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Non-Kähler C-Y 3-folds arising from singular K3 surfaces

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Abstract. In the paper, we study the blow-ups at singular points of a singular K3 surface and analyze the change in the Picard group during this process. Our aim is to produce examples of such surfaces that serve as the base of Seifert fibrations, carrying a non-Kähler complex structure with a trivial canonical bundle. These spaces can be explicitly identified up to a diffeomorphism and are natural candidates for solutions of the Hull-Strominger system.

Keywords: K3 orbifold; Seifert fibration; Non-Kähler geometry

MSC 2020 classification: Primary 32Q25, Secondary 53C56

1 Introduction

Many complex non-Kähler spaces appear as fibrations over a Kähler base. One of the first such examples was Calabi-Eckmann manifolds [3], which are principal toric bundles over a product of projective spaces. Since then, a toric bundle construction has been used to produce various special metrics in Hermitian geometry ([10, 11, 6, 7]). One particular construction, due to Goldstein and Prokushkin [10], uses a K3 surface as a base. This makes the total space of a principal 2-torus bundle over it a complex 3-fold with trivial canonical bundle.

Based on the example in [10], Fu and Yau [8, 9] constructed the first example of solution of the equations in String Theory, called Hull-Strominger system ([20], [13]), which has non-constant dilaton.

These spaces are diffeomorphic up to a finite cover to one of $S^1\sharp_{21}(S^2 \times S^3)$ or $\sharp_{20}(S^2 \times S^4)\sharp_{21}(S^3 \times S^3)$. Using Fu-Yau methods, but over singular K3 surfaces, solutions for the Hull-Strominger system were found in [7] on spaces diffeomorphic to $S^1\sharp_k(S^2 \times S^3)$ or $\sharp_r(S^2 \times S^4)\sharp_{r+1}(S^3 \times S^3)$ for all $13 \leq k \leq 21, 14 \leq r \leq 20$.

This note is the first part of a project to construct more solutions with different topology using Fu-Yau ansatz. In particular, we find examples of singular K3 surfaces with an appropriate divisors allowing to construct smooth complex threefolds, which have the similar topology, but for $3 \leq k \leq 21, 4 \leq r \leq 20$. The detailed proof of the diffeomorphism type and the problem for existence of solutions of the Hull-Strominger system on these spaces is given in a preprint by the second author [17].

2 Preliminaries

In this Section, we collect the main definitions and basic properties of the specific singular algebraic surfaces in weighted projective spaces which we'll utilize later. We focus on the properties of blow-ups of singular points. Most of the results are well-know, but we present the proofs for reader's convenience. We start with the following definition which specifies the surfaces we consider in this paper:

Definition 2.1. A *singular K3 surface* X is a complex surface with only isolated singularities which form a set $\Sigma = \{p_1, ..., p_k\}$, such that $H^1(X, \mathcal{O}_X) = 0$ and $\omega_X \cong \mathcal{O}_X$, where $\omega_X := i_*\omega_{X\setminus\Sigma}$, $i : X \setminus \Sigma \to X$ is the inclusion, and $\omega_{X\setminus\Sigma}$ is the canonical bundle sheaf of $X \setminus \Sigma$.

In particular the singular points are Du Val singularities. We'll be interested in even more specific surfaces with only A_n-singularities.

Definition 2.2. An A_n singularity in a complex surface X is a singular point $p \in X$ such that there is a neighborhood W of p, where $W \cong \mathbb{A}^2_{\mathbb{C}}/\mathbb{Z}_{n+1}$, or equivalently, $W \cong V(x^2 + y^2 + z^{n+1})$.

By the adjunction formula over weighted projective spaces (see [5] Theorem 3.3.4) for quasi smooth complete intersection of multi degree $X = X_{d_1,...,d_k} \subset \mathbb{P}(a_0, ..., a_n)$, we have

$$\omega_X = \mathcal{O}_X(\Sigma_i d_i + \Sigma_i a_i).$$

In particular, for hypersurfaces in a weighted projective 3-space, Miles Reid provided a classification of all singular K3 surfaces which contain 95 families (see [14], p. 138-140). In these cases, the only type of singularities which appear are Du Val of type A_n.

