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On the local spectral subspace preservers

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Abstract. Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on a complex Banach space X with $\dim X \geq 3$. In this paper, we characterize the maps from $\mathcal{B}(X)$ into itself which preserves the dimension of the local spectral subspace relative to $\{1\}$ of the product of operators. The form of the maps from $\mathcal{B}(X)$ into itself preserving the local spectral subspace relative to $\{1\}$ of the product of operators are also described.

Keywords: local spectral subspace; preserver problem; operator algebra

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Introduction

Let X be a complex Banach space and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X with identity I. We denote by $\mathcal{P}(X)$ the set of all idempotent operators in $\mathcal{B}(X)$, and by X^* the dual space of X. For each $x \in X$ and $f \in X^*$, we denote by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)(y) = f(y)x$ for any $y \in X$. The operator $x \otimes f$ is of rank one when x and f are nonzero and each rank one operator in $\mathcal{B}(X)$ can be written in such form. Note that $x \otimes f$ is idempotent if and only if $f(x) = 1$, and that $x \otimes f$ is nilpotent if and only if $f(x) = 0$. We denote by $\mathcal{F}_1(X), \mathcal{P}_1(X)$ and $\mathcal{N}_1(X)$ the set of all rank one operators, the set of all rank one idempotent operators and the set of all rank one nilpotent operators in $\mathcal{B}(X)$, respectively. The local resolvent of $A \in \mathcal{B}(X)$ at a point $x \in X$, denoted by $\rho_A(x)$, is the union of all open subsets U of $\mathbb{C}$ for which there exists an analytic function $f : U \to X$ such that $(A - \lambda)f(\lambda) = x$ for every $\lambda \in U$. The local spectrum of A at x is defined by $\sigma_A(x) := \mathbb{C} \setminus \rho_A(x)$ and is a (possibly empty) closed subset of $\sigma(A)$, the usual spectrum of A. In fact $\sigma_A(x) \neq \emptyset$ for all nonzero vectors $x \in X$ precisely when A has the single-valued extension property, abbreviated as SVEP. Recall that an operator A is said to have SVEP provided that for every open subset U of $\mathbb{C}$ the equation $(A - \lambda)f(\lambda) = 0$ has no non-trivial analytic solution f.

For a subset $\Omega \subset \mathbb{C}$ and an operator $A \in \mathcal{B}(X)$, the local spectral subspace A relative to Ω is

$$X_A(\Omega) := \{x \in X : \ \sigma_A(x) \subset \Omega\},$$

which is a subspace of X. For more information about these notions one can see the books [1, 7].

The set of all fixed points of A is defined by

$$F(A) := \{x \in X : Ax = x\}$$

and is a subspace of X too.

The study of maps on operator algebras preserving certain properties is a topic that attracts the attention of many authors, for example see [2, 3, 4, 5] and the references therein. In [4], Bourhim and Ransford proved that the only additive map on $\mathcal{B}(X)$ preserving the local spectrum at any vector is nothing else than the identity. Motivated by this result, Elhodaibi and Jaatit [5] showed that an additive map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies

$$X_{\phi(A)}(\{\lambda\}) = X_A(\{\lambda\}) \text{ for all } A \in \mathcal{B}(X) \text{ and } \lambda \in \mathbb{C},$$

has the form $\phi(A) = A$ for all $A \in \mathcal{B}(X)$. This result has been generalized in [3]. The authors studied the non-additive case, and gave the form of all surjective maps $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfying $X_{\phi(A)\phi(B)}(\{\lambda\}) = X_{AB}(\{\lambda\})$ for all $A, B \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$.

Recently, Taghavi and Hosseinzadeh [12] characterized all surjective maps ϕ on $\mathcal{B}(X)$ satisfying

$$\dim F(\phi(A)\phi(B)) = \dim F(AB) \text{ for all } \ A, B \in \mathcal{B}(X).$$

Motivated by these results, we determine the form of all unital surjective maps ϕ from $\mathcal{B}(X)$ into itself satisfying

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X).$$

Moreover, the form of surjective maps from $\mathcal{B}(X)$ into itself satisfying

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X),$$

have also been established.

This paper is divided into four sections. In Section 2, we collect some auxiliary Lemmas and summarize some basic properties of the local spectral subspaces. In section 3, we prove some identity principles in term of the dimension of local spectral subspaces of the product of two operators. In section 4, we state and prove our main results.

