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Some special bases of the 2–swap algebras

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Abstract. We study the algebra Σ_n induced by the action of the symmetric group S_n on $V^{\otimes n}$ when $\dim V = 2$. Our main result is that the space of symmetric elements of Σ_n is linearly spanned by the involutions of S_n.

Keywords: swap algebra, symmetric group, representations

MSC 2020 classification: Primary 20CXX, Secondary 20C30

Introduction

Let V be a vector space of finite dimension d over some field F (in our computations we will take $F = \mathbb{Q}$ the rational numbers or $\mathbb{C}$ the complex numbers).

In the classical theory of Schur–Weyl a major role is played by the action of the symmetric group S_n on n elements on the n^{th} tensor power $V^{\otimes n}$ by exchanging the tensor factors. The algebra of operators on $V^{\otimes n}$, generated by these permutations will be denoted by $\Sigma_n(d)$ and called *a d–swap algebra*. It is the algebra formed by the elements which commute with the diagonal action of $GL(V)$ or, if V is a Hilbert space, by the corresponding unitary group.

The name comes from the use, in the physics literature, to call *swap* the exchange operator $(1, 2)$ on $V^{\otimes 2}$.

In the literature on quantum information theory the states lying in $\Sigma_n(d)$ are called *Werner states* and widely used as source of examples, due to fundamental work of the physicist R. F. Werner [13].

See for instance [4], [5], [6], [7] for applications to separability, entanglements or the quantum max-cut problem.

A classical theorem, see [12], states that the corresponding algebra homomorphism $F[S_n] \to \Sigma_n(d) \subset End(V^{\otimes n})$ is injective if and only if $\dim V \geq n$.

[i] I wish to thank Felix Huber for pointing out the problem and some of the literature.

When $d = \dim V < n$ the kernel of this map is the two sided ideal of $F[S_n]$ generated by the antisymmetrizer

$$A_{d+1} := \sum_{\sigma \in S_{d+1}} \epsilon_\sigma \sigma, \quad \epsilon_\sigma \text{ the sign of the permutation.}$$

If F has characteristic 0 the algebra $F[S_n]$ decomposes as direct sum of matrix algebras indexed by partitions, corresponding to the irreducible representations of S_n. As for $\Sigma_n(d)$ only the blocks relative to partitions of height $\leq d$ survive.

In the case $d = \dim V < n$ an interesting problem is to describe a basis of $\Sigma_n(d)$ formed by permutations. In fact in the physics literature there are several examples of Hamiltonians lying in $\Sigma_n(d)$. Thus it may be convenient to express such Hamiltonian in a given special basis,

Moreover in $\Sigma_n(d)$ we have the involution $g \mapsto g^{-1}$, if V is a Hilbert space this coincides with adjuction. So we would like also to describe a basis of $\Sigma_n(d)^+$ the subspace of symmetric elements made by permutations and also a basis of self adjoint operators.

In [8] I have proved that a possible basis of $\Sigma_n(d)$ is formed by the permutations which are $d+1$–good. I recall this briefly.

By definition a permutation g, written as a string of numbers, is $d+1$–good if and only if this string does not contain a decreasing subsequence of length $d+1$.

By a beautiful Theorem of Schensted [11] this is equivalent to the fact that the pair of tableaux associated to g is of height $\leq d$.

This of course, by classical theory, is exactly the dimension of $\Sigma_n(d)$ so it is enough to prove that such permutations span $\Sigma_n(d)$ and this, in [8] is done by a *straightening algorithm* deduced from the relations.

Since the pair of tableaux associated to g^{-1} by the Robinson–Schensted correspondence is obtained by exchanging that associated to g, it follows that if g is $d+1$–good so is g^{-1}, and this gives also a basis $g + g^{-1}$, where g is $d+1$–good, for the symmetric elements.

We have also the basis $g - g^{-1}$ for the antisymmetric elements, from which we have a basis over the real numbers for self adjoint operators given by $g + g^{-1}$and $i(g - g^{-1})$, where g is $d+1$–good.

On the other hand, specially for $d = 2$, one may want to find a basis formed by simpler type of elements. For this discussion the simplest elements are the elements of order 2 (called *involutions*) which are permutations with cycles only of order 2,1 and eigenvalues only ± 1.

We call $\Sigma_n(2)$ the *n–swap algebra* and denote it simply Σ_n. It is known that $\dim \Sigma_n = C_n$ the n^{th} Catalan number, see §2.1 for a simple proof.

The list of the first 10 Catalan numbers is

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$$

Definition 1. *The set $\mathcal{S}$ of* special permutations *is formed by the involutions and also by the permutations with cycles only of order 2,1 plus a single cycle of order 3.*

The 3 cycle can be further normalised to be increasing.

The main Theorem of this paper is the following

Theorem 1. *(1) For each n the algebra Σ_n has a basis formed by special elements.*

(2) Σ_n^+ has a basis over $\mathbb{C}$ formed by involutions.

(3) The space of real and symmetric elements has a basis over $\mathbb{R}$ formed by involutions.

Notice that items (2) and (3) are equivalent and follow from (1).

In fact the involutions are symmetric and if a permutation is of the form $g = ab$ with one 3 cycle a and the rest b is a product 2 or 1 cycles its symmetrization is $(g + g^{-1}) = (a + a^{-1})b$.

If a is a 3 cycle in the algebra Σ_3, by relation (1), we have that $a + a^{-1}$ is the sum of -1 and 3 transpositions. The claim follows.

In the same way $(g - g^{-1}) = (a - a^{-1})b$ gives bases for antisymmetric elements, and together for self adjoint operators.

The dimensions of the real symmetric elements are, from $n = 1$ to $n = 10$

$$1, 2, 4, 10, 26, 76, 232, 750, 2494, 8524, \cdots$$

(see The On-Line Encyclopedia of Integer Sequences A007123 for many interesting informations on this sequence).

