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The role of the Darboux property in the core theorems of Calculus

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Abstract. In this article we explore the role of the Darboux property in proving some of the basic results of Mathematical Analysis, such as Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem or Taylor's theorem. We show that the classical hypotheses used to prove these theorems can be relaxed.

Keywords: Darboux function, Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem, Taylor's theorem.

MSC 2020 classification: Primary 97I40

1 Introduction

Some of the most fundamental theorems of calculus –namely Rolle's theorem, Lagrange's mean value theorem, Cauchy's mean value theorem and Taylor's theorem– are traditionally presented with hypotheses that are of an unmistakably similar flavor. This comes not as a surprise since each one of them can be considered as a generalization of the previous one –see for instance [1, Theorem 5.20]. Specifically, they all deal with a function $f : [a, b] \to \mathbb{R}$ to which two regularity hypotheses are imposed. One of them applies to the open interval (a, b) and the other (a weaker one) to the closed interval. This second one is continuity or, in the case of Taylor's theorem for the case of an order n approximation, being n times continuously differentiable. Still, when we look at the proofs of these results it comes to mind that it is not continuity that we are using, but the Darboux property, that is, that f maps intervals to intervals. In the case of Taylor's theorem this is even more flagrant for the great regularity imposed on f at a and b, as it does not seem to contribute more to the proof that it would do asking for mere continuity. Indeed, if f is known on (a, b) and

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asked to be continuous on $[a,b]$, there is only one way f can be defined at the endpoints a and b. Despite the fact, the usual proofs found in the literature use this premise in a crucial way, and so the hypothesis remains.

Although there are many variants and generalizations of these theorems throughout the literature –see, for instance, [2]– there are surprisingly few results in this direction. One of the few, for instance, appears in [1, Theorem 5.13], for the case of Cauchy's mean value theorem. There we find a version of this result not requiring continuity at the endpoints but, instead, the existence of the limit of the derivative at those points is assumed. Still, there seems to be no previous works where the hypothesis is reduced to the Darboux property.

In this article we will show that the Darboux property suffices as the hypothesis to be asked on the closed interval. To that end, we first introduce some notation and properties regarding oscillation limits and study their relation to the Darboux property. It is then that we move to prove, one by one, the aforementioned results.

2 Oscillation of a function

In what follows we will use the next definitions and notation. Whenever we want to consider intervals between two endpoints but we do not know which one is greater that the other we will write

$$(\!|a,b|\!) := (\min\{a,b\},\max\{a,b\}),\ [\![a,b]\!] := [\min\{a,b\},\max\{a,b\}].$$

We will denote $\mathbb{R}^+ := (0,+\infty)$. Given a set $A\subset\mathbb{R}$, $x_0\in\mathbb{R}$ and $\delta>0$ we write

$$B_A(x_0,\delta) := (x_0-\delta,x_0+\delta)\cap A,\ B_A^*(x_0,\delta) := (x_0-\delta,x_0+\delta)\cap(A\backslash\{x_0\}).$$

$\overline{A}$ will denote the closure of $A\subset\mathbb{R}$ in the usual topology of $\mathbb{R}$, A' its set of accumulation points and $\operatorname{Int}(A)$ its interior. The lateral limits will we written as $f(a^\pm) := \lim_{x\to a^\pm} f(x)$.

Definition 2.1. Let $f: A\subset\mathbb{R}\to\mathbb{R}$. We say that $l\in\mathbb{R}$ is an *oscillation limit of f at $x_0\in A'$* if there exists a sequence $(x_n)_{n\in\mathbb{N}}\subset A\backslash\{x_0\}$ such that $x_n\to x_0$ and $\lim_{n\to\infty} f(x_n)=l$. We denote by $L(f,x_0)$ the *oscillation* of f at x_0, that is, the set of oscillation limits of f at x_0.

The properties of oscillation limits are well known. In particular, we will use the next result.

Proposition 2.2. *Let $f: A\subset\mathbb{R}\to\mathbb{R}$ and $x_0\in A'$. Then $L(f,x_0)$ is closed and* $L(f,x_0)=\bigcap_{\delta\in\mathbb{R}^+}\overline{f\left(B_A^*(x_0,\delta)\right)}$.