1 Local spectral subspace properties

The first Lemma summarizes some basic properties of the local spectrum which will be used frequently.

Lemma 1. *([1, 7]) For an operator $A \in \mathcal{B}(X)$, vectors $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$, the following statements hold:*

(1) $\sigma_A(\alpha x) = \sigma_A(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha A}(x) = \alpha\sigma_A(x)$;

(2) If $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$ then $\sigma_A(x) \subset \{\lambda\}$. Furthermore if $x \neq 0$ and A has SVEP then $\sigma_A(x) = \{\lambda\}$;

(3) If $B \in \mathcal{B}(X)$ commutes with A, then $\sigma_A(Bx) \subset \sigma_A(x)$;

(4) If A has SVEP and $Ax = \alpha y$, then $\sigma_A(y) \subset \sigma_A(x) \subset \sigma_A(y) \cup \{0\}$.

The following result gives an explicit identification of the local spectral subspace of rank one operators. We denote by $\ker(A)$ and $Im(A)$ the kernel and the range of $A \in \mathcal{B}(X)$, respectively.

Lemma 2. *([4, Lemma 2.3]) Let $R \in \mathcal{F}_1(X)$ be a non-nilpotent operator and let λ be a nonzero eigenvalue of R. Then*

$$X_R(\{0\}) = \ker(R) \text{ and } X_R(\{\lambda\}) = Im(R).$$

If R is a rank one nilpotent operators we have the following Lemma.

Lemma 3. *Let $R \in \mathcal{F}_1(X)$ be a nilpotent operator and λ be a nonzero scalar. Then*

$$X_R(\{0\}) = X \text{ and } X_R(\{\lambda\}) = \{0\}.$$

Proof. It is a consequence of the fact that $\sigma_R(x) \subset \sigma(R) = \{0\}$ for all $x \in X$. QED

An operator $A \in \mathcal{B}(X)$ is a scalar operator if there exists $\alpha \in \mathbb{C}$ such that $A = \alpha I$. The next Lemma is proved in [6] which gives an important identity principle for this class of operators.

Lemma 4. *([6, proposition 2.3]) Let $A, B \in \mathcal{B}(X)$ be non-scalar operators. If $AP \in \mathcal{P}(X) \setminus \{0\}$ implies $BP \in \mathcal{P}(X) \setminus \{0\}$ for every $P \in \mathcal{P}_1(X)$, then $B = \alpha I + (1-\alpha)A$ for some $\alpha \in \mathbb{C} \setminus \{1\}$.*

Let $x \otimes f$ and $y \otimes g$ be rank one operators in $\mathcal{F}_1(X)$ for some $x, y \in X$ and $f, g \in X^*$. We define the relation $\sim$ on $\mathcal{F}_1(X)$ by $x \otimes f \sim y \otimes g$ if and only if either $\{x, y\}$ or $\{f, g\}$ is a linearly dependent set. Recall that two idempotents $P, Q \in \mathcal{P}_1(X)$ are said to be orthogonal if $PQ = QP = 0$. In this case we write $P \perp Q$.

Lemma 5. *([8, Lemma 2.12]) Let P and Q be two rank one idempotent operators. The following are equivalent:*

(1) $P \perp Q$.

(2) There exist $M, N \in \mathcal{N}_1(X)$ such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$ and $N \nsim M$.

A map $\psi : \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ preserves orthogonality if for every pair $P, Q \in \mathcal{P}_1(X)$ the relation $P \perp Q$ implies $\psi(P) \perp \psi(Q)$. If ψ is bijective and $P \perp Q \Leftrightarrow \psi(P) \perp \psi(Q)$ $P, Q \in \mathcal{P}_1(X)$, then we say that ψ preserves orthogonality in both directions. For maps that preserve rank one operators we refer to [9]. We end this section by presenting a result of bijective maps from $\mathcal{P}_1(X)$ into itself preserving the orthogonality in both directions which is proved in [10, Theorem 2.4].

Lemma 6. *Let X be an infinite-dimensional Banach space, and let $\phi : \mathcal{P}_1(X) \longrightarrow \mathcal{P}_1(X)$ be a bijective map preserving orthogonality in both directions. Then either there exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that*

$$\phi(P) = SPS^{-1} \text{ for all } P \in \mathcal{P}_1(X),$$

or there exists a bounded invertible linear or conjugate-linear operator $S : X^* \to X$ such that

$$\phi(P) = SP^*S^{-1} \text{ for all } P \in \mathcal{P}_1(X).$$

In the second case, X must be reflexive.

