

$$d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}$$

and $d(G)$ measures how close G is to being abelian. This notion is introduced by Erdös and Turan in [5] while studying symmetric groups. It is evident that $d(G) = 1$ if and only if G is abelian. A well-known result due to Gustafson [9] in 1973 states that $d(G) \leqslant 5/8$ for all finite non-abelian groups G with equality if and only if $G/Z(G) \cong C_2 \times C_2$. Also, he shows that $d(G) \leqslant (p^2+p-1)/p^3$ for all non-abelian finite p-groups G. Since then the commuting degree of groups has studied extensively by many authors. Rusin [21] in 1979 classifies all finite groups with $d(G) \geqslant 11/32$ up to central factors. Lescot [13] in 1995 studies commuting degrees in more details and shows that any two isoclinic finite groups have the same commuting degrees. Also, Guralnick and Robinson [8] in 2006 give various general upper and lower bounds for commuting degrees of finite groups. See [2, 4, 6, 10, 11, 14] for further details.

Erdös and Turan in [5] show that

$$d(G) = \frac{k(G)}{|G|}$$

for every finite group G, where $k(G)$ denotes the number of conjugacy classes of G. Hence, studying the commuting degree of finite groups is equivalent to studying the number of their conjugacy classes. Erdös and Turan in [5] show that $k(G) \geqslant \log_2 \log_2 |G|$ for any finite group G giving rise to a general lower bound for commuting degree of finite groups. Poland [19] strengthen the lower bound to $k(G) \geqslant \log_2 |G|$ for finite nilpotent groups G. Extending a result of Pyber [20], Keller in [12] establishes the best known general lower bound for $k(G)$. He shows that

$$k(G) > \epsilon \frac{\log_2 |G|}{(\log_2 \log_2 |G|)^7}$$

for all finite groups G with ϵ being an explicitly computable constant. Moreover,

$$k(G) > \epsilon \frac{\log_2 |G|}{\log_2 \log_2 |G|}$$

when G is a finite solvable group.

Miller [15, 16, 17] in 1901-1902 studied finite p-groups of order p^n and exponent p^{n-2}. Later, in 1994, Ninomiya [18] gives a complete and tidy classification of all finite p-groups of order p^n and exponent p^{n-2}. The aim of this paper is to compute the number of conjugacy classes of these groups and use it to compute their commuting degrees.

Our results are based on the following classification theorems of Ninomiya.

Theorem 1.1 (Ninomiya [18])**.** *Let G be a non-abelian p-group of odd order p^n and exponent p^{n-2}. Then G is isomorphic to one of the following groups:*

(a) $n \geqslant 3$:

- *(1)* $G_1 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = [b, c] = 1, [a, c] = b, \rangle$;

(b) $n \geqslant 4$:

- *(2)* $G_2 = \langle a, b : a^{p^{n-2}} = b^{p^2} = 1, [a, b] = a^{p^{n-3}} \rangle$;
- *(3)* $G_3 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, c] = [b, c] = 1, [a, b] = a^{p^{n-3}} \rangle$;
- *(4)* $G_4 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = [a, c] = 1, [b, c] = a^{p^{n-3}} \rangle$;
- *(5)* $G_5 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = 1, [a, c] = b, [b, c] = a^{p^{n-3}} \rangle$;
- *(6)* $G_6 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [a, b] = 1, [a, c] = b, [b, c] = a^{rp^{n-3}} \rangle$, *where r is a quadratic non-residue modulo p;*
- *(7)* $G_7 = \langle a, b, c : a^{p^{n-2}} = b^p = c^p = [b, c] = 1, [a, c] = b, [a, b] = a^{p^{n-3}} \rangle$,

(c) $n \geqslant 5$:

- *(8)* $G_8 = \langle a, b : a^{p^{n-2}} = b^{p^2} = 1, [a, b] = a^{p^{n-4}} \rangle$;

(9) $G_9 = \langle a, b : a^{p^{n-2}} = b^{p^2} = 1, [b, a] = b^p \rangle$;

(d) $n \geqslant 6$:

(10) $G_{10} = \langle a, b : a^{p^{n-2}} = 1, a^{p^{n-3}} = b^{p^2}, [a, b] = b^p \rangle$;

(e) $p^n = 3^4$:

(11) $G_{11} = \langle a, b, c : a^9 = b^3 = [a, b] = 1, a^3 = c^3, [a, c] = b, [c, b] = a^3 \rangle$.