Remark 2.1. One of the most general forms of blow-up is [19]. We will use a specific case called an embedded blow-up of a point in a surface, which is the following. Let X be a complex algebraic surface and let $p \in U \subset X$ be an affine neighborhood of p, so there is an embedding $U \hookrightarrow \mathbb{A}^3_{\mathbb{C}}$. We take $\mathbb{A}^3_{\mathbb{C}}$ and perform the blow-up at the origin, giving us $\pi : \widetilde{\mathbb{A}^3_{\mathbb{C}}} \to \mathbb{A}^3_{\mathbb{C}}$. We have a map $U \setminus \{p\} \hookrightarrow \widetilde{\mathbb{A}^3_{\mathbb{C}}}$ and we take the closure of the image and we denote that as $\widetilde{U}$. Observe that $U \setminus \{p\} \cong \widetilde{U} \setminus \pi^{-1}(p)$, so we can glue the schemes $\widetilde{U}$ and $X \setminus \{p\}$ and we denote that $\tilde{X}$.

We now will explain how the blow-up of Du Val A_n singularities appear locally. This will help us detect the divisors that appear after the blow-ups and do explicit calculations of intersections of these divisors.

Lemma 2.1. Let X be a complex surface with an A_n-type singularity at p, and let $\pi : \tilde{X} \to X$ be the blow-up of X at p as in Remark 2.1. Then we have an affine cover of the exceptional divisor $\{D_u, D_v, D_w\}$ where

$$\mathcal{O}_{\tilde{X}}(D_u) = \frac{\mathbb{C}[\frac{v}{u}, \frac{w}{u}, x]}{\left(1 + \left(\frac{v}{u}\right)^2 + \left(\frac{w}{u}\right)^{n+1} x^{n-1}\right)},$$

$$\mathcal{O}_{\tilde{X}}(D_v) = \frac{\mathbb{C}[\frac{u}{v}, \frac{w}{v}, y]}{\left(\left(\frac{u}{v}\right)^2 + 1 + \left(\frac{w}{v}\right)^{n+1} y^{n-1}\right)},$$

and

$$\mathcal{O}_{\tilde{X}}(D_v) = \frac{\mathbb{C}[\frac{u}{w}, \frac{v}{w}, z]}{\left(\left(\frac{u}{w}\right)^2 + \left(\frac{v}{w}\right)^2 + z^{n-1}\right)}.$$

Moreover, the exceptional divisor has the following forms: when $n = 1$ we have

$$\pi^{-1}(p) \cong \{u^2 + v^2 + w^2 = 0\} \subset \mathbb{P}^2,$$

and when $n > 1$ we have

$$\pi^{-1}(p) \cong \{(u + iv)(u - iv) = 0\} \subset \mathbb{P}^2.$$

Proof. Since blowing up is local, assume $p = (0, 0, 0)$ in $\mathrm{Spec}(\mathbb{C}[x, y, z])$, so that the A_n singularity is given by $x^2 + y^2 + z^{n+1} = 0$.

Let $A = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^{n+1})$ and form the graded ring $A[u, v, w] = A \otimes_{\mathbb{C}} \mathbb{C}[u, v, w]$.

Then the blow-up is

$$\mathrm{Proj}\Big(A[u, v, w]/(xv - yu, xw - zu, yw - zv)\Big),$$

which embeds in $\mathrm{Spec}(A) \times \mathbb{P}^2$. On the affine chart $D_+(u)$, writing $y = \frac{v}{u}x$ and $z = \frac{w}{u}x$ yields

$$x^2\Big(1 + \Big(\frac{v}{u}\Big)^2 + \Big(\frac{w}{u}\Big)^{n+1} x^{n-1}\Big) = 0.$$