2 Identity principles

In this section, we introduce and prove some Lemmas which are useful for the proof of the main results. The first one gives a necessary and sufficient condition for equality among two operators in terms of the dimension of local spectral subspaces.

Lemma 7. *Let $A, B \in \mathcal{B}(X)$ be such that*

$$\dim X_{AT}(\{1\}) = \dim X_{BT}(\{1\}) \textit{ for all } T \in \mathcal{F}_1(X).$$

Then $A = B$.

Proof. Assume by the way of contradiction that there is a nonzero vector $x \in X$ for which $Ax \neq Bx$. Let $f \in X^*$ be a linear functional such that $f(Ax) = 1$ and $f(Bx) \neq 1$. By considering $T = x \otimes f$, we obtain $X_{AT}(\{1\}) = \langle Ax \rangle$ and $X_{BT}(\{1\}) = \{0\}$. Then $\dim X_{AT}(\{1\}) = 1$ and $\dim X_{BT}(\{1\}) = 0$, a contradiction. QED

Lemma 8. *Let $A \in \mathcal{B}(X) \setminus \{I\}$. Then $\dim X_{AP}(\{1\}) = 0$ for every $P \in \mathcal{P}_1(X)$ if and only if $A \in \mathbb{C}I$.*

Proof. The 'if' part is easily verified since $\dim X_{\lambda P}(\{1\}) = 0$ for every $P \in \mathcal{P}_1(X)$ and for all scalar $\lambda \neq 1$. For the 'only if' part, let A be a non-scalar operator. Then there is a vector $x \in X$ such that x and Ax are linearly independent. So, we can find a linear functional f such that $f(x) = f(Ax) = 1$. Setting $P = x \otimes f$, we obtain $\dim X_{AP}(\{1\}) = 1$, a contradiction. This proves the Lemma. QED

Lemma 9. *Let $A, B \in \mathcal{B}(X)$ be non-scalar operators. If $\dim X_{AP}(\{1\}) = \dim X_{BP}(\{1\})$, for all $P \in \mathcal{P}_1(X)$, then there exists an $\alpha \in \mathbb{C} \setminus \{1\}$ such that $B = \alpha I + (1 - \alpha)A$.*

Proof. Let $P \in \mathcal{P}_1(X)$. If $AP \in \mathcal{P}_1(X)$, then $\dim X_{AP}(\{1\}) = 1$. By assumption $\dim X_{BP}(\{1\}) = 1$. Since BP is a rank one operator, $BP \in \mathcal{P}_1(X) \setminus \{0\}$. The assertion follows from Lemma 4. QED

Lemma 10. *Let $A, B \in \mathcal{B}(X)$ be non-scalar operators. If $\dim X_{AR}(\{1\}) = \dim X_{BR}(\{1\})$ for all $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$, then $A = B$.*

Proof. Let $A, B \in \mathcal{B}(X)$ two non-scalar operators such that for all $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$, we have $\dim X_{AR}(\{1\}) = \dim X_{BR}(\{1\})$. By Lemma 9, there exists a nonzero scalar α such that $B = \alpha I + (1 - \alpha)A$. Since A is a non-scalar operator, there exists a vector $x \in X$ such that x and Ax are linearly independent. Let f be a linear functional satisfying $f(x) = 0$ and $f(Ax) = 1$. By choosing $R = x \otimes f$, we get

$$\dim X_{Ax\otimes f}(\{1\}) = \dim X_{(\alpha I + (1-\alpha)A)x\otimes f}(\{1\}).$$

Since $\dim X_{Ax\otimes f}(\{1\}) = 1$ and $(\alpha I + (1 - \alpha)A)x \otimes f$ is a rank one operator, we obtain $(\alpha I + (1 - \alpha)A)x \otimes f \in \mathcal{P}_1(X) \setminus \{0\}$ which implies that $\alpha = 0$ and hence $A = B$. QED

The idea of the last Lemma in this section comes from [6, Proposition 2.7], which is a characterization of the relation $\sim$ in term of the local spectral subspaces.