On the other hand the number $I(n)$ of involutions in S_n from $n = 1$ to $n = 10$ is

$$I(n) = 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, \cdots$$

which is also equal (by the Robinson–Schensted correspondence) to the number of standard Young tableaux with n cells (O.E.I.S A000085).

So a curious fact is that these two sequences coincide up to $n = 7$.

We have thus that the involutions are a basis of the real symmetric elements for $n \leq 7$ and after that they have linear relations.

It would be interesting to understand these relations, they span a representation of S_n under conjugation. For $n = 8$ we have 14 relations and one may ask if they form the irreducible representation of S_8 associated to the partition 4,4.

The antisymmetric elements are spanned by elements of type $ab - a^{-1}b$ with a a 3 cycle, which we can assume to be in increasing order, and b an involution. There are $\binom{n}{3}$ such 3 cycles in S_n and so $\binom{n}{3}I(n-3)$ such elements.

$$\binom{n}{3}I(n-3) = 1, 4, 20, 60, 350,\ n = 3, \cdots, 7$$

so there appear 4 relations already for $n = 5$ where the dimension of Σ_5 is 42, while the number of normalised special elements is 46. We will discuss these relations in remark 5.

Remark 1. *The set of special elements has the following compatibility with the* partial traces $\mathtt{t}_i : End(V)^{\otimes n} \to End(V)^{\otimes n-1}$,

$$\mathtt{t}_i : x_1 \otimes \cdots x_i \otimes \cdots x_n \mapsto tr(x_i)x_1 \otimes \cdots x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n.$$

The map $\mathtt{t}_i$, applied to a permutation decomposed into cycles, removes i from the cycle in which it appears, and so a special element is mapped to a special element. In case i is a fixed element removes i and multiplies by the dimension of the space, in our case 2.

Notice that instead the partial trace of a 3–good element may be 3–bad as for instance

$$\mathtt{t}_4(\{3, 4, 1, 2\}) = \{3, 2, 1\}$$

where by $\{3, 4, 1, 2\}$ we mean the permutation as string and not as cycle, in cycle form $\{3, 4, 1, 2\} = (1, 3)(2, 4) \mapsto (1, 3)(2) = \{3, 2, 1\}$.

Remark 2. *Given a basis e_1, e_2, e_3, e_4 for the space $End(V)$ of linear operators on V one has the* dual basis *$f_i,\ i = 1, \cdots, 4$ for the trace form $tr(ab)$. That is 4 operators satisfying $tr(e_i f_j) = \delta^i_j$.*

Then the operator $(1, 2) : V \otimes V \to V \otimes V$ can be written as

$$(1, 2) = e_1 \otimes f_1 + e_2 \otimes f_2 + e_3 \otimes f_3 + e_4 \otimes f_4.$$

Any involution being product of elements (i, j) can the be expressed using this Formula in term of the basis.

We have some freedom in the choice of the basis. The most common is the basis by matrix units $e_{i,j}$ in which

$$(1,2) = e_{1,1} \otimes e_{1,1} + e_{1,2} \otimes e_{2,1} + e_{2,1} \otimes e_{1,2} + e_{2,2} \otimes e_{2,2}.$$

In particular in Physics are widely used the *Pauli matrices*.

$$\sigma_0 := \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \sigma_x := \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \sigma_y := \begin{vmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{vmatrix}, \sigma_z := \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

They are equal to the dual basis up to a scaling by $\frac{1}{2}$ so that:

$$(1,2) = \frac{1}{2}(\sigma_0 \otimes \sigma_0 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$

The proof of this Theorem is algorithmic. We give an algorithm which, given as input a permutation $\sigma \in S_n$, produces a linear combination of elements in S which in Σ_n equals to σ.

1 The algorithm

Usually we write the permutations in their cycle structure. Let us start with the basic antisymmetrizer which vanishes in Σ_3.

$$A = (1,2,3) + (1,3,2) - (1,2) - (1,3) - (2,3) + 1$$

$$(1,2,3) + (1,3,2) = (1,2) + (1,3) + (2,3) - 1. \tag{1}$$

First remark that in S_3 all permutations are special, moreover

$$(1,3,2) = -(1,2,3) + (1,2) + (1,3) + (2,3) - 1. \tag{2}$$

so a 3–cycle can be normalised.

In S_4 we have the 4-cycles which are not special and we have to write them as linear combination of special permutations in Σ_4.

Notice that if we can do this for a single cycle we can do it for all cycles, since permutations of the same cycle structure are conjugate and clearly the space spanned by special permutations is closed under conjugation.

For any $n > 3$ we have the natural embedding of S_3 in S_n as the permutations on the first 3 elements. This induces an embedding of the basic antisymmetrizer

A in the algebra of the symmetric group of S_n which we denote by An. This element vanishes in the swap algebra Σ_n, and in $\mathbb{C}[S_n]$ generates the ideal of relations for Σ_n. Thus in Σ_4 we have the vanishing of

$$(2,4)A4 = (4,2,3,1) + (3,4,2,1) - (4,2,1) - (3,1)(4,2) - (3,4,2) + (2,4)$$

$$(3,4)A4(2,4) = (2,3,1) + (4,1)(2,3) - (2,3,4,1) - (4,2,3,1) - (2,3) + (3,4,2)$$
$$A4(3,4) = (2,3,4,1) + (3,4,2,1) - (1,2)(3,4) - (3,4,1) - (3,4,2) + (3,4).$$

We then have, in Σ_4, $0 = A4(3,4) - (3,4)A4(2,4)$ that is

$$\begin{aligned} -2(2,3,4,1) &= (3,4,2,1) + (4,2,3,1) - (1,2)(3,4) - (3,4,1) \\ &-2(3,4,2) + (3,4) - (2,3,1) - (4,1)(2,3) + (2,3) \end{aligned} \tag{3}$$