Theorem 1.2 (Ninomiya [18])**.** *Let G be a non-abelian 2-group of order 2^n and exponent 2^{n-2}. Then G is isomorphic to one of the following groups:*

(a) $n \geqslant 4$:

(1) $G_1 = \langle a, b : a^{2^{n-2}} = b^4 = 1, [a, b] = a^{2^{n-3}} \rangle$;
(2) $G_2 = \langle a, b, c : a^{2^{n-2}} = c^2 = [a, c] = [b, c] = 1, a^{2^{n-3}} = b^2, [b, a] = a^2 \rangle$;
(3) $G_3 = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, c] = [b, c] = 1, [b, a] = a^2 \rangle$;
(4) $G_4 = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, c] = [a, b] = 1, [b, c] = a^{2^{n-3}} \rangle$;
(5) $G_5 = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, b] = [b, c] = 1, [a, c] = b \rangle$;

(b) $n \geqslant 5$:

(6) $G_6 = \langle a, b : a^{2^{n-2}} = b^4 = 1, [b, a] = a^2 \rangle$;
(7) $G_7 = \langle a, b : a^{2^{n-2}} = b^4 = 1, [a, b] = a^{2^{n-3}-2} \rangle$;
(8) $G_8 = \langle a, b : a^{2^{n-2}} = 1, a^{2^{n-3}} = b^4, [b, a] = a^2 \rangle$;
(9) $G_9 = \langle a, b : a^{2^{n-2}} = b^4 = 1, [a, b] = b^2 \rangle$;
(10) $G_{10} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, c] = [b, c] = 1, [a, b] = a^{2^{n-3}} \rangle$;
(11) $G_{11} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, c] = [b, c] = 1, [a, b] = a^{2^{n-3}-2} \rangle$;
(12) $G_{12} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, b] = 1, [c, a] = a^2, [b, c] = a^{2^{n-3}} \rangle$;
(13) $G_{13} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a, b] = [b, c] = 1, [a, c] = a^{-2}b \rangle$;
(14) $G_{14} = \langle a, b, c : a^{2^{n-2}} = b^2 = [a, b] = [b, c] = 1, a^{2^{n-3}} = c^2, [a, c] = a^{-2}b \rangle$;
(15) $G_{15} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [b, c] = 1, [a, b] = a^{2^{n-3}}, [a, c] = a^{2^{n-3}-2} \rangle$;
(16) $G_{16} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = 1, [a, b] = [b, c] = a^{2^{n-3}}, [a, c] = a^{2^{n-3}-2} \rangle$;
(17) $G_{17} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [b, c] = 1, [a, b] = a^{2^{n-3}}, [a, c] = b, \rangle$;
(18) $G_{18} = \langle a, b, c : a^{2^{n-2}} = b^2 = 1, b = c^2, [a, b] = a^{2^{n-3}}, [a, c] = a^{-2}b \rangle$;

(c) $n \geqslant 6$:

(19) $G_{19} = \langle a, b : a^{2^{n-2}} = b^4 = 1, [a,b] = a^{2^{n-4}} \rangle$;

(20) $G_{20} = \langle a, b : a^{2^{n-2}} = b^4 = 1, [a,b] = a^{2^{n-4}-2} \rangle$;

(21) $G_{21} = \langle a, b : a^{2^{n-2}} = 1, a^{2^{n-3}} = b^4, [a,b] = b^2 \rangle$;

(22) $G_{22} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a,b] = 1, [a,c] = a^{2^{n-4}}b, [b,c] = a^{2^{n-3}} \rangle$;

(23) $G_{23} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [a,b] = 1, [a,c] = a^{2^{n-4}-2}b, [b,c] = a^{2^{n-3}} \rangle$;

(24) $G_{24} = \langle a, b, c : a^{2^{n-2}} = b^2 = c^2 = [b,c] = 1, [a,b] = a^{2^{n-3}}, [a,c] = a^{2^{n-4}-2} \rangle$;

(25) $G_{25} = \langle a, b, c : a^{2^{n-2}} = b^2 = [b,c] = 1, a^{2^{n-3}} = c^2, [a,b] = a^{2^{n-3}}, [a,c] = a^{2^{n-4}-2} \rangle$;

(d) $n = 5$:

(26) $G_{26} = \langle a, b, c : a^8 = b^2 = [b,c] = 1, c^2 = a^4, [a,b] = a^4, [a,c] = b \rangle$.