3 The Darboux Property

Definition 3.1. Let $I \subset \mathbb{R}$ be an interval. We say a function $f : I \to \mathbb{R}$ is a *Darboux function*, that it *satisfies the Darboux property* or that it satisfies the *intermediate value property* if for every interval $[a, b] \subset I$ we have that $[\![f(a), f(b)]\!] \subset f([a, b])$. That is, if f reaches every value between $f(a)$ and $f(b)$ between a and b.

It is immediate to show that this definition is equivalent to the following statement.

Lemma 3.2. *Let $I \subset \mathbb{R}$ be an interval. $f : I \subset \mathbb{R} \to \mathbb{R}$ is a Darboux function if and only if for every interval $J \subset I$ we have that $f(J)$ is also an interval.*

In this way, the topological nature of the Darboux property is clear: a function is a Darboux function if it preserves connectedness. Many functions we usually work with satisfy the intermediate value property. The Intermediate Value Theorem establishes that a continuous function defined on an interval is a Darboux functions and, Darboux's Theorem, that the same goes for the derivative of a function.

Albeit simple, this notion is not in general comfortable to work with. Among other things, the Darboux property is not closed under addition, not even in the case one of the functions is continuous [3].

From now on we will assume that $a, b \in \mathbb{R}$, $a < b$, so $[a, b]$ is a nondegenerate interval.

Proposition 3.3. *Let $f : [a, b] \to \mathbb{R}$ be a Darboux function, $x_0 \in [a, b]$. Then $L(f, x_0)$ is a closed interval and $f(x_0) \in L(f, x_0)$.*

Proof. We already know that $L(f, x_0)$ is closed. On the other hand, given $\delta \in \mathbb{R}^+$, we have that $B_{[a,b]}(x_0, \delta)$ is an interval, so $f(B_{[a,b]}(x_0, \delta))$ is an interval containing $f(x_0)$. Thus, if $f(B_{[a,b]}(x_0, \delta))$ is a nondegenerate interval,

$$\overline{f(B^*_{[a,b]}(x_0, \delta))} \supset \overline{f(B_{[a,b]}(x_0, \delta))\backslash\{f(x_0)\}} = \overline{f(B_{[a,b]}(x_0, \delta))}.$$

Taking into account that

$$f(B^*_{[a,b]}(x_0, \delta)) \subset f(B_{[a,b]}(x_0, \delta)),$$

we have that

$$\overline{f(B^*_{[a,b]}(x_0, \delta))} = \overline{f(B_{[a,b]}(x_0, \delta))}.$$

On the other hand, if $f(B_{[a,b]}(x_0, \delta))$ is degenerate,

$$f(B_{[a,b]}(x_0, \delta)) = \{f(x_0)\} = f(B^*_{[a,b]}(x_0, \delta)) = \overline{f(B^*_{[a,b]}(x_0, \delta))}.$$

In either case,

$$\overline{f(B^*_{[a,b]}(x_0,\delta))}=\overline{f(B_{[a,b]}(x_0,\delta))}$$

is an interval. Thus,

$$L(f,x_0)=\bigcap_{\delta\in\mathbb{R}^+}\overline{f(B^*_{[a,b]}(x_0,\delta))}=\bigcap_{\delta\in\mathbb{R}^+}\overline{f(B_{[a,b]}(x_0,\delta))}\ni f(x_0)$$

is also an interval since it is an intersection of intervals (the intersection of convex sets is convex). $\boxed{QED}$

The next proposition is the key to extend the Darboux property beyond the open interval.

Proposition 3.4. *Let $f:[a,b]\to\mathbb{R}$ be a Darboux function on (a,b). Then f is a Darboux function on $[a,b]$ if and only if $f(a)\in L(f,a)$ and $f(b)\in L(f,b)$.*

Proof. If f is a Darboux function on $[a,b]$, then, by Proposition 3.3, we have that $f(a)\in L(f,a)$ and $f(b)\in L(f,b)$.

