An isomorphism between projective models of toric and hyperplane graphic arrangements

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Abstract. This paper presents a bridge between the theories of wonderful models associated with toric arrangements and wonderful models associated with hyperplane arrangements. In a previous work, the same authors noticed that the model of the toric arrangement of type A_{n-1} associated with the minimal building set is isomorphic to the one of the hyperplane arrangement of type A_n associated again with the minimal building set; it is natural to ask if there exist similar isomorphisms between other families of arrangements. The aim of this paper is to study one such family, namely the family of arrangements defined by graphs. The main result states that there is indeed an isomorphism between the model of the toric arrangement defined by a graph Γ and the model of the hyperplane arrangement defined by the cone of Γ, provided that a suitable building set is chosen.

1 Introduction

In this paper we highlight a bridge between wonderful models of toric arrangements and wonderful models of subspace arrangements. In particular we point out some cases where the wonderful model of a toric arrangement is isomorphic to the wonderful model of an associated hyperplane arrangement. In these cases both the involved arrangements are graphic arrangements.

1.1   Historical sketches

A *subspace arrangement* is a collection of subspaces in a vector space V. Analogously, a *toric arrangement* is a collection of subtori in an algebraic torus T. In both cases, in this paper we will assume that the base field is $\mathbb{C}$. We say that an arrangement is divisorial if all its elements have codimension 1. A *wonderful model* for an arrangement is a variety whose big open dense set is isomorphic to the complement of the arrangement and whose boundary is a divisor with normal crossings. The construction of a model associated with an arrangement depends on the choice of a *building set*, i.e. a particular subset of the lattice of intersections of the arrangement that satisfies some combinatorial properties. More precise definitions will be provided in Section 2.

The construction of wonderful models of subspace arrangements was first described in the seminal papers [12; 13]. The initial motivation was the study of Drinfeld's construction in [18] of special solutions of the Knizhnik-Zamolodchikov equations with some prescribed asymptotic behavior, but it was soon pointed out that the models are geometric objects with their own great interest. For instance in the case of a complexified root arrangement of type A_n the minimal model coincides with the moduli spaces of stable curves of genus 0 with $n+2$ marked points.

In [13] De Concini and Procesi showed, using a description of the cohomology rings of the projective wonderful models to give an explicit presentation of a Morgan algebra, that the mixed Hodge numbers and the rational homotopy type of the complement of a complex subspace arrangement depend only on the intersection lattice (viewed as a ranked poset). The cohomology rings of the models of complex subspace arrangements were also studied in [24; 36] and, in the real case, in [19; 33]. The case of arrangements associated with complex reflection groups was studied from different point of views in [27] and in [6].

The connections between the geometry of these models and the Chow rings of matroids were pointed out first in [22] and then in [1], where they also played a crucial role in the study of some relevant log-concavity problems. The relations with toric and tropical geometry were enlightened for instance in [2; 16; 21].

The study of toric arrangements started in [29]. In [15] and [14] the role of toric arrangements as a link between partition functions and box splines is pointed out. In [7], it was shown, extending the results in [4; 5] and [31], that in the divisorial case the data needed in order to state the presentation of the rational cohomology ring of the complement $\mathcal{M}(\mathcal{A})$ of a toric arrangement $\mathcal{A}$ is fully encoded in the poset given by all the connected components of the intersections of the layers. It follows that in the divisorial case the combinatorics of this poset determines the rational homotopy of $\mathcal{M}(\mathcal{A})$.

One of the motivations for the construction of *projective wonderful models of*

a toric arrangement $\mathcal{A}$ in [9], in addition to the interest in their own geometry, was that they could be an important tool to explore the generalization of the above mentioned results to the non-divisorial case.

Indeed the presentation of the cohomology ring of these models described in [10] was used in [30] to construct a Morgan differential algebra which determines the rational homotopy type of $\mathcal{M}(\mathcal{A})$. We notice that these models, and therefore their associated Morgan algebras, depend not only on the initial combinatorial data, but also on some choices. In [11] a new differential graded algebra was constructed as a direct limit of the above mentioned differential Morgan algebras: it has a presentation which depends only on a set of initial discrete data extracted from $\mathcal{A}$, and it can be used to prove that in the non-divisorial case the rational homotopy type of $\mathcal{M}(\mathcal{A})$ depends only on these data.