From the vanishing of $(2,4)A4$ we deduce:

$$(4,2,3,1) + (3,4,2,1) = (4,2,1) + (3,1)(4,2) + (3,4,2) - (2,4)$$

Substituting in (3) we deduce:

$$\begin{aligned} &2(1,2,3,4) = \\ &(1,4)(2,3) + (1,2)(3,4) - (1,3)(4,2) - (1,4,2) \\ &+(1,3,4) + (2,3,4) + (1,2,3) + (2,4) - (3,4) - (2,3). \end{aligned} \tag{4}$$

The term $(4,2,1)$ is not normalised, but it can be rewritten using Formula (2).

$$-(4,2,1) = (1,2,4) - (1,2) - (1,4) - (2,4) + 1$$

$$\begin{aligned} &2(1,2,3,4) = \\ &(1,4)(2,3) + (1,2)(3,4) - (1,3)(4,2) + (1,2,4) \\ &+(1,3,4) + (2,3,4) + (1,2,3) - (1,2) - (1,4) - (3,4) - (2,3) + 1. \end{aligned} \tag{5}$$

Since all 4 cycles are conjugate we deduce that statement (1) is true for S_4.

Now notice the following general fact: consider two cycles (a,A), (a,B) of lengths h,k respectively where A and B are strings of integers of lengths $h-1, k-1$ respectively and disjoint. Then their product is the cycle of length $h+k-1$:

$$(a,B)(a,A) = (a,A,B), \quad e.g.\ (1,2,3)(1,5,4,6) = (1,5,4,6,2,3). \tag{6}$$

Thus take a cycle of length $p > 4$ and, up to conjugacy we may take

$$\mathsf{c}_p := (1,2,3,4,5,\dots,p) = (1,5,\dots,p)(1,2,3,4). \tag{7}$$

In Σ_p we have thus that $2\mathsf{c}_p$ equals $(1,5,\dots,p)$ times the expression of Formula (5).

But then applying again Formula (6) we see that the resulting formula is a sum of permutations on p elements which are **not** full cycles.

By iterating then the operation on the cycles of length ℓ with $4 \le \ell \le p-1$ we have a preliminary.

Proposition 2. *The cycle c_p (formula (7)) is a linear combination in Σ_p of elements which contain only cycles of length 1,2,3.*

Hence Σ_n is spanned by permutations which contain only cycles of length 1,2,3.

Example For $p = 5,6$ we have for $2\mathsf{c}_p$ the formula obtained from Formula (5):

$$\begin{aligned}&2(1,2,3,4,5) \overset{(7)}{=} 2(1,5)(1,2,3,4) =\\&(1,4,5)(2,3) + (1,2,,5)(3,4) - (1,3,5)(4,2) - (1,4,2,5) + (1,3,4,5)\\&+ (3,4,2)(1,5) + (1,2,3,5) + (2,4)(1,5) - (3,4)(1,5) - (2,3)(1,5)\end{aligned}$$

In the previous formula appear three 4-cycles, for which we can apply Formula (5) (see in the appendix, the expanded Formula (10)).

Notice that the final Formula must be invariant under conjugation by powers of the cycle, but this only up to the relations in Σ_n.

$$2(1,2,3,4,5,6) \overset{(7)}{=} 2(1,5,6)(1,2,3,4) = \tag{8}$$

$$(1,4,5,6)(2,3) + (1,2,5,6)(3,4) - (1,3,5,6)(4,2) - (1,4,2,5,6) + (1,3,4,5,6)$$

$$+(3,4,2)(1,5,6) + (1,2,3,5,6) + (2,4)(1,5,6) - (3,4)(1,5,6) - (2,3)(1,5,6).$$

Of course in the previous formulas appear 4-cycles, for which we can apply Formula (5), and then 5-cycles, for which we can apply the final formula developed before. Notice now that in Formula (8) all terms are either special or can be expanded into a linear combination of special elements, using the formulas of 4 and 5–cycles, except the term $(3,4,2)(1,5,6)$.

Remark 3. *In order to prove Theorem 1 using Proposition 2 it is enough to prove that, in S_6, a permutation of type $3,3$ can be developed as linear combination of special elements.*

Proof. We apply recursively this reduction to a product of k disjoint 3-cycles. If k is even we replace them all and if odd we remain with only one 3-cycle which can be normalized if necessary using Formula (2). QED

The computation in S_6 in principle is similar to that in S_4 but now we have to handle a priori many more relations and I had to be assisted by the software "Mathematica" in order to discover the needed relations.

Let me sketch what I did.

1.1 The computation

A set of relations for Σ_6 can be obtained from the antisymmetrizer $A6$ by multiplication to the left and right by the 720 permutations. Actually it is not necessary to use all permutations since there are 36×6 pairs which stabilyze $A6$ up to sign

Finally using these reductions we have 2400 relations each a sum of 6 permutations, of which 3 even and 3 odd.

Each 6–cycle c, appearing in these relations, needs to be developed by using the appropriate conjugate of Formula (8) by the permutation which has as string the same form of the cycle c and which conjugates the standard 6 cycle into c.

So a 6 cycle is replaced, using a conjugate of formula (8), by a permutation of type $3,3$ plus a sum of special terms. In this way we obtain 2400 relations which, by inspection contain either 0, 2 or 3 permutations of type $3,3$ and with the remaining terms special.

The ones with 0, 2 permutations of type $3,3$ are linear combinations of special permutations and cannot be used.

Remain 360 relations containing 3 permutations of type $3,3$, arising from relations with two 6–cycles and one permutations of type $3,3$.

Remark 4. *There are 40 permutations of type* $(3,3)$ *but using Formula* (1) *we can normalise these elements.*

If a 3 cycle (a,b,c) *is not strictly increasing (up to cyclic equivalence) we can replace it by a strictly increasing cycle introducing a sign and adding some special permutations applying a conjugate of formula* (1).