Lemma 11. *Let $A_1, A_2 \in \mathcal{B}(X)$ be linearly independent rank one operators. Then the following are equivalent:*

(1) $A_1 \sim A_2$;

(2) There is a rank one operator $B \in \mathcal{B}(X)$ such that $\{B, A_i\}$ is linearly independent for $i = 1, 2$, and for each $T \in \mathcal{B}(X)$

$$\dim X_{\zeta A_i T}(\{1\}) = 0, \text{ for all } i = 1, 2, \text{ and } \zeta \in \mathbb{C} \Longrightarrow \dim X_{BT}(\{1\}) = 0.$$

Proof. Let $A_1 = x \otimes f$ and $A_2 = y \otimes g$ for some $x, y \in X$ and $f, g \in X^*$. Assume that $A_1 \sim A_2$, then there exists a nonzero scalar λ such that $y = \lambda x$ or $g = \lambda f$. If $g = \lambda f$, we can write $A_2 = y \otimes f$. Note $B = A_1 + A_2$, since A_1 and A_2 are linearly independent, B is linearly independent of A_i , $i = 1, 2$. Moreover, if for every $T \in \mathcal{B}(X)$ and $\zeta \in \mathbb{C}$, $\dim X_{\zeta A_i T}(\{1\}) = 0$ for $i = 1, 2$. Then $\zeta f(Tx) \neq 1$ and $\zeta f(Ty) \neq 1$. Which implies that $f(Tx) = f(Ty) = 0$. Hence $f(T(x + y)) = 0$. Thus $\dim X_{BT}(\{1\}) = 0$. In the case where x and y are linearly dependent, the same process leads to the same conclusion.
Conversely, in order to obtain a contradiction assume that $A_1 = x \otimes f$ and $A_2 = y \otimes g$ are rank one operators such that x, y and f, g are linearly independent, and there exists a rank one operators $B = u \otimes k$ which satisfies the second condition. We show that k is a linear combination of f and g. If it is not the case, we can find a vector $z \in X$ such that $k(z) \neq 0$, and $f(z) = g(z) = 0$. Moreover, we can find an operator $T \in \mathcal{B}(X)$ satisfying $Tu \neq 0$ and $Tx, Ty, Tu \in < z >$. It follows that $f(Tx) = g(Ty) = 0$ and $k(Tu) = 1$. Which leads to $\dim X_{A_i T}(\{1\}) = 0, i = 1, 2$ and $\dim X_{BT}(\{1\}) = 1$, a contradiction. With a similar reasoning we show that u is a linear combination of x and y. So, we can write $B = (\lambda x + \mu y) \otimes (\alpha f + \beta g)$. Consider η and ν any complex numbers. Since f and g are linearly independent, we can find $w_1, w_2 \in X$ such that $f(w_1) = 0$, $g(w_1) = \eta$, $f(w_2) = \nu$, and $g(w_2) = 0$. Since x and y are linearly independent we can also find an operator $T \in \mathcal{B}(X)$ such that $Tx = w_1$ and $Ty = w_2$. Then

$$0 = k(Tu) = (\alpha f + \beta g)(\lambda Tx + \mu Ty) = \alpha\mu\nu + \beta\lambda\eta$$

which implies that $\alpha\mu = \beta\lambda = 0$. Thus B is multiple either of A_1 or A_2, a contradiction. QED

3 Main results and proof

The first promised main results of this section is the following Theorem.

Theorem 1. *Let X be an infinite dimensional Banach space. If a surjective unital map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \quad \text{for all} \quad A, B \in \mathcal{B}(X), \tag{1}$$

then there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = SAS^{-1}$ for all $A \in \mathcal{B}(X)$.

Proof. Assume that ϕ is a surjective map from $\mathcal{B}(X)$ into itself satisfies (1). We divide the proof into several steps.

Step 1. $\phi(0) = 0$ and ϕ is injective.

Assume that $\phi(0) \neq 0$, then there exists $x \in X$ such that $\phi(0)x \neq 0$. Let f be a linear functional satisfying $f(\phi(0)x) = 1$. Since ϕ is surjective, there exists $A \in \mathcal{B}(X)$ such that $\phi(A) = x \otimes f$. By hypothesis and Lemma 2, we obtain

$$\begin{aligned} 0 &= \dim X_0(\{1\}) \\ &= \dim X_{0A}(\{1\}) \\ &= \dim X_{\phi(0)x\otimes f}(\{1\}) = 1. \end{aligned}$$

This contradiction proves that $\phi(0) = 0$.

Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. By (1), we have

$$\begin{aligned} \dim X_{AT}(\{1\}) &= \dim X_{\phi(A)\phi(T)}(\{1\}) \\ &= \dim X_{\phi(B)\phi(T)}(\{1\}) \\ &= \dim X_{BT}(\{1\}), \end{aligned}$$

for all $T \in \mathcal{B}(X)$. Thus by Lemma 7 $A = B$, which proves that ϕ is injective. Thus ϕ is bijective, since it is assumed to be surjective. Moreover ϕ^{-1} satisfies the equation (1) too.

Step 2. ϕ preserves rank one operators in both directions.

Let $R \in \mathcal{B}(X)$ be a rank one operator. Assume $\phi(R)$ is of rank greater than 2. Then there exist linearly independent vectors y_1 and y_2 in the range of $\phi(R)$. Let $x_1, x_2 \in X$ such that $\phi(R)(x_1) = y_1$ and $\phi(R)(x_2) = y_2$. Note that x_1 and x_2 are linearly independent too. We can find a finite rank operator $B \in \mathcal{B}(X)$ satisfying $By_1 = x_1$ and $By_2 = x_2$. Since ϕ is surjective, there exists $A \in \mathcal{B}(X)$ such that $\phi(A) = B$. Thus, we have $B\phi(R)(x_1) = x_1$ and $B\phi(R)(x_2) = x_2$ which imply that

$$x_1, x_2 \in X_{B\phi(R)}(\{1\})$$

and so

$$\dim X_{AR}(\{1\}) = \dim X_{B\phi(R)}(\{1\}) \geq 2.$$

Which contradicts the fact that $\dim X_{AR}(\{1\}) \leq 1$. Therefore, the rank of $\phi(R)$ is less than 1. Since $\phi(0) = 0$, we have $\phi(R)$ of rank one.
Since ϕ^{-1} satisfies the equality (1), we conclude that ϕ preserves rank one operators in both directions.

Step 3. ϕ preserves rank one idempotent operators in both directions.

Let $x \in X$ and $f \in X^*$ be such that $f(x) = 1$. By the previous step, we can find $y \in X$ and $g \in X^*$ such that $\phi(x \otimes f) = y \otimes g$. Thus by hypothesis, we have

$$\begin{aligned} 1 &= \dim X_{x\otimes f}(\{1\}) \\ &= \dim X_{y\otimes g}(\{1\}). \end{aligned}$$

Which implies that $g(y) = 1$. Then $\phi(x \otimes f) = y \otimes g$ is a rank one idempotent operator.

Step 4. $\phi(\beta I) = \gamma(\beta)I$ for all $\beta \in \mathbb{C}$, where γ is bijective function on $\mathbb{C}$.

By Lemma 8 and the first step, the assertion is true. Moreover since ϕ is bijective, we conclude that γ is bijective and this completes the proof.

Step 5. $\phi(\beta P) = \delta(\beta)\phi(P)$ for every rank one idempotent P and for all $\beta \in \mathbb{C}$, where δ is a bijective function on $\mathbb{C}$.
To do that, consider a rank one idempotent $P \in \mathcal{P}_1(X)$ and a nonzero scalar $\beta \in \mathbb{C}$. It follows by (1), that $\dim X_{\phi(\beta P)\phi(\beta^{-1}I)}(\{1\}) = 1$. This holds only if $\phi(\beta P)\phi(\beta^{-1}I) \in \mathcal{P}_1(X)$. By the previous step, $\phi(\beta P)\phi(\beta^{-1}I) = \gamma(\beta^{-1})\phi(\beta P)$. It remains to show that $\gamma(\beta^{-1})\phi(\beta P) = \phi(P)$ for all $\beta \in \mathbb{C}\setminus\{0\}$. Since $\phi(I) = I$, it is enough to prove the last equality for all $\beta \in \mathbb{C} \setminus \{0, 1\}$. Assume that $\gamma(\beta^{-1})\phi(\beta P) \neq \phi(P)$ for some $\beta \in \mathbb{C} \setminus \{0, 1\}$. Since both operators are rank one idempotent, we can write $\gamma(\beta^{-1})\phi(\beta P) = x \otimes f$ and $\phi(P) = y \otimes g$ with $f(x) = g(y) = 1$. In order to obtain a contradiction, let us discuss the following two cases.
Case 1. x and y are linearly dependent.
Without loss of generality, we can assume that $x = y$. Let us prove that f and g are also linearly dependent. If is not true, let $u \in X$ be such that $g(u) = 1$ and $f(u) = \gamma(\beta^{-1})$. We have