We intend to prove the following results.

Theorem A. *The number of conjugacy classes of the group $G_i(p)$ of odd order p^n is given in Table I, for $i = 1, \ldots, 11$.*

Theorem B. *The number of conjugacy classes of the group $G_i(2)$ of order 2^n is given in Table II, for $i = 1, \ldots, 26$.*

2 Preliminary results

In this section, we shall present a series of theorems and lemmas in order to prove our results.

Theorem 2.1 (Ahmad, Magidin, and Morse [1, Theorem 1.2(ii)])**.** *Let G be a 2-generator p-group of nilpotency class 2. If G has order p^n and has derived subgroup of order p^γ, then G has*

$$p^{n-2\gamma-1}\left(p^{\gamma+1} + p^\gamma - 1\right)$$

conjugacy classes.

Remark. Every group with two at least distinct abelian maximal subgroups is nilpotent of class at most two.

Lemma 2.2. *Let G be a finite group and H be a subgroup of G such that $G = HZ(G)$. Then $k(G) = k(H)[Z(G) : H \cap Z(G)]$.*

In the sequel, we consider p-groups having an abelian maximal or second-maximal subgroup.

Lemma 2.3 ([3, Lemma 1.1])**.** *Suppose G is a non-abelian finite p-group with an abelian maximal subgroup. Then $|G| = p|Z(G)||G'|$.*

Lemma 2.4. *Let G be a non-abelian p-group of order p^n with an abelian maximal subgroup. If $|Z(G)| = p^m$, then*

$$k(G) = p^{m-1}(p^{n-m-1} + p^2 - 1).$$

Proof. Let M be an abelian maximal subgroup of G. Since M is a normal subgroup of G and $C_G(g) = M$ for all $g \in M \setminus Z(G)$, M contains $p^m + (p^{n-1} - p^m)/p = p^{n-2} + p^m - p^{m-1}$ conjugacy classes of G. On the other hand, as M is an abelian maximal subgroup of G, we must have that $C_G(g) = \langle Z(G), g\rangle$ is a subgroup of G of order p^{m+1} for all $g \in G \setminus M$, which implies that $G \setminus M$ contains $(p^n - p^{n-1})/p^{n-m-1} = p^{m+1} - p^m$ conjugacy classes. Therefore,

$$k(G) = (p^{n-2} + p^m - p^{m-1}) + (p^{m+1} - p^m) = p^{n-2} + p^{m+1} - p^{m-1},$$

as required. QED

Lemma 2.5. *Let G be a non-abelian 2-generated p-group of order p^n with $[G : Z(G)] = p^4$. Suppose G has an abelian second-maximal subgroup N but no abelian maximal subgroups. If the centralizers of non-central elements of N are maximal subgroups of G, then*