We then are reduced to 10 normalised permutations of type $(3,3)$.

We have several relations involving these normalised permutations plus special elements and we have to eliminate in one relation all permutations of type $(3,3)$ *except one, thus obtaining the desired relation.*

1.2 The useful relations

Surprisingly in order to obtain the desired relation only the following 2 suffice:

$$(5,6,1)(3,4)A6 =$$
$$(1,2,4,3,5,6)+(1,4,3,2,5,6) - (2,5,6,1)(3,4) - (4,3,5,6,1) - (5,6,1)(4,3,2)+(5,6,1)(3,4)$$

$$(6,1)(4,5,3)A6 =$$
$$(1,2,4,5,3,6) + (1,4,5,3,2,6) - (2,6,1)(4,5,3) - (4,5,3,6,1) - (6,1)(4,5,3,2) + (6,1)(4,5,3)$$

In Formula (8) the contribution to the expansion of $2(1,2,3,4,5,6)$ of an element of type $3,3$ is $+(3,4,2)(1,5,6)$.

Therefore the contributions of type $3,3$ of the 4 cycles of length 6 appearing in the previous Formulas are obtained by conjugating $(3,4,2)(1,5,6)$ with the permutation which has as string the same form of the cycle.

We obtain:

$$2\,(1,2,4,3,5,6) = (3,5,4)(2,6,1) + \cdots, \quad 2\,(1,4,3,2,5,6) = (2,5,3)(4,6,1) + \cdots$$
$$2\,(1,2,4,5,3,6) = (5,3,4)(2,6,1) + \cdots, \quad 2\,(1,4,5,3,2,6) = (3,2,5)(4,6,1) + \cdots$$

So, by Formula (1), the previous elements can be written in Σ_6 as

$$2\,(1,2,4,3,5,6) = -(3,4,5)(1,2,6) + \cdots, \quad 2\,(1,4,3,2,5,6) = -(2,3,5)(1,4,6) + \cdots$$
$$2\,(1,2,4,5,3,6) = (3,4,5)(1,2,6) + \cdots, \quad 2\,(1,4,5,3,2,6) = -(2,3,5)(1,4,6) + \cdots$$

where the $\cdots$ represent special elements.

Therefore the previous 2 relations multiplied by 2 are of the form

$$\begin{gathered} -(3,4,5)(1,2,6) - (2,3,5)(1,4,6) + 2(2,3,4)(1,5,6) + \cdots \\ (3,4,5)(1,2,6) - (2,3,5)(1,4,6) - 2(3,4,5)(1,2,6) + \cdots \end{gathered} \tag{9}$$

Subtracting the second from the first one has the desired Formula:

$$0 = 2(2,3,4)(1,5,6) + \cdots$$

a relation with a single permutation $2(2,3,4)(1,5,6)$ of type $(3,3)$ and the remaining elements special.

This gives the desired expression, which is explicited in the Appendix.

2 Comments

2.1 The Formula $\dim \Sigma_n = C_n$

The n^{th} Catalan number is $\frac{1}{n+1}\binom{2n}{n}$. By the *hook Formula* it is easily seen that this is the dimension of the irreducible representation of S_{2n} associated to a Young diagram with two rows of length n. This in turn appears as the isotypic component of the $SL(2)$ invariants in $V^{\otimes 2n}$, $\dim V = 2$.

By identifying V with V^* as $SL(2)$ representations we have an $SL(2)$ linear isomorphism between $End(V^{\otimes n}) = V^{\otimes n} \otimes V^{*\otimes n}$ and $V^{\otimes 2n}$ which induces a linear isomorphism with the respective invariants

$$\Sigma_n = End_{SL(V)} V^{\otimes n} \simeq (V^{\otimes 2n})^{SL(V)}.$$

2.2 Several bases

Several bases formed by special elements can be obtained from Theorem 1, so a main problem is to describe a *best one* by combinatorial means.

The main advantage of the special elements is that their eigenvalues are ± 1 and the two primitive 3-roots of 1. They also are *local* in the sense that involve only 2 tensor factors at a time, and at most a single instance of 3 tensor factors.

One should compare the complexity of the algorithm to express a permutation as linear combination of special ones to that of the algorithm to express a permutation as linear combination of 3-good ones.

Some remarks on the algorithm to express a permutation as linear combination of 3-good ones.

(1) Permutations are ordered lexicographically.

(2) The 3-good permutations on n elements are in number C_n the n^{th} Catalan number.

(3) The last 3-good permutation is $n, 1, 2, 3, \cdots, n-1$.

The algorithm takes a permutation σ and checks recursively if there is a string of 3 elements decreasing. If there is not one the permutation is 3-good. Otherwise as soon as one encounters one such sequence, by applying the antisymmetrizer on these elements one obtains that σ is equivalent to a sum with signs of 5 permutations which are lexicographically less than σ. This means that if we have already developed the previous permutations as linear combination of 3-good ones we immediately obtain the developments for σ. Notice that in this development the coefficients are all integers.

2.3 $d \geq 3$

One may ask the same question for $\Sigma_n(d)$ and $d \geq 3$. The first problem is:

Determine the minimum $m = m(d)$ so that $\Sigma_{m+1}(d)$ is spanned by the permutations which are NOT $m+1$–cycles.

This number m has also other interesting interpretations (see [1] page 331 for the interesting history of this question).

The same m is the maximum degree of the generators of invariants of $d \times d$ matrices.

It is also the minimum degree for which, given an associative algebra R over a field of characteristic 0, in which every element x satisfies $x^d = 0$ one has $R^{m(d)} = 0$.

The known estimates for $m(d)$ are the lower bound $m(d) \geq \binom{d+1}{2}$ due to Kuzmin, see [3] or [2] and the upper bound $m(d) \leq d^2$ due to Razmyslov, see [10] or [1]. Kuzmin conjectures that $m(d) = \binom{d+1}{2}$ which has been verified only for $d \leq 4$.