$$\dim X_{(u\otimes g)(x\otimes g)}(\{1\}) = \dim X_{u\otimes g}(\{1\}) = 1,$$

which implies that

$$\dim X_{\phi^{-1}(u\otimes g)P}(\{1\}) = 1. \tag{2}$$

On the other hand

$$\dim X_{(u\otimes g)(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = \dim X_{(\gamma(\beta^{-1})^{-1}u\otimes f)}(\{1\}),$$

which gives that

$$\dim X_{\phi^{-1}(u\otimes g)\beta P}(\{1\}) = 1. \tag{3}$$

Combining (2) and with (3), we get a contradiction since $\beta \notin \{0,1\}$.
Case 2. x and y are linearly independent. If $f(y) \neq 0$, let $h \in X^*$ be such that $h(x) = (\gamma(\beta^{-1})^{-1} f(y))^{-1}$ and $h(y) = 1$. We have

$$\dim X_{(y\otimes h)(y\otimes g)}(\{1\}) = \dim X_{y\otimes g}(\{1\}) = 1.$$

Which implies that

$$X_{\phi^{-1}(y\otimes h)P}(\{1\}) = 1. \tag{4}$$

On the other hand

$$\dim X_{(y\otimes h)(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = \dim X_{f(y)^{-1}y\otimes f}(\{1\}) = 1$$

thus, by hypothesis

$$\dim X_{\phi^{-1}(y\otimes h)\beta P}(\{1\}) = 1. \tag{5}$$

Combining (4) and (5), we get a contradiction because $\beta \notin \{0,1\}$. With a similar reasoning, we get a contradiction when $g(x) \neq 0$. Suppose that $f(y) = g(x) = 0$, and let $A = y \otimes g + \gamma(\beta^{-1}) x \otimes f$. Then

$$\dim X_{Ay\otimes g}(\{1\}) = 1 \text{ and } \dim X_{A(\gamma(\beta^{-1})^{-1}x\otimes f)}(\{1\}) = 1.$$

Thus, we obtain

$$\dim X_{\phi^{-1}(A)P}(\{1\}) = 1 \text{ and } \dim X_{\phi^{-1}(A)\beta P}(\{1\}) = 1,$$

which is a contradiction, since $\beta \notin \{0,1\}$.

Define for $\beta \in \mathbb{C} \setminus \{0\}$, $\delta(\beta) = \gamma(\beta^{-1})^{-1}$ and $\delta(0) = 0$. This completes the proof of this step.

Step 6. ϕ preserves rank one nilpotent operators in both directions ($N \in \mathcal{N}_1(X) \Leftrightarrow \phi(N) \in \mathcal{N}_1(X)$). Moreover $\phi(\beta N) = \delta(\beta)\phi(N)$ for every $N \in \mathcal{N}_1(X)$ and for all $\beta \in \mathbb{C}$.

Let N be a rank one non-nilpotent operator. For $x \in X$ and $f \in X^*$ with $f(x) \neq 0$ let us write $N = x \otimes f$. By (1) and Step 4, we have

$$1 = \dim X_{f(x)^{-1}N}(\{1\}) = \dim X_{\gamma(f(x)^{-1})\phi(N)}(\{1\}).$$

By Step 2, $\phi(N)$ is a rank one operator, consequently $\gamma(f(x)^{-1})\phi(N)$ is a rank one idempotent. Therefore, $\phi(N)$ is not nilpotent. Since ϕ^{-1} satisfies (1), we conclude that ϕ preserves rank one nilpotent operators in both directions.

For the second part, let N be a rank one nilpotent operator and $\beta \in \mathbb{C}$. Then we have

$$\begin{aligned}\dim X_{\beta NP}(\{1\}) &= \dim X_{\phi(\beta N)\phi(P)}(\{1\}) \\ &= \dim X_{\phi(N)\phi(\beta P)}(\{1\}) \\ &= \dim X_{\delta(\beta)\phi(N)\phi(P)}(\{1\}),\end{aligned}$$

for every $P \in \mathcal{P}_1(X)$. By Lemma 4, there exists a scalar $\alpha \neq 1$ such that $\phi(\beta N) = \alpha I + (1-\alpha)\delta(\beta)\phi(N)$. Since $\phi(N)$ is a rank one operator, then $\alpha = 0$. Thus $\phi(\beta N) = \delta(\beta)\phi(N)$, as desired.