$$k(G) = p^{n-5}(p^3 + p^2 - 1).$$

Proof. First observe that $N/Z(G)$ is non-cyclic for otherwise $N = \langle Z(G), g\rangle$ for some $g \in G$ and hence $C_G(g)$ is an abelian maximal subgroup of G contradicting the hypothesis. Let $N/Z(G) = \langle xZ(G), yZ(G)\rangle$. Then $C_G(x)$ and $C_G(y)$ are distinct maximal subgroups of G, which implies that $N = C_G(x) \cap C_G(y)$ is a normal subgroup of G. Also, the $p+1$ cyclic subgroups $\langle x_1Z(G)\rangle, \ldots, \langle x_{p+1}Z(G)\rangle$ of $N/Z(G)$ yield $p+1$ distinct maximal subgroups $C_G(x_1), \ldots, C_G(x_{p+1})$ of G. Since G is generated by two elements it follows that every maximal subgroup of G is equal to $C_G(x_i)$ for some $1 \leqslant i \leqslant p+1$. From the hypothesis, we know that N contains $p^{n-4} + (p^{n-2} - p^{n-4})/p = p^{n-3} + p^{n-4} - p^{n-5}$ conjugacy classes of G. On the other hand, if $g \in G \setminus N$, then g belongs to a maximal subgroup of G so that $g \in C_G(x_i)$ for some $1 \leqslant i \leqslant p+1$. If $C_G(g)$ is a maximal subgroup of G, then $C_G(g) = C_G(x_j)$ for some $1 \leqslant j \leqslant p+1$, from which it follows that $C_G(x_j)$ is an abelian maximal subgroup of G, a contradiction. Thus $C_G(g) = \langle Z(G), x_i, g\rangle$ is a second maximal subgroup of G. Accordingly, the number of conjugacy classes of G in $G \setminus N$ is equal to $(p^n - p^{n-2})/p^2 = p^{n-2} - p^{n-4}$. Therefore, the number of conjugacy classes of G is equal to

$$k(G) = (p^{n-3} + p^{n-4} - p^{n-5}) + (p^{n-2} - p^{n-4}) = p^{n-2} + p^{n-3} - p^{n-5},$$

as required. QED

3 Proof of our theorems

We are now in the position to prove our main theorems.

Proof of Theorem A. Let p be an odd prime and $G := G_i(p)$. The center and derived subgroup of $G_i(p)$ is computed in Table I.

It is not difficult to see that G has an abelian maximal subgroup $\langle a, b\rangle$, $\langle a, b^p\rangle$, $\langle a, c\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$, $\langle a^p, b, c\rangle$, $\langle a, b^p\rangle$, $\langle a, b\rangle$ for $i = 1, 2, 3, 4, 5, 6, 7, 9, 11$, respectively. Hence, by Lemma 2.4, we obtain $k(G)$.

If $i = 8$ and $n \geqslant 6$, then G is a 2-generated p-group of nilpotency class 2 so that, by Theorem 2.1, we get $k(G) = p^{n-5}(p^3 + p^2 - 1)$.

Now, assume that either $i = 8$ with $n = 5$, or $i = 10$. By Lemma 2.3, G has no abelian maximal subgroups. A simple verification shows that $\langle Z(G), a^p, b^p\rangle$ is an abelian subgroup of G such that the centralizer of its non-central elements are maximal subgroups of G. Indeed, $C_G(a^{pi}b^{pj}) = \langle Z(G), a^i b^j, a^p, b^p\rangle$ is a maximal subgroup of G for any $0 \leqslant i, j < p$ with $(i, j) \neq (0, 0)$ as G is nilpotent of class $\leqslant 3$, $z' := [b, a]^p \in Z(G)$, and

$$(a^i b^j)^p = a^{pi} b^{pj} [b, a]^{ij\binom{p}{2}} z = a^{pi} b^{pj} z'^{ij(p-1)/2} z$$

for some $z \in Z(G)$. Therefore, by Lemma 2.5, $k(G) = p^{n-5}(p^3 + p^2 - 1)$. The proof is complete. □