Let us assume that f is a Darboux function on (a,b) and $f(b)\in L(f,b)$. Let us verify that for every interval $I\subset(a,b]$ such that $b\in I$ $f(I)$ is also an interval. Assume, on the contrary, that $I\subset(a,b]$ is an interval but $f(I)$ is not. Then I is nondegenerate and, since f is a Darboux function on (a,b), $f(I\backslash\{b\})$ is an interval, so $b\in I$. Let $r\in\mathbb{R}^+$ be such that $(b-r,b)\subset I\backslash\{b\}$. Hence, either $f(b)>\sup f(I\backslash\{b\})$ or $f(b)<\inf f(I\backslash\{b\})$. Let us assume the first case as the second is analogous. In that case, there exists $\varepsilon\in\mathbb{R}^+$ such that $f(b)-f(x)\geqslant\varepsilon$ for every $x\in I\backslash\{b\}$, that is, $f(I\backslash\{b\})\subset(-\infty,f(b)-\varepsilon]$. Now,

$$L(f,b)=\bigcap_{\delta\in\mathbb{R}^+}\overline{f(B^*_{[a,b]}(b,\delta))}=\bigcap_{\delta<r}\overline{f(b-\delta,b)}\subset(-\infty,f(b)-\varepsilon].$$

Thence, $f(b)\notin L(f,b)$ and we arrive to a contradiction.

Analogously, if $f(a)\in L(f,a)$, for every interval $I\subset[a,b)$ such that $a\in I$ we have that $f(I)$ is an interval. Thus, if $f(a)\in L(f,a)$ and $f(b)\in L(f,b)$, for every interval $I\subset[a,b]$, dividing it into two if necessary, we conclude that $f(I)$ is an interval. $\boxed{QED}$

Corollary 3.5. *Let $f,g:[a,b]\to\mathbb{R}$ be such that f and g are Darboux functions on $[a,b]$, $f+g$ is a Darboux function on (a,b) and f is continuous at a and b. Then, $f+g$ is a Darboux function on $[a,b]$.*

Proof. By Proposition 3.4, it is enough to verify that $f(a)+g(a)\in L(f+g,a)$ and $f(b)+g(b)\in L(f+g,b)$. We check this for b, as for a is analogous. Since g

is a Darboux function on $[a,b]$, there exists $(x_n) \subset (a,b)$ such that $x_n \to b$ and $g(x_n) \to g(b)$. Hence, by the continuity of f at b, we have that $f(x_n)+g(x_n) \to f(b)+g(b)$, so $f(b)+g(b) \in L(f+g,b)$. QED

Remark 3.6. Although we have assumed that f is continuous at a and b, we could have assumed that f is continuous at a and g continuous at b.

4 Mean value theorems

Before proving a general version Rolle's Theorem we will show that a positive derivative implies the function is increasing. Usually, this is shown to be a consequence of the mean value theorem, but it is not necessary to arrive to this result.

Theorem 4.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable from the right on (a,b). If $f'_+ \geq$ $(>)$ $(\leq)$ $(<)$ $(=)$ 0 on (a,b), then f is increasing (strictly increasing) (decreasing) (strictly decreasing) (constant) on $[a,b]$. We have also the analogous result for derivatives from the left.*

Proof. Assume, in the first place, that $f'_+ > 0$ in (a,b). Observe that f cannot be constant on any nondegenerate compact interval $[x,y] \subset [a,b]$, for in that case $f' = 0$ on (x,y), contradicting the hypothesis. Assume that f is not strictly increasing on $[a,b]$. Then there exist $\widetilde{x}, \widetilde{y} \in [a,b]$, $\widetilde{x} < \widetilde{y}$ such that $f(\widetilde{x}) \geqslant f(\widetilde{y})$. We have two cases: either $f(\widetilde{x}) > f(\widetilde{y})$ and in this case we take $x = \widetilde{x}$, $y = \widetilde{y}$; or $f(\widetilde{x}) = f(\widetilde{y})$. In this second case, since f cannot be constant on $[\widetilde{x}, \widetilde{y}]$, there exists $z \in (\widetilde{x}, \widetilde{y})$ such that $f(z) \neq f(\widetilde{x}) = f(\widetilde{y})$, so, either $f(z) > f(\widetilde{y})$ and we take $x = z$, $y = \widetilde{y}$, or $f(z) < f(\widetilde{x})$ and we take $x = \widetilde{x}$, $y = z$. In any case, we have that $f(x) > f(y)$. Since f is continuous at x, there exists $\delta \in \mathbb{R}^+$ such that $f(t) > f(y)$ for every $t \in [x, x+\delta)$.