Analogous expressions hold on $D_+(v)$ and $D_+(w)$. The exceptional divisor corresponds to $x = 0$ (or $y = 0$, or $z = 0$), so on $D_+(u)$ the defining equation becomes

$$1 + \Big(\frac{v}{u}\Big)^2 + \Big(\frac{w}{u}\Big)^{n+1} x^{n-1} = 0.$$

For $n = 1$, setting $x = 0$ gives

$$1 + \Big(\frac{v}{u}\Big)^2 + \Big(\frac{w}{u}\Big)^2 = 0,$$

which in homogeneous coordinates is equivalent to

$$u^2 + v^2 + w^2 = 0.$$

For $n > 1$, the term involving x^{n-1} drops out, and similar arguments yield

$$1 + \Big(\frac{v}{u}\Big)^2 = 0,$$

so that in projective coordinates the exceptional divisor is

$$u^2 + v^2 = (u + iv)(u - iv) = 0.$$

QED

Remark 2.2. With the notation above in the case where A_n has $n > 1$, the ideal sheaf $\widetilde{(v)}$ corresponds to $(n-1)C$ where C is one of the irreducible components in the exceptional divisor. We compute $\widetilde{(v)} \cdot C' = i^*_{C'}\widetilde{(v)} = \mathcal{O}_{C'}(1)$ and $C' \cong \mathbb{P}^1_{\mathbb{C}}$, so $(n-1)C \cdot C' = 1$ meaning that $(n-1)C$ is the minimal multiple of C to be Cartier, this also applies to C', that is $(n-1)C'$ is Cartier. See Remark 3.1.

Du Val singularities are characterized by preserving the canonical bundle by the pullback of a blow-up. It is important for us to have this feature because this ensures that after any partial resolution of a singular K3 surface with a Du Val singularity we still have a K3 surface.

Lemma 2.2. Let X be a K3 surface with an A_n singularity at $p \in X$, and let $\pi : \tilde{X} \to X$ be a blow-up at p as in Remark 2.1. Then $K_{\tilde{X}} = \pi^* K_X = 0$, where K_X denotes the canonical divisor of X.

Proof. The proof is in [18] in subsection (1.9) in pages 351 and 352 for a more general case. Essentially we have the expression $K_{\tilde{X}} = \sigma^* K_X + (n-k)E$, and in our case $n = k = 2$. QED

Lemma 2.3. Let X be a complex algebraic surface with at worst finite A_n singularities. Let $\pi : \tilde{X} \to X$ be a blow-up as in Remark 2.1 at a singular point $p \in X$. Then we have the exact sequence

$$0 \to Pic(X) \underset{\pi^*}{\to} Pic(\tilde{X}) \underset{i_B^*}{\to} Pic(B) \to 0$$

where $B := \pi^{-1}(p)$ is the exceptional divisor of the blow-up $\pi : \tilde{X} \to X$, and $i_B : B \to \tilde{X}$ is the inclusion of the exceptional divisor. Moreover, when the singularity at $p \in X$ is of type A_n with $n > 1$ we have

$$Pic(\tilde{X}) \cong Pic(X) \oplus \mathbb{Z}^2,$$

and when it is of type A_1 we have

$$Pic(\tilde{X}) \cong Pic(X) \oplus \mathbb{Z}.$$

Proof. In the case that the singularity at the point $p \in X$ is of type A_n with $n > 1$ we have that B is two projective lines $C \cup C'$ intersecting transversally at a point, so $Pic(B) = \{(\mathcal{O}_C(n), \mathcal{O}_{C'}(m) | n, m \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z}$. The morphism i_B^* is the pullback of the inclusion $i_B : B \to \tilde{X}$, and π^* is of the blow-up map projection. We can write the map $i_B^*(D) = (D \cdot C, D \cdot C')$.