2.4 Transpositions

Let us remark a simple fact

Proposition 3. *The identity plus all transpositions give linearly independent operators in all $\Sigma_n(d)$.*

Proof. Of course it is enough to prove this when $d = 2$, we do it by induction. Assume we have a relation

$$0 = a \cdot I_n + \sum_{i<j} a_{i,j}(i,j)$$

with I_n the identity on $V^{\otimes n}$. Apply the partial trace $\mathtt{t}_1 = \mathtt{t}$ on the first factor as in [9]

$$\mathtt{t} : X_1 \otimes X_2 \otimes \ldots \otimes X_{n-1} \otimes X_n \mapsto tr(X_1)X_2 \otimes \ldots \otimes X_{n-1} \otimes X_n.$$

As proved in that paper if S_{n-1} is the subgroup of S_n fixing 1 we have

$$\mathtt{t}(\sigma) = 2\sigma, \ \forall \sigma \in S_{n-1} \quad \mathtt{t}(\tau(1,i)) = \tau, \ \forall \tau \in S_{n-1}$$

So we have in $V^{\otimes n-1}$.

$$0 = b \cdot I_{n-1} + 2 \sum_{1<i<j} a_{i,j}(i,j), \quad b = 2a + \sum_{1<j} a_{1,j}.$$

By induction we have $b = 0,\ a_{i,j} = 0,\ \forall 1 < i < j$.

So the relation is among I_n and the $n-1$ transpositions $(1, i),\ i = 2, n$. Apply the partial trace $\mathtt{t}_n$ on the last factor obtaining in $V^{\otimes n-1}$ that $a_{1,j} = 0,\ \forall j < n$ and finally a relation among I_n and $(1, n)$ which does not exist.

QED

3 Appendix A *explicit formulas*

In the formula for $p = 5$ we multiply by 2

$$4(1,2,3,4,5) \overset{(7)}{=} 4(1,5)(1,2,3,4) =$$
$$2(1,4,5)(2,3) + 2(1,2,5)(3,4) - 2(1,3,5)(4,2) - 2(1,4,2,5) + 2(1,3,4,5)$$
$$+ 2(3,4,2)(1,5) + 2(1,2,3,5) + 2(2,4)(1,5) - 2(3,4)(1,5) - 2(2,3)(1,5)$$

and develop the 4–cycles we obtain

$$-2(1,4,2,5) + 2(1,3,4,5) + 2(1,2,3,5) =$$
$$(1,2)(3,5) + (1,2)(4,5) - (1,3)(2,5) + (1,3)(4,5) - (1,4)(2,5) - (1,4)(3,5) - (1,5)(2,4) + (1,5)(3,4)$$
$$+(1,5)(2,3) + (2,4,5) + (1,2,4) + (1,3,5) + (3,4,5) + (1,3,4) + (1,3,5) + (2,3,5) + (1,2,3)$$
$$-(2,3) - 2(4,5) - (3,4) - 2(1,2) - (2,4) - (1,3) - (3,5) - (1,5) + 3.$$

So the formula

$$4(1,2,3,4,5) \overset{(7)}{=} 4(1,5)(1,2,3,4) = \tag{10}$$
$$(1,2)(3,5) + (1,2)(4,5) - (1,3)(2,5) + (1,3)(4,5) - (1,4)(2,5) - (1,4)(3,5) + (1,5)(2,4) - (1,5)(3,4)$$
$$-(2,3)(1,5) + (2,4,5) + (1,2,4) + (1,3,5) + (3,4,5) + (1,3,4) + (1,3,5) + (2,3,5) + (1,2,3)$$
$$-(2,3) - 2(4,5) - (3,4) - 2(1,2) - (2,4) - (1,3) - (3,5) - (1,5) + 3$$
$$+2(1,4,5)(2,3) + 2(1,2,5)(3,4) - 2(1,3,5)(4,2) + 2(2,3,4)(1,5)$$

Remark 5. *If we conjugate the previous Formula with the 5 powers of the cycle* $(1,2,3,4,5)$ *we obtain 5 different formulas for the same element, thus we get 4 relations as predicted in page 42.*

Notice that instead Formula (5) *for the 4–cycle, is invariant under these conjugations and there are no relations.*

The formula (8) for $p = 6$

$$8(1,2,3,4,5,6) \overset{(7)}{=} 8(1,5,6)(1,2,3,4) = \tag{11}$$

$$4(1,4,5,6)(2,3){+}4(1,2,5,6)(3,4){-}4(1,3,5,6)(4,2){-}4(1,4,2,5,6){+}4(1,3,4,5,6)$$

$$+4(3,4,2)(1,5,6){+}4(1,2,3,5,6){+}4(2,4)(1,5,6){-}4(3,4)(1,5,6){-}4(2,3)(1,5,6).$$