Consider the set

$$X = \{t \in [x,y] \ : \ f(s) < f(x), \ s \in [t,y]\}.$$

$y \in X$ and X is bounded from below by x, so there exists $c = \inf X$. By the continuity of f, $f(c) = f(x)$, so $a < x \leqslant c < y$. Hence, we have that $f(t) < f(x)$ for every $t \in (c,y]$. Thus, for every $t \in (c,y]$,

$$0 \geqslant \frac{f(t)-f(c)}{t-c}. \tag{4.1}$$

Since $c \in (x,y)$, there exists $f'_+(c)$ and, taking the limit in inequality (4.1) when $t \to c^+$, we have that $f'_+(c) \leqslant 0$, which is a contradiction.

Assume now the case where $f'_+ \geqslant 0$ in (a,b). Let $g_\varepsilon(x) := f(x) + \varepsilon\, x$ for $\varepsilon \in \mathbb{R}^+$ fixed. $(g_\varepsilon)'_+ = f'_+ + \varepsilon > 0$, so, using what we have already proven, we deduce that g_ε is strictly increasing, that is, given $x, y \in [a,b]$, $x < y$ we have that

$$g_\varepsilon(x) = f(x) + \varepsilon\, x < f(y) + \varepsilon\, y = g_\varepsilon(y).$$

Since ε was fixed arbitrarily, taking the limit $\varepsilon \to 0^+$ in $f(x) + \varepsilon\, x < f(y) + \varepsilon\, y$ we have that $f(x) \leqslant f(y)$, as we wanted to prove.

The results for $f'_- > 0$ and $f'_- \geqslant 0$ are proven in an analogous way. Those for the inequalities $<$ and $\leqslant$ are obtained changing f by $-f$. The result for the equality is obtained by combining those with $\leqslant$ and $\geqslant$. QED

Now we are ready to prove Rolle's Theorem.

Theorem 4.2 (Rolle's theorem). *Let $[a,b] \subset \mathbb{R}$ be a nondegenerate interval, $f : [a,b] \to \mathbb{R}$ a Darboux function on $[a,b]$ differentiable on (a,b). If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Assume, by contradiction, that $f'(x) \neq 0$ for every $x \in (a,b)$. Then, by Darboux's Theorem, f' has strictly positive or negative sign on (a,b) (for, otherwise, it would vanish at some point). Assume, without loss of generality, that $f' > 0$. That is, f is increasing on (a,b) —just apply Theorem 4.1 to every $[x,y] \subset (a,b)$. Thus, either $f(a^+)$ exists, and then $L(f,a) = \{f(a^+)\}$, or it diverges to infinity, and therefore $L(f,a) = \emptyset$. On the other hand, since f is a Darboux function, by Proposition 3.3, $f(a) \in L(f,a)$, so $L(f,a) = \{f(a^+)\}$ and $f(a) = f(a^+)$, that is, f is continuous at a. With the same reasoning we conclude that f is continuous at b. Since f is increasing on (a,b) and continuous on $[a,b]$, it is increasing on $[a,b]$. But f is not constant for, otherwise, $f'(x) = 0$ for every $x \in (a,b)$, so we conclude that $f(b) > f(a)$, which contradicts the hypothesis. QED

Applying Theorem 4.2 we obtain Lagrange's mean value theorem.

Theorem 4.3 (Lagrange's mean value theorem). *Let $f : [a,b] \to \mathbb{R}$ be a Darboux function on $[a,b]$ differentiable on (a,b). Then, there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let

$$g(x) = -\frac{f(b) - f(a)}{b - a}(x - a),$$

and

$$h(x) = f(x) + g(x).$$

Since f and g are Darboux functions on $[a,b]$, h is a Darboux function on (a,b) (for $f+g$ is continuous on (a,b)) and g is continuous at a and b, we can assert, by Corollary 3.5, that h is a Darboux function on $[a,b]$. It is now enough to apply Theorem 4.2 to h. $\boxed{QED}$

Similarly, we obtain Cauchy's (or extended) mean value theorem.