Firstly, we prove that i_B^* is surjective. Let $p \in W \subset X$ such that $W \cong V(xy + z^{n+1})$ and p is the origin. In $\pi^{-1}(W) \subset W \times \mathbb{P}^2$, we have two lines $\ell := (0, y, 0; 0 : 1 : 0)$ and $\ell' := (x, 0, 0; 1 : 0 : 0)$ which each intersect the components C, C' respectively at the points $(0, 0, 0; 0 : 1 : 0)$ and $(0, 0, 0; 1 : 0 : 0)$. These points are not rationally equivalent in B, so the lines ℓ, ℓ' are not rationally equivalent. We can take their projective closure, since we have an embedding $\tilde{X} \subset \mathbb{P}^N$. The projective closure are projective lines $\bar{\ell}, \bar{\ell}'$ in $\tilde{X}$. Both of these closures are isomorphic to $\mathbb{P}^1$. This is because ℓ lies in an affine open set of $\mathbb{P}^N$, and an affine open is dense in $\mathbb{P}^N$. So $\bar{\ell}$ embeds to a projective line in $\mathbb{P}^N$. Locally in $\pi^{-1}(W)$ these lines do not pass trough the singular point in B, but their point at infinity might pass through another A_m singularity in $\tilde{X}$. In that case we just take $\mathcal{L} := m\bar{\ell}$ and $\mathcal{L}' := m'\bar{\ell}'$, so $\mathcal{L}, \mathcal{L}'$ are Cartier, and $\mathcal{L} \cdot C = 1$, $\mathcal{L}' \cdot C' = 1$. Thus $i_B^*(\mathcal{L}) = (1, 0)$ and $i_B^*(\mathcal{L}') = (0, 1)$.

Secondly, we prove that π^* is the kernel of i_B^*. So let $\widetilde{\mathcal{L}} \in Pic(\tilde{X})$ such that $i_B^*(\widetilde{\mathcal{L}}) = \mathcal{O}_{\widetilde{X}}$. This means that we can find a Cartier divisor (U_k, f_k) (such that

$D_w, D_v, D_u \in \{U_k\}$ and otherwise $U_k \cap B = \emptyset$) that is associated to $\widetilde{\mathcal{L}}$ such that the pullback

$$(U_k \cap C, i_C^* f_k) = (U_k \cap C, f_k \circ i_C) = (U_k \cap C, \alpha_k)$$

where $\alpha_k \in \mathbb{C}^\times$. We define now a Cartier divisor $\mathcal{L}$ over X over the cover $\{\pi(U_k), W \mid U_k \neq D_w, D_v, D_u\}$ by $(\pi(U_k), g_k)$ where $g_k(q) = f_k(q)$ for $q \neq p$ and $g_k(p) = \alpha_k$. Thus, we get $\pi^*\mathcal{L} \cong \widetilde{\mathcal{L}}$ since the transition functions over every open set contained in $\tilde{X} \setminus B$ are the same as in $X \setminus \{p\}$, and a transition function over $U \subset \tilde{X}$ containing B is constant in B.

Now, consider the case when the singularity at the point $p \in X$ is of type A_1. The ideal sheaf generated by w in $\tilde{X}$, $\mathcal{I}_{(w)}$ is, locally, two rationally equivalent lines $\ell = \{(x, 0, 0; 1 : 0 : 0)\}$ and $\ell' = \{(0, y, 0; 0 : 1 : 0)\}$ i.e. $\mathcal{I}^*_{(w)} = \ell + \ell'$ as Weil divisors. Now, when we compute the intersection product $\mathcal{I}^*_{(w)} \cdot E$, E being the exceptional divisor, we get $2 = \mathcal{I}^*_{(w)} \cdot E = (\ell + \ell') \cdot C = \ell \cdot C + \ell' \cdot C$, so that $\ell \cdot C = \ell' \cdot C = 1$. Note, ℓ's point at infinity may be a singular point, so there is a minimal integer such that $\mathcal{L} := m\bar{\ell}$ is a Cartier divisor over $\tilde{X}$. The exact same argument follows as before.

We want to explain the splitting of the group

$$Pic(\tilde{X}) \cong Pic(X) \oplus Pic(B).$$

We can identify the generators of group $Pic(B)$ in $Pic(\tilde{X})$ with $\mathcal{L}, \mathcal{L}' \in Pic(\tilde{X})$ as defined previously, so we have the maps

$$Pic(B) \stackrel{\sim}{\to} \langle \mathcal{L}, \mathcal{L}' \rangle \to Pic(\tilde{X}) \stackrel{i_B^*}{\to} Pic(B).$$