Table I. Groups of odd orders

G	$Z(G)$	G'	$k(G)$	Cond.
G_1	$\langle a^p, b\rangle \cong C_{p^{n-3}} \times C_p$	$\langle [a,c]\rangle \cong C_p$	$p^{n-3}(p^2+p-1)$	
G_2	$\langle a^p, b^p\rangle \cong C_{p^{n-3}} \times C_p$	$\langle a^{p^{n-3}}\rangle \cong C_p$	$p^{n-3}(p^2+p-1)$	
G_3	$\langle a^p, c\rangle \cong C_{p^{n-3}} \times C_p$	$\langle a^{p^{n-3}}\rangle \cong C_p$	$p^{n-3}(p^2+p-1)$	
G_4	$\langle a\rangle \cong C_{p^{n-2}}$	$\langle a^{p^{n-3}}\rangle \cong C_p$	$p^{n-3}(p^2+p-1)$	
G_5	$\langle a^p\rangle \cong C_{p^{n-3}}$	$\langle a^{p^{n-3}}, [a,c]\rangle \cong C_p \times C_p$	$p^{n-4}(2p^2-1)$	
G_6	$\langle a^p\rangle \cong C_{p^{n-3}}$	$\langle a^{p^{n-3}}, [a,c]\rangle \cong C_p \times C_p$	$p^{n-4}(2p^2-1)$	
G_7	$\langle a^p\rangle \cong C_{p^{n-3}}$	$\langle a^{p^{n-3}}, [a,c]\rangle \cong C_p \times C_p$	$p^{n-4}(2p^2-1)$	
G_8	$\langle a^{p^2}\rangle \cong C_p$	$\langle a^p\rangle \cong C_{p^2}$	p^3+p^2-1	$n=5$
	$\langle a^{p^2}\rangle \cong C_{p^{n-4}}$	$\langle a^{p^{n-4}}\rangle \cong C_{p^2}$	$p^{n-3}(p^3+p^2-1)$	$n>5$
G_9	$\langle a^p, b^p\rangle \cong C_{p^{n-3}} \times C_p$	$\langle b^p\rangle \cong C_p$	$p^{n-3}(p^2+p-1)$	
G_{10}	$\langle a^{p^2}\rangle \cong C_{p^{n-4}}$	$\langle b^p\rangle \cong C_{p^2}$	$p^{n-5}(p^3+p^2-1)$	
G_{11}	$\langle a^3\rangle \cong C_3$	$\langle a^3, [a,c]\rangle \cong C_3 \times C_3$	17	

Proof of Theorem B. Let $G := G_i(2)$. The center and derived subgroup of $G_i(2)$ is computed in Table II.

It is not difficult to see that G has an abelian maximal subgroup $\langle a, b^2\rangle$, $\langle a, c\rangle$, $\langle a, c\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$, $\langle a, b^2\rangle$, $\langle a, b^2\rangle$, $\langle a, b^2\rangle$, $\langle a, b^2\rangle$, $\langle a, c\rangle$, $\langle a, c\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$, $\langle a^2, b\rangle$, $\langle a, b\rangle$, $\langle a, b\rangle$ for $i = 1$, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 21, 22, 23, respectively. Hence, Lemma 2.4 can be applied to compute $k(G)$.

If $i = 17$, then by Lemma 2.3, Lemma 2.5, and the fact that the centralizers of non-central elements of $\langle a^2, b\rangle$ are maximal subgroups of G, we obtain $k(G) = 11 \cdot 2^{n-5}$.

If $i = 19$, then G is a 2-generated 2-group of nilpotency class 2. Hence, by Theorem 2.1, we get $k(G) = 11 \cdot 2^{n-5}$.

Now, assume that $i \in \{15, 16, 18, 20, 24, 25\}$. First observe that we can rewrite G_{16} and G_{20} as

$$G_{16} = \langle a, b, c : a^{2^{n-2}} = b^2 = [b,c] = 1, c^2 = [a,b] = a^{2^{n-3}}, [a,c] = a^{2^{n-3}-2}\rangle$$

by replacing c by ac and

$$G_{20} = \langle a, b, c : a^{2^{n-2}} = c^4 = 1, b = c^2, [a,c] = a^{2^{n-4}-2}\rangle$$

by replacing (b^2, b) by (b, c). From Table II and Lemma 2.3, one can verify that G has no abelian maximal subgroups. However, G has a maximal subgroup $M := \langle a, b\rangle$ whose all maximal subgroups are abelian. Notice that $a^b = a^{2^{n-3}+1}$ so that $\langle a, b\rangle = \langle a\rangle \rtimes \langle b\rangle \cong C_{2^{n-2}} \rtimes C_2$ with $N_1 := \langle a\rangle$, $N_2 := \langle a^2, b\rangle$, and $N_3 := \langle a^2, ab\rangle$ being the three abelian maximal subgroups of $\langle a, b\rangle$. Let $M^* = N_1 \cap N_2 \cap N_3 = \langle a^2\rangle$. Then $[M : M^*] = 4$ for $N_1 \cap N_2 \cap N_3 = N_1 \cap N_2$. In particular, $M^* = Z(M)$. If $n = 5$, then we apply GAP [7] to compute the number of conjugacy classes of G. Hence assume that $n \geqslant 6$.