Theorem 4.4 (Cauchy's mean value theorem). *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable on (a,b) and such that the function*

$$h(x) := (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$$

is a Darboux function on $[a,b]$. Then, there is $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Remark 4.5. Following Corollary 3.5, in order to guarantee that the function h occurring in Theorem 4.4 is a Darboux function, it would be enough to ask f or g to be continuous at a and b.

It is only natural to wonder whether Theorem 4.4 holds if we change the hypothesis *"h is a Darboux function"* by *"f and g are Darboux functions"*. The following example shows that this is not the case.

Example 4.6. Let

$$f(x) := \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \in \left(0, \dfrac{6}{\pi}\right], \\ 0, & x = 0, \end{cases} \qquad g(x) := \begin{cases} \sin\left(\dfrac{1}{x}\right) + \dfrac{\pi}{6}x, & x \in \left(0, \dfrac{6}{\pi}\right], \\ 1, & x = 0. \end{cases}$$

f and g are differentiable on $\left(0, \frac{6}{\pi}\right]$ and, given that $L(f,0) = L(g,0) = [-1,1]$, f and g are Darboux functions. Also, for every $c \in \left(0, \frac{6}{\pi}\right)$, we have that

$$\left(g\left(\frac{6}{\pi}\right) - g(0)\right) f'(c) - \left(f\left(\frac{6}{\pi}\right) - f(0)\right) g'(c) = -\frac{\pi}{12} \neq 0,$$

so the conclusion of Theorem 4.4 does not hold.

5 Taylor's Theorem

The path to prove Taylor's theorem is less straightforward. This was to be expected as the usual hypotheses in its statement include *"f is n times continuously differentiable on $[a,b]$"*, something we want to downgrade to *"f is a Darboux function on $[a,b]$"*. The proof of Taylor's Theorem will rest on Lemma 5.1 below.

Lemma 5.1. *Let $f:[a,b]\to\mathbb{R}$ be a Darboux function on $[a,b]$ and differentiable on $[a,b)$. Then, for every $c\in(0,+\infty)$ the function $g(x)=f(x)+c\,f'(x)(b-x)$, $x\in[a,b)$ satisfies $L(f,b)\subset L(g,b)$.*

Proof. Since f is a Darboux function, $f(b)\in L(f,b)$, which also is an interval.

Case 1: $L(f,b)=\{f(b)\}$. Let $(x_n)_{n\in\mathbb{N}}\subset(a,b)$ be such that $x_n\to b$. We now show that $f(x_n)\to f(b)$. First, observe that there cannot exist a subsequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ converging to a limit $L\in\mathbb{R}$, $L\neq f(b)$, as this would imply $L\in L(f,b)$. Likewise, we cannot have a subsequence $f(x_{n_k})\to\pm\infty$, because, given that f satisfies the Darboux property, this would imply that $L(f,b)$ contains either $[f(b),\infty)$ or $(-\infty,f(b)]$. Thus, any subsequence of $(f(x_n))_{n\in\mathbb{N}}$ either fails to converge in $[-\infty,\infty]$ (with the topology of a compact interval) or converges to $f(b)$.

Suppose, for contradiction, that $f(x_n)\not\to f(b)$. Then there exists $\varepsilon\in\mathbb{R}^+$ such that $|f(x_n)-f(b)|\geq\varepsilon$ for infinitely many n. Therefore, $f(x_n)\in[-\infty,f(b)-\varepsilon]\cup[f(b)+\varepsilon,\infty]$ for infinitely many n. Since $[-\infty,f(b)-\varepsilon]\cup[f(b)+\varepsilon,\infty]$ is compact in $[-\infty,\infty]$, there exists a subsequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ converging to some $L\in[-\infty,f(b)-\varepsilon]\cup[f(b)+\varepsilon,\infty]$, contradicting the earlier conclusion. Thus, we must have $f(x_n)\to f(b)$.