Note that the composition of all these arrows give us the identity of $Pic(B)$, so the sequence splits. The other case, where $Pic(B)$ is generated by only one element, follows by the same argument. QED

This lemma also proves that any partial resolution of A_n singularities via blow-ups $\pi : \tilde{X} \to X$ has an exact sequence $0 \to Pic(X) \to Pic(\tilde{X}) \to Pic(\pi^{-1}(\Sigma)) \to 0$ where $\Sigma \subset X$ is the set of singular points. In addition, we have isomorphisms $Pic(\tilde{X}) \cong Pic(X) \oplus Pic(\pi^{-1}(\Sigma)) \cong Pic(X) \oplus \mathbb{Z}^N$ where N is the sum the n's in the A_n of the singularities that we are blowing-up. It also points in the direction of a later theorem because an increase in the rank of the Picard group means that b_2 will increase by the same amount.

The following lemma is more a feature of the theory, but it will justify some of the intersections of divisors later.

Lemma 2.4. Let X be a complex algebraic surface with at worst a finite number A_n singularities. Let $\pi : X_{sm} \to X$ be a minimal desingularization of the surface. Then for $\mathcal{L} \in Pic(X)$, D the Weil divisor of a smooth curve on X, and $\tilde{D}$ the proper preimage of D with respect to π:

$$\mathcal{L} \cdot D = \pi^* \mathcal{L} \cdot \tilde{D}.$$

Proof. We have an inclusion morphism $i_{\tilde{D}} : \tilde{D} \to X_{sm}$, the inclusion morphism $i_D : D \to X$, and the restriction of π over $\tilde{D}$, say $\rho : \tilde{D} \to D$ which is an isomorphism since D is smooth. Note that $\pi \circ i_{\tilde{D}} = i_D \circ \rho$.

Now, we have

$$\pi^* \mathcal{L} \cdot \tilde{D} = i^*_{\tilde{D}}(\pi^* \mathcal{L}) = (\pi \circ i_{\tilde{D}})^* \mathcal{L} = (i_D \circ \rho)^* \mathcal{L} = \rho^*(i^*_D \mathcal{L}) \cong \mathcal{L} \cdot D,$$

and the last isomorphism is given by the fact that ρ is an isomorphism. QED

Corollary 2.1. Let X be a complex K3 orbisurface with at worst finite A_n singularities say $\{A_{n_1}, ..., A_{n_j}\}$. Then

$$\chi_{top}(X) = 24 - \sum_{i=1}^{j} n_i,$$

moreover

$$b_0 = b_4 = 1, \ b_1 = b_3 = 0, \text{ and } b_2 = 22 - \sum_{i=1}^{j} n_i.$$

Proof. Firstly, observe that $\chi(X, \mathcal{O}_X) = 2$ by the fact that Dolbeault isomorphism is still true for orbifolds [1]. Indeed, $H^2(X, \mathcal{O}_X) \cong$
$H^0(X, \omega_X) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, and $H^1(X, \mathcal{O}_X) = 0$ by definition of K3 surface. So we have $2 = \frac{1}{12}\left(\chi_{top}(X) + \sum_i n_i\right)$.

Now, the Hodge numbers are $h^{i,j} = dim H^j(X, \Omega^i_X)$, $h^{i,j} = h^{j,i}$, and $b_k = \sum_{i+j=k} h^{i,j}$. Thus, $b_1 = h^{0,1} + h^{1,0} = 2h^{0,1} = 2dim H^1(X, \mathcal{O}_X) = 0$ and by Poincaré duality $b_1 = b_3$. Similarly, $b_0 = dim H^0(X, \mathcal{O}_X) = 1$ and $b_0 = b_4$. Finally, $\chi_{top}(X) = 2 + b_2$, thus $b_2 = 22 - \sum_i n_i$. QED

3 Main result

In Miles Reid's list of singular 95 K3 surfaces (see [14], p. 138-140), the singularities determine the rank of the Picard group of the smooth resolution of the generic member. In this section, we'll use an example which has largest such group.