Let $g \in M \setminus Z(G)$. A simple verification shows that $C_G(g)$ is a maximal subgroup of G if and only if $g \in (\langle a^2\rangle \setminus \langle a^{2^{n-3}}\rangle) \cup \langle a^{2^{n-4}}\rangle b$. To show this, first assume that $g = a^i$ for some $0 \leqslant i < 2^{n-2}$. Since $C_G(a) = \langle a\rangle$, the number i is even. Then $\langle a, b\rangle = C_G(a^2) \subseteq C_G(g)$ so that $C_G(g) = \langle a, b\rangle$ is a maximal subgroup of G provided that $g \notin Z(G)$. Next assume that $g = a^i b$ for some $0 \leqslant i < 2^{n-2}$. Since $\langle a^2, a^i b\rangle \subseteq C_G(g)$ and $\langle a^2, a^i b\rangle$ has four cosets $\langle a^2, a^i b\rangle$, $a\langle a^2, a^i b\rangle$, $c\langle a^2, a^i b\rangle$, and $ac\langle a^2, a^i b\rangle$ in G, we observe that $C_G(g)$ is a maximal subgroup of G if and only if either $c \in C_G(g)$ or $ac \in C_G(g)$. If $c \in C_G(g)$ (resp. $ac \in C_G(g)$), then $[c, a^i b] = 1$ (resp. $[ac, a^i b] = 1$) and this holds if and only if $2^{n-3} \mid i$ (resp. $2^{n-4} \mid i$ but $2^{n-3} \nmid i$). Hence $C_G(g)$ is a maximal subgroup of G if and only if $2^{n-4} \mid i$. Thus M contains

$$\begin{aligned}
&|Z(G)| + \frac{1}{2}|(\langle a^2\rangle \setminus Z(G)) \cup \langle a^{2^{n-4}}\rangle b| + \frac{1}{4}|M \setminus (\langle a^2\rangle \cup \langle a^{2^{n-4}}\rangle b)| \\
&\quad = 2 + \frac{(2^{n-3} - 2) + 4}{2} + \frac{2^{n-1} - (2^{n-3} + 4)}{4} \\
&\quad = 2^{n-3} + 2^{n-4} - 2^{n-5} + 2
\end{aligned}$$

conjugacy classes. Note that the above arguments show that $M_1 := C_G(a^2) = \langle a, b\rangle$, $M_2 := C_G(b) = \langle a^2, b, c\rangle$, and $M_3 := C_G(a^{2^{n-4}} b) = \langle a^2, b, ac\rangle$ are maximal subgroups of G containing the subgroup $\langle a^2, b\rangle$ of G of index 4, which implies that $G = C_G(a^2) \cup C_G(b) \cup C_G(a^{2^{n-4}} b)$. Thus $|C_G(g)| \geqslant 8$ for all $g \in G$.

Now, let $g \in G \setminus M$. Since $C_M(g)$ is a proper subgroup of M, we must have $C_M(g) \subseteq N_i$ for some $1 \leqslant i \leqslant 3$ so that $C_G(g) = C_{N_i}(g)\langle g\rangle$ with $g^2 \in N_i$. If $|C_G(g)| > 8$, then $M^* \cap C_G(g) \supset Z(G)$ as

$$|M^* C_G(g)| = |M^* C_{N_i}(g)\langle g\rangle| \leqslant |N_i\langle g\rangle| = 2|N_i| < |G|$$

and consequently

$$|M^* \cap C_G(g)| > \frac{|M^*||C_G(g)|}{|G|} = \frac{|C_G(g)|}{8} \geqslant 2 = |Z(G)|$$

by Table II. Let $x \in M^* \cap C_G(g) \setminus Z(G)$. Then

$$C_G(x) \supseteq \langle N_1, N_2, N_3, C_G(g)\rangle = \langle M, C_G(g)\rangle = G$$

as $N_1, N_2, N_3, C_G(g)$ are abelian, a contradiction. Therefore, $|C_G(g)| = 8$ and consequently $G \setminus M$ has $(2^n - 2^{n-1})/2^{n-3} = 4$ conjugacy classes. It follows that

$$k(G) = 2^{n-3} + 2^{n-4} - 2^{n-5} + 6.$$

Finally, if $i = 26$, then $|G| = 32$ and consequently $k(G) = 11$ by GAP. The proof is complete. □