Using Theorem 4.3, for every $n\in\mathbb{N}$ there exists $y_n\in(x_n,b)$ such that $f'(y_n)=(f(b)-f(x_n))/(b-x_n)$. Hence, $y_n\to b$ and, for the same reasons that guarantee that $f(x_n)\to f(b)$, we have that $f(y_n)\to f(b)$. Given that $0<\frac{b-y_n}{b-x_n}<1$,

$$\begin{aligned}\lim_{n\to\infty}g(y_n)&=\lim_{n\to\infty}[f(y_n)+c\,f'(y_n)(b-y_n)]\\&=\lim_{n\to\infty}\left[f(y_n)+c\,(f(b)-f(x_n))\frac{b-y_n}{b-x_n}\right]=f(b).\end{aligned}$$

Therefore, $f(b)\in L(g,b)$.

Case 2: $L(f,b)$ is a nondegenerate interval. Let $y_0\in\operatorname{Int}(L(f,b))$. Then there exist $\alpha,\beta\in L(f,b)$ such that $\alpha<y_0<\beta$. Let $(x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}},(z_n)_{n\in\mathbb{N}}\subset(a,b)$, be such that $x_n,y_n,z_n\to b$, $f(x_n)\to y_0$, $f(y_n)\to\alpha$, $f(z_n)\to\beta$. Let $N\in\mathbb{N}$ be such that

$$|\alpha - f(y_n)|\,,\ |y_0 - f(c_n)|\,,\ |\beta - f(z_n)| < \frac{\min\{y_0-\alpha, \beta - y_0\}}{2},$$

for every $n \geqslant N$. Then, $f(y_n) < f(x_n) < f(z_n)$ for every $n \geqslant N$. Taking subsequences if necessary, we can assume that $f(y_n) < f(x_n) < f(z_n)$ for every $n \in \mathbb{N}$. Now, for a given $n \in \mathbb{N}$, we define $k_n := \inf\{k \in \mathbb{N} \ :\ z_k > x_n\}$. Let

$$\begin{aligned} \widetilde{x}_n :=& \sup\{x \in [x_n, z_{k_n}] \ :\ f(x) = f(x_n)\}, \\ \widetilde{z}_n :=& \inf\{x \in [x_n, z_{k_n}] \ :\ f(x) = f(z_{k_n})\}. \end{aligned}$$

By the definition of k_n, $\widetilde{x}_n$, $\widetilde{z}_n$ and the continuity of f on $[a,b)$ we have that $\widetilde{x}_n < \widetilde{z}_n$, $f(\widetilde{x}_n) = f(x_n)$, $f(\widetilde{z}_n) = f(z_{k_n})$ and

$$f([\widetilde{x}_n, \widetilde{z}_n]) = [f(\widetilde{x}_n), f(\widetilde{z}_n)] = [f(x_n), f(z_n)].$$

Using Theorem 4.3, for every $n \in \mathbb{N}$ there exists $b_n \in (\widetilde{x}_n, \widetilde{z}_n)$ such that $f'(b_n) = (f(\widetilde{z}_n) - f(\widetilde{x}_n))/(\widetilde{z}_n - \widetilde{x}_n) > 0$. Thus,

$$\begin{aligned} g(b_n) =& f(b_n) + c\,f'(b_n)(b-b_n) \\ =& f(b_n) + c\,\frac{f(\widetilde{z}_n) - f(\widetilde{x}_n)}{\widetilde{z}_n - \widetilde{x}_n}(b-b_n) > f(\widetilde{x}_n) = f(x_n). \end{aligned}$$

Furthermore, we have that $(b_n)_{n\in\mathbb{N}} \to b$. We repeat the process and define, for a given $n \in \mathbb{N}$, $j_n := \inf\{j \in \mathbb{N} \ :\ y_j < x_n\}$. Let

$$\begin{aligned} \widehat{x}_n :=& \sup\{x \in [y_{j_n}, x_n] \ :\ f(x) = f(x_n)\}, \\ \widehat{y}_n :=& \inf\{x \in [x_n, y_{j_n}] \ :\ f(x) = f(y_{j_n})\}. \end{aligned}$$