We start with:

Theorem 3.1. Let X be an integral $\mathbb{Q}$-Cartier complex orbisurface with only A_n isolated singularities, let $\pi : \tilde{X} \to X$ be a chain of blow-ups of singular points as in Remark 2.1 with one A_n with $n > 1$, and let $\bar{H} \in Pic(X) \otimes \mathbb{Q}$ be an ample divisor and keep the notation $\bar{H}$ for its pullbacks throughout the chain of blow-ups. Then there is an ample rational divisor $\mathcal{E} \in \mathrm{Pic}(\tilde{X}) \otimes \mathbb{Q}$, and non-zero $\mathcal{D}, \mathcal{D}' \in \mathrm{Pic}(\tilde{X})$ such that $\mathcal{D}, \mathcal{D}'$ are primitive with respect to $\mathcal{E}$, i.e. $\mathcal{D} \cdot \mathcal{E} = 0, \mathcal{D}' \cdot \mathcal{E} = 0$.

Remark 3.1. Note that since we are assuming that X is $\mathbb{Q}$-Cartier we have that for any Weil divisor D in X there is a $k \in \mathbb{Z}$ such that $\mathcal{O}_X(nD)$ is an invertible sheaf or a Cartier divisor since X is assumed to be integral, so we will denote $\mathcal{O}_X(nD)$ just as nD. This means that the groups $Cl(X) \otimes \mathbb{Q} \cong Pic(X) \otimes \mathbb{Q}$. We call the elements of $Pic(X) \otimes \mathbb{Q}$ rational divisors.

Proof. We only need to check the case when the last blow-up $\varphi : \tilde{X} \to Y$ in the chain of blow-ups $\pi : \tilde{X} \to X$ is of an A_n singularity with $n > 1$. We consider the divisors to be of the form:

$$\begin{aligned}
\mathcal{E} &:= \frac{k(an-b)(bn-a)}{(n-1)\bar{\mathcal{E}} \cdot \bar{H}}(\varphi^*\bar{\mathcal{E}}) - aC - bC', \\
\mathcal{D} &:= \bar{H} - k(bn-a)C, \\
\mathcal{D}' &:= \bar{H} - k(an-b)C';
\end{aligned}$$

where n is the index in A_n, $a = mn + c - 1$, $b = c + m$, with $k, m \in \mathbb{Z}^+$ to be determined later, C and C' are the irreducible curves in the exceptional divisor of the blow-up, and $\bar{\mathcal{E}}$ is an ample divisor in Y.

We first check that $\mathcal{E}$ is indeed an ample divisor via the Nakai–Moishezon ampleness criterion which says that $\mathcal{E}$ is ample if $\mathcal{E}^2 > 0$ and $\mathcal{E} \cdot D > 0$ for any irreducible curve D in $\tilde{X}$. Any irreducible curve D will either be the proper preimage of a curve in Y making $\mathcal{E} \cdot D = q(\varphi^*\bar{\mathcal{E}} \cdot D) = q(\bar{\mathcal{E}} \cdot \varphi_* D) > 0$ where q is the rational number in front of $\varphi^*\bar{\mathcal{E}}$ in $\mathcal{E}$, or it will be C or C' making $\mathcal{E} \cdot D = \mathcal{E} \cdot C$. The expression $q(\bar{\mathcal{E}} \cdot \varphi_* D) > 0$ is true because we assumed that $\bar{\mathcal{E}}$ is ample and q is a positive number. So we only have to check for the other cases, and if $\mathcal{E}^2 > 0$ we get:

$$
\begin{aligned}
\mathcal{E} \cdot C &= -aC^2 - bC \cdot C' \\
&= \frac{an-b}{n-1} = \frac{n(mn-1)+c(n-1)-m}{n-1} > 0 \\
\mathcal{E} \cdot C' &= -aC \cdot C' - bC'^2 \\
&= \frac{bn-a}{n-1} = \frac{c(n-1)+1}{n-1} > 0 \\
\mathcal{E}^2 &= \frac{k^2(an-b)^2(bn-a)^2\bar{\mathcal{E}}^2}{(n-1)^2(\bar{\mathcal{E}} \cdot \bar{H})^2} - \left(\frac{a(an-b)+b(bn-a)}{n-1}\right) > 0,
\end{aligned}
$$

the last inequality is because we can choose k to be big enough.