1.2   A bridge between the two families of models

It is well-known that a root system defines both a hyperplane arrangement and a toric arrangement, so it is natural to study the projective wonderful models associated with them. In [25], the authors noticed that the model of the toric arrangement of type A_{n-1} associated with the minimal building set is isomorphic to the one of the hyperplane arrangement of type A_n associated with the minimal building set. In this work we extend this result by studying a wider family of arrangements, namely the graphic arrangements (see Definitions 6 and 7). In particular, given a graph Γ we consider the toric arrangement defined by Γ and the hyperplane arrangement defined by the cone of Γ (see Definition 5). It turns out that, by choosing two convenient building sets, the wonderful models associated with these two arrangements are isomorphic. The proof that we present relies on two key observations:

(1) both their complements can be realised as the complement of the same projective hyperplane arrangement in a suitable projective space;

(2) the constructions of both the toric and the hyperplane models can be translated as the construction of the model associated with this projective hyperplane arrangement, with the only difference being the order of the blowups of the elements of the building set.

1.3   Structure of the paper

The paper is structured as follows: in Section 2 we recall the definitions and constructions related to projective models associated with hyperplane and toric arrangements; in Section 3 we introduce the graphic arrangements and study the construction of the models associated with the toric and hyperplane

graphic arrangements, focusing our attention on the choice of the building set; Section 4 is devoted to our main result, namely the proof of the isomorphism between the model of the toric arrangement associated with a graph Γ and the one of the hyperplane arrangement associated with the cone of Γ; Section 5 concludes the paper by showing how our main result can be applied to particular families of graphic arrangements to uncover interesting equalities between known combinatorial objects.

Remark 1. We will use the two superscripts $(\ )^T$ and $(\ )^H$ to denote the objects relative to the fields of toric and hyperplane arrangements respectively.

2   Compact models of hyperplane and toric arrangements

Both hyperplane/subspace arrangements and toric ones can be seen as special cases of *arrangements of subvarieties*. Following [28], in this section we recall the main definitions and constructions of projective wonderful models in this more general setting, and afterwards we translate them in the context of the hyperplane and toric arrangements.

Definition 1 (see [28, Definition 2.1]). Let X be a non-singular algebraic variety. An *arrangement of subvarieties* of X is a finite set Λ of non-singular closed connected subvarieties properly contained in X such that

(1) for every two $\Lambda_i, \Lambda_j \in \Lambda$, either $\Lambda_i \cap \Lambda_j$ is a disjoint union of elements of Λ or $\Lambda_i \cap \Lambda_j = \emptyset$;

(2) if $\Lambda_i \cap \Lambda_j \neq \emptyset$, the intersection is *clean*, i.e. it is non-singular and for every $y \in \Lambda_i \cap \Lambda_j$ we have the following conditions on the tangent spaces:

$$\mathrm{T}_y(\Lambda_i \cap \Lambda_j) = \mathrm{T}_y(\Lambda_i) \cap \mathrm{T}_y(\Lambda_j).$$

We denote by $\mathcal{M}(\Lambda)$ the complement of the arrangement Λ in X, i.e.

$$\mathcal{M}(\Lambda) \coloneqq X \setminus \bigcup_{\Lambda_i \in \Lambda} \Lambda_i.$$

If for every two $\Lambda_i, \Lambda_j \in \Lambda$ the intersection $\Lambda_i \cap \Lambda_j$ is either empty or connected, the arrangement is called *simple*. For the rest of this paper we will only deal with simple arrangements, therefore from now on this hypothesis will always be implicitly assumed.

The key ingredient for the construction of the model is the notion of building set, of which we give two definitions (see [28, Definition 2.2]).

Definition 2. Let Λ be an arrangement of subvarieties. A subset $\mathcal{G} \subseteq \Lambda$ is a *building set* for Λ if for every $L \in \Lambda \backslash \mathcal{G}$ the minimal elements (w.r.t. the inclusion) of the set $\{G \in \mathcal{G} \mid L \subset G\}$ intersect transversally and their intersection is L. These minimal elements are called the $\mathcal{G}$-*factors* of L.

Definition 3. Let $\mathcal{G}$ be a set of connected, closed, non-singular subvarieties of a variety X and let $\Lambda(\mathcal{G})$ be the set of all the connected components of all the possible non-empty intersections of elements of $\mathcal{G}$ (i.e. the arrangement of subvarieties *induced* by $\mathcal{G}$). We say that $\mathcal{G}$ has the property of being *building* if it is building for $\Lambda(\mathcal{G})$ according to Definition 2.