Step 7. ϕ preserves the orthogonality of rank one idempotent operators in both directions.

First, we show that $R_1 \sim R_2 \Leftrightarrow \phi(R_1) \sim \phi(R_2)$ for every operators $R_1, R_2 \in \mathcal{N}_1(X) \cup \mathcal{P}_1(X)$. Indeed, if $R_1 \sim R_2$ by Lemma 11 there is a rank one operator B linearly independent of R_1 and linearly independent of R_2, and for every $T \in \mathcal{B}(X)$

$$\dim X_{R_iT}(\{1\}) = 0, \quad i = 1,2 \Rightarrow \dim X_{BT}(\{1\}) = 0.$$

Hence $\phi(B)$ is a rank one operator linearly independent of $\phi(R_i)$, $i = 1, 2$. Indeed, if it is not then there exists a scalar α such that $\phi(B) = \alpha\phi(R_i) = \delta(\beta)\phi(R_i) = \phi(\beta R_i)$. Which contradicts the fact that B and R_i are linearly independent for $i = 1, 2$. By the surjectivity of ϕ and (1) we get $\phi(R_1) \sim \phi(R_2)$.

Let P, Q be rank one idempotent operators such that $PQ = QP = 0$. Then by Lemma 5 there exist $M, N \in \mathcal{N}_1(X)$ such that $P \sim N$, $P \sim M$, $Q \sim N$, $Q \sim M$ and $N \nsim M$. Therefore $\phi(M), \phi(N) \in \mathcal{N}_1(X)$ such that $\phi(P) \sim \phi(N)$, $\phi(P) \sim \phi(M)$, $\phi(Q) \sim \phi(N)$, $\phi(Q) \sim \phi(M)$ and $\phi(N) \nsim \phi(M)$. Thus $\phi(P)\phi(Q) = \phi(Q)\phi(P) = 0$. So ϕ preserves the orthogonality of rank one idempotent operators in both directions.

Combining Step 3, Step 6, and Lemma 6 we infer that ϕ takes one of the following forms:

(1) There exists a bounded invertible linear or conjugate-linear operator $S : X \to X$ such that

$$\phi(P) = SPS^{-1}, \ P \in \mathcal{P}_1(X).$$

(2) X is reflexive, and there exists a bounded invertible linear or conjugate-linear operator $S : X^* \to X$ such that

$$\phi(P) = SP^*S^{-1}, \ P \in \mathcal{P}_1(X).$$

Let N be a nilpotent operator of rank one, suppose that the first form occurs. Since $\dim X_A(\{1\}) = \dim X_{TAT^{-1}}(\{1\})$ for every operator of finite rank A and every bijective operator $T \in \mathcal{B}(X)$. Then for every idempotent operator of rank one P, we have

$$\begin{aligned}\dim X_{\phi(N)\phi(P)}(\{1\}) &= \dim X_{NP}(\{1\})\\ &= \dim X_{SNPS^{-1}}(\{1\})\\ &= \dim X_{SNS^{-1}SPS^{-1}}(\{1\})\\ &= \dim X_{SNS^{-1}\phi(P)}(\{1\}).\end{aligned}$$

By Lemma 9 and Step 3 we obtain $\phi(N) = \alpha I + (1-\alpha)SNS^{-1}$ for some $\alpha \in \mathbb{C} \setminus \{1\}$. Step 6, implies that $\phi(N)$ is a nilpotent operator of rank one. Hence $\alpha = 0$, as desired. If the second form occurs, then with a similar way we can prove that $\phi(N) = SN^*S^{-1}$.

Step 8. ϕ takes the desired forms.
Assume that there exists a bounded invertible linear or conjugate-linear operator S such that

$$\phi(P) = SPS^{-1},\ P \in \mathcal{P}_1(X).$$

We discuss the following two cases.
Case 1. $\phi(A) = SAS^{-1}$ for every non-scalar operator $A \in \mathcal{B}(X)$.