Now we show that $\mathcal{D}$ and $\mathcal{D}'$ are primitive with respect to $\mathcal{E}$. This follows from:

$$
\begin{aligned}
\mathcal{E} \cdot \mathcal{D} &= \mathcal{E} \cdot \bar{H} - k(bn-a)\mathcal{E} \cdot C \\
&= \frac{k(an-b)(bn-a)}{(n-1)\bar{\mathcal{E}} \cdot \bar{H}} \bar{\mathcal{E}} \cdot \bar{H} - \frac{k(bn-a)(an-b)}{n-1} = 0, \\
\mathcal{E} \cdot \mathcal{D}' &= \mathcal{E} \cdot \bar{H} - k(an-b)\mathcal{E} \cdot C' \\
&= \frac{k(an-b)(bn-a)}{n-1} - k(an-b)\frac{(bn-a)}{n-1} = 0.
\end{aligned}
$$

QED

The case of the blow-up of an A_1 singularity can be found in [7].

Corollary 3.1. Let X be a singular K3 surface in Miles Reid's 95 list such that it has isolated singularities of type A_1, A_7, A_{10}. Then for every k where $2 \leq k \leq 18$ there is a partial resolution $\pi : \tilde{X} \to X$ such that $b_2(\tilde{X}) = b_2(X) + k$. Moreover, for any partial resolution involving an A_n singularity with $n > 1$, there are two generators in the integral cohomology $H^2(\tilde{X}, \mathbb{Z})$ given by the divisors $\mathcal{D}, \mathcal{D}'$ from Theorem 3.1; and for a partial resolution involving only A_1 singularities, there is one generator in the integral cohomology $H^2(\tilde{X}, \mathbb{Z})$ given by the divisor $\mathcal{D}$ from Theorem 3.1.

QED

The surface in Corollary 3.1, as seen in [14] (No.76, p. 139), is given by a generic polynomial of degree 30 in the weighted projective space $\mathbb{P}(5, 6, 8, 11)$. Corollary 3.1 follows directly from previous statements, firstly, $b_2(\tilde{X}) = b_2(X) + k$ follows from Corollary 2.1. Secondly, the existence of one divisor $\mathcal{D}$ in the case with only A_1 singularities, or two $\mathcal{D}, \mathcal{D}'$ in the case with at least one A_n, with

$n > 1$, singularity is part of the statement of Theorem 3.1. Finally, the fact that the divisors have representatives in the integral cohomology $H^1(\tilde{X}, \mathbb{Z})$ follows from the existence of the exponential sequence in surfaces with cyclic singularities, thus in cohomology the long exact sequence we have a natural c_1 injective map (injective since $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$) from $H^1(\tilde{X}, \mathcal{O}^{\times}_{\tilde{X}}) \cong Pic(\tilde{X})$ to $H^1(\tilde{X}, \mathbb{Z})$.

Remark: The divisors $\mathcal{D}$ and $\mathcal{D}'$ are necessary to construct specific Seifert principal bundles over X which have smooth total space and admit a geometric structure necessary for a solution for the Hull-Strominger system. Such bundles are classified by their first Chern classes $c_1(Y_k/X'_k)$ [2, 15, 16]. This construction was used in [7] for a different singular K3 surface to produce solutions of the Hull-Strominger system for certain 6-manifolds. In a similar way as in [7], one can show that the total space of Seifert 2-torus bundles of the surface in Corollary 3.1 has a topology of $S^1\sharp_k(S^2 \times S^3)$ or $\sharp_r(S^2 \times S^4)\sharp_{r+1}(S^3 \times S^3)$ for $3 \leq k \leq 22$ and $4 \leq k \leq 22$. As in [7], the proof is based on the classification of the simply-connected 6-manifolds with free S^1 action.

In [7], the examples were only for $13 \leq k \leq 22$ and $14 \leq r \leq 22$. A construction of solutions for the Hull-Strominger system on our examples is left for a forthcoming paper [17] and details of the proof of the diffeomorphism type.

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