A similar definition of building set can be given in a purely combinatorial setting, namely in the context of meet-semilattices (see [20]).

Remark 2. Definition 3 remarks that being building is an intrinsic property of the combinatorics of a set of subvarieties. In fact, while the induced arrangement is unique for a chosen set of subvarieties $\mathcal{G}$, there may exist different building sets for a chosen arrangement Λ—however, if $\mathcal{G}$ is building for Λ, then $\mathcal{G}$ is also building according to Definition 3 since in this case $\Lambda(\mathcal{G}) = \Lambda$.

Finally, starting from a non-singular variety X, an arrangement of subvarieties Λ and a building set $\mathcal{G}$ for Λ, consider the locally closed embedding

$$\mathcal{M}(\Lambda) \longrightarrow \prod_{G \in \mathcal{G}} \mathrm{Bl}_G X \tag{1}$$

where $\mathrm{Bl}_G X$ is the blowup of X along G.

Definition 4 (see [28, Definition 1.1]). The closure of the image of the morphism (1) is the *wonderful model* associated with X, Λ and $\mathcal{G}$ and it is denoted by $\mathcal{Y}(X, \Lambda, \mathcal{G})$. If Λ is the arrangement induced by $\mathcal{G}$ we will simply write $\mathcal{Y}(X, \mathcal{G})$.

Theorem 1 (see [28, Theorem 1.3]). *Let $\mathcal{G}$ be a building set in an algebraic variety X. Let us order the elements $G_1, \ldots, G_m$ of $\mathcal{G}$ in such a way that for every $1 \le k \le m$ the set $\mathcal{G}_k := \{G_1, \ldots, G_k\}$ is building. Then if we set $X_0 := X$ and $X_k := \mathcal{Y}(X, \mathcal{G}_k)$ for $1 \le k \le m$, we have*

$$X_k = \mathrm{Bl}_{\widetilde{G_k}} X_{k-1},$$

where $\widetilde{G_k}$ denotes the dominant transform of G_k in X_{k-1}. In particular $\mathcal{Y}(X, \mathcal{G})$ is obtained for $k = m$.

Remark 3. (1) Any total ordering of the elements of a building set $\mathcal{G} = \{G_1, \ldots, G_m\}$ which refines the ordering by inclusion, that is $i < j$ if $G_i \subset G_j$, satisfies the condition of Theorem 1.

(2) It follows from the preceding constructions that $\mathcal{M}(\Lambda)$ is the open dense set of $\mathcal{Y}(X, \Lambda, \mathcal{G})$ and that the boundary of $\mathcal{Y}(X, \Lambda, \mathcal{G})$ is the union of the non-singular irreducible divisors D_G provided by the transforms of every $G \in \mathcal{G}$. The intersection of any subset of these divisors is non-empty if and only if the corresponding subset of $\mathcal{G}$ is $\mathcal{G}$-nested (see Definition 9). If this intersection is non-empty, then it is transversal. (See also [28, Theorem 1.2].)

2.1 Models for hyperplane arrangements

Let $V = \mathbb{C}^n$ as a vector space. An arrangement of subspaces $\mathcal{A}$ is a finite family of linear subspaces in V. We point out that $\mathcal{A}$ is central, i.e. 0 belongs to every subspace of $\mathcal{A}$, and we can assume that $\mathcal{A}$ is essential, i.e. $\bigcap_{H \in \mathcal{A}} H = \{0\}$.

The poset of intersections of $\mathcal{A}$ is the set of all the possible intersections of the elements of $\mathcal{A}$, including the whole space V obtained as the empty intersection, partially ordered by reverse inclusion. It is denoted by $L(\mathcal{A})$ and it is a lattice (because $\mathcal{A}$ is central).

Since $\mathcal{A}$ is central, its projectivization $\overline{\mathcal{A}}$ is well-defined as an arrangement of projective subspaces in $\mathbb{P}(V) = \mathbb{P}^{n-1}$. Given a building set $\mathcal{G}$ for $L(\mathcal{A})$, a projective wonderful model associated with $\mathcal{A}$ is the closure of the image of the locally closed embedding

$$\mathcal{M}(\overline{\mathcal{A}}) \longrightarrow \mathbb{P}(V) \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$$

(see [13]) and we denote this model as $\mathcal{Y}^H(\mathcal{A}, \mathcal{G})$. This construction is equivalent to the one in Definition 4 by noticing that $L(\overline{\mathcal{A}})$ forms an arrangement of subvarieties in $\mathbb{P}^{n-1}$ according to Definition 1.