So expanding the terms containing 4 and 5-cycles: we finally have

$$8(1,2,3,4,5,6) \overset{(7)}{=} 8(1,5,6)(1,2,3,4) = \tag{12}$$

$5-2(1,2)-3(1,3)+(1,6)+(2,3)-2(1,4)(2,3)-3(1,6)(2,3)-3(2,4)+2(1,3)(2,4)-(1,3)(2,6)-$
$(1,4)(2,6)+(3,4)-2(1,2)(3,4)-3(1,6)(3,4)-2(2,5)(3,4)+2(1,6)(2,5)(3,4)-2(1,5)(2,6)(3,4)-$
$2(3,5)+2(2,4)(3,5)-2(1,6)(2,4)(3,5)-(3,6)+(1,2)(3,6)-(1,4)(3,6)-2(1,5)(3,6)+2(1,5)(2,4)(3,6)+$
$(1,6)(4,2)-4(1,6)(4,5)-2(2,3)(4,5)+2(1,6)(2,3)(4,5)-2(4,6)+(1,2)(4,6)+(1,3)(4,6)-$
$2(1,5)(2,3)(4,6)-2(5,6)+2(1,3)(5,6)-2(2,3)(5,6)+2(1,4)(2,3)(5,6)+2(2,4)(5,6)-2(1,3)(2,4)(5,6)-$
$2(3,4)(5,6)+2(1,2)(3,4)(5,6)+(1,2,3)+(1,2,4)+2(3,4)(1,2,5)-2(1,2,6)+2(3,4)(1,2,6)+$
$2(3,5)(1,2,6)+2(4,5)(1,2,6)+(1,3,4)+2(1,3,5)-2(2,4)(1,3,5)+2(1,3,6)-2(2,4)(1,3,6)-$
$2(2,5)(1,3,6)+2(4,5)(1,3,6)+2(2,3)(1,4,5)+2(1,4,6)+2(2,3)(1,4,6)-2(2,5)(1,4,6)-2(3,5)(1,4,6)+$
$2(1,6)(2,3,5)+(2,3,6)+2(1,6)(2,4,5)+(2,4,6)+2(3,4)(2,5,6)+2(1,6)(3,4,5)+(3,4,6)+2(3,5,6)-$
$2(2,4)(3,5,6)+2(2,3)(4,5,6)+4(1,5,6)(2,3,4)$

Hence the final formula for the permutation of type 3,3 in term of special elements obtained by the method explained in section 1.2 is:

$$8(4,3,2)(5,6,1) = \tag{13}$$

$(1,2)-2(1,3)+(1,5)+4(1,6)-3(2,3)+2(1,5)(2,3)-2(1,6)(2,3)-3(2,4)+2(1,5)(2,4)-2(1,6)(2,4)+$
$7(2,5)-2(1,3)(2,5)-4(1,4)(2,5)-6(1,6)(2,5)-2(2,6)-2(1,3)(2,6)+4(1,4)(2,6)+2(1,5)(2,6)+$
$(3,4)-2(1,5)(3,4)-2(1,6)(3,4)-4(2,5)(3,4)-4(3,5)+4(1,4)(3,5)+4(2,4)(3,5)+4(2,6)(3,5)-$
$4(1,4)(2,6)(3,5)-4(3,6)+4(1,2)(3,6)-4(1,4)(3,6)-4(2,5)(3,6)+4(1,4)(2,5)(3,6)-2(4,5)+$
$2(1,2)(4,5)+2(1,3)(4,5)+4(2,3)(4,5)+4(1,3)(2,6)(4,5)+4(3,6)(4,5)-4(1,2)(3,6)(4,5)-2(1,2)(4,6)+$
$2(1,3)(4,6)-4(1,3)(2,5)(4,6)+4(1,2)(3,5)(4,6)+6(5,6)-4(1,2)(5,6)-4(4,5)(1,2,3)+4(5,6)(1,2,3)+$
$(1,2,5)-4(3,6)(1,2,5)-2(1,2,6)+4(4,5)(1,2,6)+2(1,3,6)+4(2,5)(1,3,6)-4(4,5)(1,3,6)-$
$4(3,5)(1,4,2)+4(5,6)(1,4,2)+4(2,5)(1,4,3)-4(5,6)(1,4,3)-4(2,6)(1,4,5)+4(3,6)(1,4,5)-(1,5,2)-$
$4(1,5,6)+8(3,4)(1,5,6)+2(1,6,3)-2(1,6,5)+(2,3,5)+2(2,3,6)-4(4,5)(2,3,6)+4(1,6)(2,4,3)-$
$(2,4,5)+4(1,6)(2,4,5)-(2,5,3)+4(1,6)(2,5,3)+(2,5,4)-2(2,5,6)+2(1,4)(2,5,6)+2(2,6,4)-$
$4(3,5)(2,6,4)-2(1,4)(2,6,5)+(3,4,5)-4(1,6)(3,4,5)+2(3,4,6)-(3,5,4)+2(1,2)(3,5,6)-$
$2(1,4)(3,5,6)+4(2,5)(3,6,4)-2(1,2)(3,6,5)+2(1,4)(3,6,5)-2(4,5,6)-2(4,6,5)$

3.1 The corresponding rules of substitution:

In writing an algorithm to transform any given permutation into a sum of special elements one thus uses the following *substitutional rules*. From:

$$2(a,b,c,d) = \tag{14}$$

$$(a,d)(b,c)+(a,b)(c,d)-(a,c)(d,b)-(a,d,b)+(a,c,d)$$
$$+(b,c,d)+(a,b,c)+(b,d)-(c,d)-(b,c).$$

and:

$$(a,B)(a,A)=(a,A,B), \quad e.g.\ (1,2,3)(1,5,4,6)=(1,5,4,6,2,3) \tag{15}$$

for a cycle $\mathsf{c} := (a,b,c,d,A)$ of length $n+4$, where A is of length $n>0$, we have

$$\mathsf{c} := (a,b,c,d,A) = (a,A)(a,b,c,d) = \tag{16}$$

$$(a,d,A)(b,c)+(a,b,A)(c,d)-(a,c,A)(d,b)-(a,d,b,A)$$
$$+(a,c,d,A)+(b,c,d)+(a,b,c,A)+(b,d)-(c,d)-(b,c).$$

This formula now contains only cycles of length $< n+4$.