By definition of j_n, $\widehat{x}_n$, $\widehat{y}_n$ and the continuity of f we have that $\widehat{x}_n < \widehat{y}_n$, $f(\widehat{y}_n) = f(y_{j_n})$, $f(\widehat{x}_n) = f(x_n)$ and

$$f([\widehat{y}_n, \widehat{x}_n]) = [f(\widehat{y}_n), f(\widehat{x}_n)] = [f(y_{j_k}), f(x_n)].$$

By Theorem 4.3, for every $n \in \mathbb{N}$ there exists $a_n \in (\widehat{x}_n, \widehat{y}_n)$ such that $f'(a_n) = (f(\widehat{y}_n) - f(\widehat{x}_n))/(\widehat{y}_n - \widehat{x}_n) < 0$. Thus,

$$\begin{aligned} g(a_n) =& f(a_n) + c\,f'(a_n)(b-a_n) \\ =& f(a_n) + c\,\frac{f(\widehat{y}_n) - f(\widehat{x}_n)}{\widehat{y}_n - \widehat{x}_n}(b-a_n) < f(\widehat{x}_n) = f(x_n). \end{aligned}$$

Furthermore, we have that $(a_n)_{n\in\mathbb{N}} \to b$.

We now build a sequence $(c_n)_{n\in\mathbb{N}}$ in the following way. Given $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $a_n < b_m$. Since g is a Darboux function on $[a_n, b_m]$ (because $g = h'$ where $h(x) = (1+c)\int_{x_1}^{x} f(y)\,\mathrm{d}\,y + c\,f(x)(b-x)$ for $x \in [x_1, b)$) and $g(a_n) < f(x_n) < g(b_m)$, we deduce that there exists $c_n \in (a_n, b_m)$ such that $g(c_n) = f(x_n)$. Then, $(c_n)_{n\in\mathbb{N}} \to b$ and $g(c_n) \to y_0$.

We conclude that $y_0 \in L(g, b)$ and, thus, $\operatorname{Int}(L(f, b)) \subset L(g, b)$. Since $L(g, b)$ is closed, $L(f, b) \subset L(g, b)$, and the proof is finished. QED

In order to state Taylor's theorem we first review the definition of Taylor's polynomial and reminder.

Definition 5.2. Let $n \in \mathbb{N}$, $f : A \subset \mathbb{R} \to \mathbb{R}$ n times differentiable on $x_0 \in A$. We define *Taylor's polynomial of degree n centered at x_0* as the polynomial

$$\begin{aligned} P_{n,x_0}(x) =& \sum_{k=0}^{n} \frac{f^{k)}(x_0)}{k!}(x-x_0)^k \\ =& f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots + \frac{f^{n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$

We define *Taylor's reminder of order n at x_0* as the function

$$R_{n,x_0}(x) = f(x) - P_{n,x_0}(x), \ x \in A.$$

That is, $f(x) = P_{n,x_0}(x) + R_{n,x_0}(x)$ for every $x \in A$.

Theorem 5.3 (Taylor's Theorem). *Let $n \in \{0, 1, \dots\}$; $[a, b] \subset \mathbb{R}$; $x, x_0 \in [a, b]$; $f : [a, b] \to \mathbb{R}$ a Darboux function on $[a, b]$ which is $n+1$ times differentiable on (a, b) and n times differentiable at x_0; and $h \in \mathcal{C}^1([\![x_0, x]\!], \mathbb{R})$ such that $h'(y) \neq 0$ for every $y \in (\!|x_0, x|\!)$. Then there exists $c_{x_0,x} \in (\!|x_0, x|\!)$ such that*

$$R_{n,x_0}(x) = \frac{f^{n+1)}(c_{x_0,x})}{n!}(x - c_{x_0,x})^n \frac{h(x) - h(x_0)}{h'(c_{x_0,x})}.$$

Proof. Let

$$g(y) = \begin{cases} \displaystyle\sum_{k=0}^{n} \frac{f^{k)}(y)}{k!}(x-y)^k, & y \in [\![x_0, x]\!] \backslash \{x\}, \\ f(x), & y = x. \end{cases}$$

Let us check that g is a Darboux function. If $x \in (a, b)$ then this is straightforward because f is $n+1$ times differentiable on (a, b), so g would be continuous.