2.2 Models for toric arrangements

Let $T = (\mathbb{C}^*)^n$ be an algebraic complex torus and let $X^*(T)$ be its group of characters. For $\chi \in X^*(T)$, let $x_\chi \colon T \to \mathbb{C}^*$ be the corresponding character on T. A *layer* in T is a subvariety of T of the form

$$\mathcal{K}(\Gamma, \phi) \coloneqq \{t \in T \mid x_\chi(t) = \phi(\chi) \text{ for all } \chi \in \Gamma\} \tag{2}$$

where $\Gamma < X^*(T)$ is a split direct summand and $\phi \colon \Gamma \to \mathbb{C}^*$ is a homomorphism. A toric arrangement $\mathcal{A}$ is a (finite) set of layers $\{\mathcal{K}_1, \dots, \mathcal{K}_r\}$ in T.

The poset of layers of $\mathcal{A}$ is the set of all the connected components of the possible intersections of the elements of $\mathcal{A}$, including the whole torus T obtained as the empty intersection, partially ordered by reverse inclusion. It is denoted by $\mathcal{C}(\mathcal{A})$.

Following [9], in order to construct a projective wonderful model associated with $\mathcal{A}$, first we embed the torus T in a suitable toric variety X_Δ with associated fan Δ. In particular, a toric variety X_Δ is good for $\mathcal{A}$ if every layer of $\mathcal{C}(\mathcal{A})$ has an equal sign basis with respect to the fan Δ.

Let $\mathcal{C}(\overline{\mathcal{A}})$ be the set of the closures of the layers of $\mathcal{C}(\mathcal{A})$ in X_Δ; it can be proven that this set forms an arrangement of subvarieties in X_Δ according to Definition 1; therefore, given a building set $\mathcal{G}$ for $\mathcal{C}(\overline{\mathcal{A}})$, we can define a projective wonderful model associated with $\mathcal{A}$ according to Definition 4. We denote this model as $\mathcal{Y}^T(\mathcal{A}, \mathcal{G})$, where the dependence on X_Δ is left implicit.

3 Graphic arrangements and their models

In this section we present the hyperplane and toric arrangements associated with some graphs and introduce the ingredients to build the projective wonderful models that we are going to study in Section 4.

We identify a graph with the pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is its set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is its set of edges. We assume all graph to be simple, undirected and without loops. Moreover we will consider only graphs with at least two vertices.

A key concept that will be widely used in this work is that of the cone of a graph. For convenience we report here the definition.

Definition 5. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph with $\mathcal{V} = [n] \coloneqq \{1, \ldots, n\}$. The *cone* of Γ, denoted with $\widehat{\Gamma}$, is a graph with vertices $\widehat{\mathcal{V}} = [n] \cup \{0\}$ and edges

$$\widehat{\mathcal{E}} = \mathcal{E} \cup \{(0, j) \mid j \in [n]\}.$$

In the following definitions, let $v = (1, \ldots, 1)$ of suitable length.

Definition 6. Identify $\mathbb{C}^{n-1}$ with $\mathbb{C}^n/(\mathbb{C}v)$. Given a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = [n]$, we define the associated hyperplane arrangement $\mathcal{A}^H(\Gamma)$ in $\mathbb{C}^{n-1}$ as the set of hyperplanes $\{H_{ij}\}_{(i,j)\in\mathcal{E}}$ where $H_{ij} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j\}/(\mathbb{C}v)$.

Definition 7. Identify $(\mathbb{C}^*)^{n-1}$ with $(\mathbb{C}^*)^n/(\mathbb{C}^*v)$. Given a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = [n]$, we define the associated toric arrangement $\mathcal{A}^T(\Gamma)$ in $(\mathbb{C}^*)^{n-1}$ as the set of layers $\{\mathcal{K}_{ij}\}_{(i,j)\in\mathcal{E}}$ where $\mathcal{K}_{ij} = \{(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \mid t_i t_j^{-1} = 1\}/(\mathbb{C}^*v)$.