By applying recursively these rules we arrive to a linear combination in which no cycles of length > 3 appear. Now the only reduction to be made is if there are pairs of 3-cycles. For these we finally repeat the final rule:

$$8(d,c,b)(e,f,a) = \tag{17}$$

$(a,b)-2(a,c)+(a,e)+4(a,f)-3(b,c)+2(a,e)(b,c)-2(a,f)(b,c)-3(b,d)+2(a,e)(b,d)-2(a,f)(b,d)+7(b,e)-2(a,c)(b,e)-4(a,d)(b,e)-6(a,f)(b,e)-2(b,f)-2(a,c)(b,f)+4(a,d)(b,f)+2(a,e)(b,f)+(c,d)-2(a,e)(c,d)-2(a,f)(c,d)-4(b,e)(c,d)-4(c,e)+4(a,d)(c,e)+4(b,d)(c,e)+4(b,f)(c,e)-4(a,d)(b,f)(c,e)-4(c,f)+4(a,b)(c,f)-4(a,d)(c,f)-4(b,e)(c,f)+4(a,d)(b,e)(c,f)-2(d,e)+2(a,b)(d,e)+2(a,c)(d,e)+4(b,c)(d,e)+4(a,c)(b,f)(d,e)+4(c,f)(d,e)-4(a,b)(c,f)(d,e)-2(a,b)(d,f)+2(a,c)(d,f)-4(a,c)(b,e)(d,f)+4(a,b)(c,e)(d,f)+6(e,f)-4(a,b)(e,f)-4(d,e)(a,b,c)+4(e,f)(a,b,c)+(a,b,e)-4(c,f)(a,b,e)-2(a,b,f)+4(d,e)(a,b,f)+2(a,c,f)+4(b,e)(a,c,f)-4(d,e)(a,c,f)-4(c,e)(a,d,b)+4(e,f)(a,d,b)+4(b,e)(a,d,c)-4(e,f)(a,d,c)-4(b,f)(a,d,e)+4(c,f)(a,d,e)-(a,e,b)-4(a,e,f)+8(c,d)(a,e,f)+2(a,f,c)-2(a,f,e)+(b,c,e)+2(b,c,f)-4(d,e)(b,c,f)+4(a,f)(b,d,c)-(b,d,e)+4(a,f)(b,d,e)-(b,e,c)+4(a,f)(b,e,c)+(b,e,d)-2(b,e,f)+2(a,d)(b,e,f)+2(b,f,d)-4(c,e)(b,f,d)-2(a,d)(b,f,e)+(c,d,e)-4(a,f)(c,d,e)+2(c,d,f)-(c,e,d)+2(a,b)(c,e,f)-2(a,d)(c,e,f)+4(b,e)(c,f,d)-2(a,b)(c,f,e)+2(a,d)(c,f,e)-2(d,e,f)-2(d,f,e)$

until we arrive at a linear combination of special elements.

4 Appendix B

4.1 The center

Given a partition $\lambda \vdash n$ denote by C_λ the sum of all permutations with cycles type λ.

In $\mathbb{C}[S_n]$ these elements form a basis of its center but in Σ_n they are linearly dependent, so it is interesting to extract from this list a basis for the center. This is done in the next Proposition 4.

In fact both Σ_{2n} and Σ_{2n+1} decompose into $n+1$ matrix blocks corresponding to the partitions of n resp. $n+1$ with at most 2–rows of lengths $h+k, h \mid 2h+k = n$.

Therefore the center of Σ_n has dimension $n+1$ with basis the central idempotents, unity of each block.

The dual of such a partition is of the form $2^h 1^k$ with $2h+k=n$, resp $2h+k=n+1$.

Proposition 4. *The $n+1$ elements $C_\lambda,\ \lambda = 2^h 1^k$ form a basis for the center of Σ_n.*

Proof. By our formulas it is clear that the various $C_\lambda,\ \lambda \vdash n$ are linear combinations of those C_μ such that the dual $\hat{\mu}$ is the cycle partition of a special element. Now for C_3 the $2\binom{n}{3}$ 3 cycles are naturally grouped into $\binom{n}{3}$ pairs $(a,b,c),(a,c,b)$ and to their sum we apply Formula (1) so that

$$C_3 = \frac{3\binom{n}{3}}{\binom{n}{2}} C_2 - \binom{n}{3} C_1 = (n-2)C_2 - \binom{n}{3} C_1, \quad \frac{3\cdot 2\cdot (n-2)!}{3!\cdot (n-3)!} = n-2.$$

Similarly

$$C_{3,2^k,1^h} = aC_{2^{k+1},1^{h+1}} - bC_{2^k,1^{h+3}} = (k+1)(h+1)C_{2^{k+1},1^{h+1}} - \binom{h+3}{3} C_{2^k,1^{h+3}}.$$

This proves that the $n+1$ elements $C_\lambda,\ \lambda = 2^h 1^k$ span the center of Σ_n but, since the center has dimension $n+1$ they are a basis.

The two coefficients a, b can be computed as follows. In general if we write a partition as $\lambda := 1^{h_1} 2^{h_2} 3^{h_3} \cdots k^{h_k}$, with h_i the number of parts of length i, we have that the number of permutations with cycles of length given by λ is $n!/z_\lambda,\ z_\lambda = \prod i^{h_i} \cdot h_i!$.

In order to compute a, b notice that in $S_n,\ n = 2k+h$ the number of permutations of type $2^k, 1^h$ is $\frac{n!}{2^k\cdot k!\cdot h!}$, those of type $3, 2^k, 1^h$ is $\frac{n!}{3\cdot 2^k\cdot k!\cdot h!}$.