Let us consider now the case $x = b$ (the case $x = a$ would be analogous). Define, for every $j \in \{0, 1, \dots, n\}$,

$$g_j(y) := \begin{cases} \displaystyle\sum_{k=0}^{j} \frac{f^{k)}(y)}{k!}(b-y)^k, & y \in [x_0, b), \\ f(b), & y = b. \end{cases}$$

Observe that $g_0 = f|_{[x_0,b)}$ and, for $j \in \{0, 1, \dots, n-1\}$ and $y \in [x_0, b)$,

$$\begin{aligned} g_j'(y)(b-y) =& f'(y)(b-y) + \sum_{k=1}^{j} \left[\frac{f^{k+1)}(y)}{k!}(b-y)^{k+1} - \frac{f^{k)}(y)}{(k-1)!}(b-y)^k \right] \\ =& \frac{f^{j+1)}(y)}{j!}(b-y)^{j+1}. \end{aligned}$$

Hence,

$$g_{j+1}(y) = \begin{cases} g_j(y) + \dfrac{1}{j+1} g_j'(y)(b-y), & y \in [x_0, b), \\ f(b), & y = b. \end{cases}$$

Since f is a Darboux function, $f(b) \in L(f, g)$ so, using Lemma 5.1 for $c = 1$, $f(b) \in L(g_1, b)$ and, therefore, g_1 is a Darboux function. Applying Lemma 5.1 iteratively we conclude that all g_j are Darboux functions and, in particular, $g_n = g$.

Since h is continuous on $[\![x_0, x]\!]$, g and h satisfy the hypothesis of Theorem 4.4 on $[\![x_0, x]\!]$ –see Remark 4.5, and we deduce that there exists $c_{x_0,x} \in (x_0, x)$ such that

$$g(x) - g(x_0) = g'(c_{x_0,x}) \frac{h(x) - h(x_0)}{h'(c_{x_0,x})}.$$

On the other hand, for $y \in (\!|x_0, x|\!)$, we have that

$$g(y) = f(y) + \sum_{k=1}^{n} \frac{f^{k)}(y)}{k!}(x-y)^k \quad \text{and} \quad g'(y) = \frac{f^{n+1)}(y)(x-y)^n}{n!}.$$

Hence,

$$g(x) - g(x_0) = \frac{f^{n+1)}(c_{x_0,x})}{n!}(x - c_{x_0,x})^n \frac{h(x) - h(x_0)}{h'(c_{x_0,x})}.$$

By the definition of g we have that $g(x) - g(x_0) = f(x) - P_{n,x_0}(x) =: R_{n,x_0}(x)$, which ends the result. QED

The above version of Taylor's theorem leaves open the choice of the function h and thus the type of remainder to use. The following corollary recovers some of the most frequently used remainders. For more expressions of the remainder see [4].

Corollary 5.4 (Taylor's remainder). *Under the hypothesis of Theorem 5.3,*

(1) If $h(y)=(x-y)^{n+1}$,

$$R_{n,\,x_0}(x)=\frac{f^{n+1)}(c_{x_0,\,x})}{(n+1)!}(x-x_0)^{n+1}\ \textit{(Lagrange's form)};$$

(2) If $h(y)=y-x_0$,

$$R_{n,\,x_0}(x)=\frac{f^{n+1)}(c_{x_0,\,x})}{n!}(x-c_{x_0,\,x})^{n}(x-x_0)\ \textit{(Cauchy's form)};$$

(3) If $f^{n+1)}$ *is continuous on* (a,b) *and*

$$h(y)=\int_{x_0}^{y}\frac{f^{n+1)}(z)}{n!}(x-z)^n\,\mathrm{d}\,z,$$

$$R_{n,\,x_0}(x)=\int_{x_0}^{x}\frac{f^{n+1)}(z)}{n!}(x-z)^n\,\mathrm{d}\,z\ \textit{(integral form)}.$$

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