Let A be a non-scalar operator. For every $R \in \mathcal{P}_1(X) \cup \mathcal{N}_1(X)$, we have

$$\begin{aligned}\dim X_{\phi(A)\phi(R)}(\{1\}) &= \dim X_{AR}(\{1\})\\ &= \dim X_{SARS^{-1}}(\{1\})\\ &= \dim X_{SAS^{-1}SRS^{-1}}(\{1\})\\ &= \dim X_{SAS^{-1}\phi(R)}(\{1\}).\end{aligned}$$

Now by Lemma 10, Step 3, and Step 6 we obtain $\phi(A) = SAS^{-1}$ for all non-scalar operators $A \in \mathcal{B}(X)$.
Case 2. $\phi(\beta I) = S(\beta I)S^{-1}$.

Let $x \otimes f$ be an idempotent operator of rank one for some $x \in X$ and $f \in X^*$, and let β be a nonzero complex number. By Step 4 and Case 1, we have

$$1 = \dim X_{\beta I.\beta^{-1}x\otimes f}(\{1\}) = \dim X_{\gamma(\beta)S\beta^{-1}x\otimes fS^{-1}}(\{1\}).$$

If S is linear, we obtain

$$1 = \dim X_{\gamma(\beta)\beta^{-1}Sx\otimes fS^{-1}}(\{1\}).$$

If S is conjugate linear, we obtain

$$1 = \dim X_{\gamma(\beta)\overline{\beta^{-1}}Sx\otimes fS^{-1}}(\{1\}).$$

Which imply that $\gamma(\beta) = \beta$ or $\overline{\beta}$. Therefore, $\phi(\beta I) = \gamma(\beta)I = S(\beta I)S^{-1}$.
The second part can not occur because by a similar way, we may show that $\phi(A) = SA^*S^{-1}$ for every $A \in \mathcal{B}(X)$. Such a form does not satisfy the condition (1) and this completes the proof. QED

The following is finite-dimensional part of our main result, however, here we obtain similar conclusions without assuming unitarity of ϕ. By $M_n(\mathbb{F})$ we denote the algebra of all $n \times n$ matrices with entries in $\mathbb{F}$.

Theorem 2. *Let $n \geq 3$. Then a surjective map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfies that*

$$\dim X_{\phi(A)\phi(B)}(\{1\}) = \dim X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X), \tag{6}$$

if and only if there exists a field automorphism $\xi : \mathbb{F} \to \mathbb{F}$, a function $k : M_n(\mathbb{F}) \to \mathbb{F}^*$ and an invertible matrix $T \in M_n(\mathbb{F})$ such that $\phi(A) = k(A)TA_\xi T^{-1}$ for all $A \in M_n(\mathbb{F})$, where $A_\xi = [\xi(a_{ij})]$ if $A = [a_{ij}]$.

Proof. Applying [10, Theorem 2.3] and [11, Theorem 4.7] and using similar argument as above, we get the desired result. QED

The following is a consequence of Theorem 1.

Theorem 3. *Let X be an infinite dimensional Banach space with* $\dim X \geq 3$. *Then a surjective map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies*

$$X_{\phi(A)\phi(B)}(\{1\}) = X_{AB}(\{1\}) \text{ for all } A, B \in \mathcal{B}(X) \tag{7}$$

if and only if $\phi(A) = \pm A$ for all $A \in \mathcal{B}(X)$.

Proof. It is not difficult to see that $\phi(I) = \pm I$. Assume that $\phi(I) = I$. By Theorem 1, there exists an invertible operator $S \in \mathcal{B}(X)$ such that $\phi(A) = SAS^{-1}$ for all $A \in \mathcal{B}(X)$. It remains to prove that, S is a scalar operator. Indeed, ϕ satisfies the equation (7) then it satisfies the equation (1). Again by Theorem 1 we have $\phi(A) = \pm SAS^{-1}$ for all $A \in \mathcal{B}(X)$. Let $x \in X$ be a nonzero vector, then there exists $f \in X^*$ such that $f(x) = 1$. Take $A = x \otimes f$ and note that

$$\langle Ax \rangle = X_A(\{1\}) = X_{\phi(A)\phi(I)}(\{1\}) = X_{SAS^{-1}}(\{1\}) = \langle SAx \rangle.$$

Consequently, Ax and SAx are linearly dependent for every $x \in X$. Then S is a scalar operator.
If $\phi(I) = -I$, a similar discussion by changing ϕ by $-\phi$ leads to prove that Ax and SAx are linearly dependent for every $x \in X$. Then S is a scalar operator in this case too, which completes the proof. QED

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