Table II. Groups of even orders

G	$Z(G)$	G'	$k(G)$	Cond.
G_1	$\langle a^2, b^2\rangle \cong C_{2^{n-3}} \times C_2$	$\langle a^{2^{n-3}}\rangle \cong C_2$	$5 \cdot 2^{n-3}$	
G_2	$\langle a^{2^{n-3}}, c\rangle \cong C_2 \times C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_3	$\langle a^{2^{n-3}}, c\rangle \cong C_2 \times C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_4	$\langle a\rangle \cong C_{2^{n-2}}$	$\langle a^{2^{n-3}}\rangle \cong C_2$	$5 \cdot 2^{n-3}$	
G_5	$\langle a^2, c\rangle \cong C_{2^{n-3}} \times C_2$	$\langle [a, c]\rangle \cong C_2$	$5 \cdot 2^{n-3}$	
G_6	$\langle a^{2^{n-3}}, b^2\rangle \cong C_2 \times C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_7	$\langle a^{2^{n-3}}, b^2\rangle \cong C_2 \times C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_8	$\langle b^2\rangle \cong C_4$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_9	$\langle a^2, b^2\rangle \cong C_{2^{n-3}} \times C_2$	$\langle b^2\rangle \cong C_2$	$5 \cdot 2^{n-3}$	
G_{10}	$\langle a^2, c\rangle \cong C_{2^{n-3}} \times C_2$	$\langle a^{2^{n-3}}\rangle \cong C_2$	$5 \cdot 2^{n-3}$	
G_{11}	$\langle a^{2^{n-3}}, c\rangle \cong C_2 \times C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_{12}	$\langle a^{2^{n-4}} b\rangle \cong C_4$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_{13}	$\langle a^{2^{n-3}}, a^2[a, c]\rangle \cong C_2 \times C_2$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_{14}	$\langle a^{2^{n-3}}, a^2[a, c]\rangle \cong C_2 \times C_2$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_{15}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$5 \cdot 2^{n-5} + 6$	
G_{16}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$5 \cdot 2^{n-5} + 6$	
G_{17}	$\langle a^4\rangle \cong C_{2^{n-4}}$	$\langle a^{2^{n-3}}, [a, c]\rangle \cong C_2 \times C_2$	$11 \cdot 2^{n-5}$	
G_{18}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	14 $5 \cdot 2^{n-5} + 6$	$n = 5$ $n > 5$
G_{19}	$\langle a^4\rangle \cong C_{2^{n-4}}$	$\langle a^{2^{n-4}}\rangle \cong C_4$	$11 \cdot 2^{n-5}$	
G_{20}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle a^2\rangle \cong C_{2^{n-3}}$	$5 \cdot 2^{n-5} + 6$	
G_{21}	$\langle a^2\rangle \cong C_{2^{n-3}}$	$\langle b^2\rangle \cong C_4$	$7 \cdot 2^{n-4}$	
G_{22}	$\langle a^2 b\rangle \cong C_{2^{n-3}}$	$\langle [a, c]\rangle \cong C_4$	$7 \cdot 2^{n-4}$	
G_{23}	$\langle (ac)^2\rangle \cong C_4$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	$2^{n-2} + 6$	
G_{24}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	$5 \cdot 2^{n-5} + 6$	
G_{25}	$\langle a^{2^{n-3}}\rangle \cong C_2$	$\langle [a, c]\rangle \cong C_{2^{n-3}}$	$5 \cdot 2^{n-5} + 6$	
G_{26}	$\langle a^4\rangle \cong C_2$	$\langle b, [a, b]\rangle \cong C_2 \times C_2$	11	

Following Theorems A and B and the fact that $d(G) = k(G)/|G|$ for any finite group G, we have indeed computed the commuting degrees of groups under considerations.

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