Each of the $\binom{n}{3}$ pairs of 3 cycles is multiplied by $\frac{(n-3)!}{2^k\cdot k!\cdot h!}$ permutations of cycles type $2^k, 1^h$, getting a total sum of $3\binom{n}{3}\frac{(n-3)!}{2^k\cdot k!\cdot h!}$ of permutations of cycles type $2^{k+1}, 1^{h+1}$ and minus $\binom{n}{3}\frac{(n-3)!}{2^k\cdot k!\cdot h!}$ of permutations of cycles type $2^k, 1^{h+3}$. So

$$a = \frac{3\binom{n}{3}\frac{(n-3)!}{2^k\cdot k!\cdot h!}}{\frac{n!}{2^{k+1}\cdot (k+1)!\cdot (h+1)!}}, \quad b = \frac{\binom{n}{3}\frac{(n-3)!}{2^k\cdot k!\cdot h!}}{\frac{n!}{2^k\cdot k!\cdot (h+3)!}}$$

$$a = \frac{3\frac{n!}{6(n-3)!}\frac{(n-3)!}{2^k\cdot k!\cdot h!}}{\frac{n!}{2(k+1)(h+1)2^k\cdot(k)!\cdot h!}} = (k+1)(h+1),\ 2k+h = n-3.$$

$$b = \frac{\binom{n}{3}\frac{(n-3)!}{2^k\cdot k!\cdot h!}}{\frac{n!}{2^k\cdot k!\cdot(h+3)!}} = \binom{h+3}{3}$$

QED

Of course there are several combinatorial questions one can ask, about the multiplication table of the previous elements or how to write them in terms of the central idempotents, finally how to write a general C_λ as linear combination of these central elements. Some of these questions can be answered as developments of symmetric functions in 2 variables.

Let us discuss how to compute the coefficients for a general expansion

$$\lambda \vdash n, \quad C_\lambda = \sum_{k=0}^{[\frac{n}{2}]} a_{\lambda,k} C_{2^k,1^{n-2k}}.$$

Given $\lambda := 1^{h_1}2^{h_2}3^{h_3}\cdots k^{h_k} \vdash m,\ \mu := (k+1)^p \vdash n$ define

$$\lambda + \mu := 1^{h_1}2^{h_2}3^{h_3}\cdots k^{h_k}(k+1)^{h_{k+1}} \vdash m+n, \quad h_{k+1} = p. \tag{18}$$

We have

$$z_{\lambda+\mu} = z_\lambda z_\mu.$$

Each permutation π of $[1, m+n]$ of type $\lambda+\mu$ decomposes this set as $[1, m+n] = A \cup B,\ |A| = m,\ |B| = n$ and π restricted to A is of type λ while on B of type μ. With obvious notations we may write

$$C_{\lambda+\mu:} = \sum_{A\subset[1,m+n]=A\cup B,\ |A|=m} C_\lambda(A)C_\mu(B)$$

$$= \sum_A \Big(\sum_{i=0}^{[\frac{m}{2}]} a_{\lambda,i} C_{2^i,1^{m-2k}}(A)\Big)\Big(\sum_{j=0}^{[\frac{n}{2}]} a_{\mu,j} C_{2^j,1^{n-2j}}(B)\Big). \tag{19}$$

In this last sum the permutations of type $2^\ell 1^{m+n-2\ell}$ arise from the contributions $a_{\lambda,i}C_{2^i,1^{m-2i}}(A)a_{\mu,j}C_{2^j,1^{m-2j}}(B)$ where $i+j=\ell$. For each of these products we have

$$a_{\lambda,i} m!/z_{2^i,1^{m-2i}} a_{\mu,j} n!/z_{2^j,1^{n-2j}}$$

The choice of k is $i \leq \min([\frac{m}{2}], \ell)$. The total is

$$B_{\lambda+\mu,\ell} := \sum_{0 \leq i \leq \min([\frac{m}{2}],\ell)} a_{\lambda,i} m!/z_{2^i,1^{m-2i}} a_{\mu,(\ell-i)} n!/z_{2^{(\ell-i)},1^{n-2(\ell-i)}}$$

thus from Formula (19)

$$a_{\lambda+\mu,\ell}(m+n)!/z_{\lambda+\mu} = \binom{m+n}{m} B_{\lambda+\mu,\ell}.$$

$$a_{\lambda+\mu} = z_\lambda \sum_{0 \leq i \leq \min([\frac{m}{2}],\ell)} a_{\lambda,i} a_{\mu,(\ell-i)} \frac{z_\mu}{z_{2^i,1^{m-2i}} z_{2^{(\ell-i)},1^{n-2(\ell-i)}}}.$$

$$\frac{z_\mu}{z_{2^i,1^{m-2i}} z_{2^{(\ell-i)},1^{n-2(\ell-i)}}} = \frac{(k+1)^p p!}{2^i i!(m-2i)! 2^{(\ell-i)}(\ell-i)!(n-2(\ell-i))!}.$$

Next take a permutation of cycle type $p^\ell \vdash p \cdot \ell$. Consider first the decompositions of $[1, p \cdot \ell]$ into ℓ subsets $A_1, \cdots, A_\ell$ each of length p. For such a decomposition we have a contribution to C_{p^ℓ} of

$$C_p(A_1)C_p(A_2)\cdots C_p(A_\ell) = \prod \sum_{k=0}^{[\frac{p}{2}]} a_{p,k} C_{2^k,1^{n-2k}}$$

continuing as before one obtains an explicit polynomial in the elements $a_{p,k}$ with rational coefficients giving an expression for C_{p^ℓ}. So the last task is to give an explicit formula for the elements $a_{p,k}$.

This is done recursively starting from Formula (16)

$$\mathsf{c}_p := (1,2,3,4,A) = (1,A)(1,2,3,4) =$$

$$(1,4,A)(2,3) + (1,2,A)(3,4) - (1,3,A)(4,2) - (1,4,2,A)$$
$$+(1,3,4,A) + (2,3,4) + (1,2,3,A) + (2,4) - (3,4) - (2,3) + 3C_3.$$

We have $\sum_{\sigma \in S_p} \sigma \mathsf{c}_p \sigma^{-1} = p \cdot C_p$ the same sum applied to the terms on the right hand side gives

$$p \cdot C_p = 2(p-2)C_{2,p-2} + (p-1)C_{1,p-1} + 3(p-3)!C_{1^{p-3},3} - 2(p-2)!C_{1^{p-2},2}$$

This is the basis for a recursion.

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