Notice that $\mathcal{K}_{ij}$ is a layer according to (2)—in fact, in this case we have that $\mathcal{K}_{ij} = \mathcal{K}(\Gamma_{ij}, \phi_{ij})$, where

- Γ_{ij} is the subgroup of $X^*((\mathbb{C}^*)^{n-1}) = \mathbb{Z}^{n-1} = \mathbb{Z}^n/(\mathbb{Z}v)$ generated by $e_{ij}/(\mathbb{Z}v)$, where $e_{ij} \in \mathbb{Z}^n$ is the vector that has 1 in the i-th coordinate, -1 in the j-th coordinate, and 0 elsewhere;

- ϕ_{ij} is the trivial homomorphism that maps every element of Γ_{ij} to $1 \in \mathbb{C}^*$.

An arrangement that arises from a graph as in Definition 6 or Definition 7 is also called *graphic arrangement*.

Example 1. The (hyperplane or toric) arrangement of type A_{n-1} is a graphic arrangement associated with K_n, i.e. the complete graph on n vertices. Indeed, any graphic arrangement can be viewed as a subarrangement of it.

For the remainder of the section, let Γ be a graph on n vertices. Notice that by construction the poset of intersections $L(\mathcal{A}^H(\Gamma))$ and the poset of layers $\mathcal{C}(\mathcal{A}^T(\Gamma))$ are isomorphic and, in fact, we can provide a further characterization. Consider the poset of set partitions of $[n]$ ordered by refinement and denote it by Π_n.

Proposition 1. *Given a subset $I \subseteq [n]$, let $\Gamma(I)$ be the subgraph of Γ induced by I. Let $\Pi_n(\Gamma) \subseteq \Pi_n$ be the subposet such that $\pi = \{\pi_1, \ldots, \pi_k\} \in \Pi_n(\Gamma)$ iff $\Gamma(\pi_\ell)$ is connected for each $\ell = 1, \ldots, k$. The poset $L(\mathcal{A}^H(\Gamma))$ (and consequently $\mathcal{C}(\mathcal{A}^T(\Gamma))$) is isomorphic to $\Pi_n(\Gamma)$.*

Proof. Let $\pi = \{\pi_1, \ldots, \pi_k\} \in \Pi_n(\Gamma)$. We define $H_\pi \in L(\mathcal{A}^H(\Gamma))$ as

$$H_\pi = \bigcap_{\ell=1}^{k} \left(\bigcap_{e_{ij} \in \mathcal{E}(\Gamma(\pi_\ell))} H_{ij} \right) \tag{3}$$

where $\mathcal{E}(\Gamma(\pi_\ell))$ is the set of edges of $\Gamma(\pi_\ell)$. Since $\pi \in \Pi_n(\Gamma)$, this definition is equivalent to

$$H_\pi = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_i = x_j \text{ if exists } \ell \text{ s.t. } \{i, j\} \subseteq \pi_\ell\}/(\mathbb{C}v).$$

On the other hand, let $H \in L(\mathcal{A}^H(\Gamma))$. It is described by a set of equations

$$\begin{cases} x_{i_1} = \cdots = x_{i_s}, \\ \cdots \\ x_{j_1} = \cdots = x_{j_t}, \end{cases}$$

where the indices in each row belong to disjoint subsets of $[n]$. Notice that by definition two indices can belong to the same row of equations only if there is a path in Γ connecting the corresponding vertices. Define the partition

$$\pi_H = \{\{i_1, \ldots, i_s\}, \ldots, \{j_1, \ldots, j_t\}\}$$

(eventually completed with singletons corresponding to the variables not appearing in the equations) which, by the previous remark, belongs to $\Pi_n(\Gamma)$.

Now it is an exercise to prove that the two maps

$$\begin{array}{ccc} L(\mathcal{A}^H(\Gamma)) & \longrightarrow & \Pi_n(\Gamma) \\ H & \longmapsto & \pi_H \end{array} \quad \text{and} \quad \begin{array}{ccc} \Pi_n(\Gamma) & \longrightarrow & L(\mathcal{A}^H(\Gamma)) \\ \pi & \longmapsto & H_\pi \end{array}$$

are inverse of each other. $\boxed{QED}$

Remark 4. We can compute the codimension of any $H \in L(\mathcal{A}^H(\Gamma))$ from π_H:

$$\mathrm{codim}(H) = \sum_{B \in \pi_H} \left(\#(B) - 1 \right).$$

We now study some particular projective wonderful models associated with graphic arrangements, starting with the choice of the building set.

Definition 8. Let $\mathcal{G}$ be the subset of either $\mathcal{C}(\mathcal{A}^T(\Gamma))$ or $L(\mathcal{A}^H(\Gamma))$, depending on the context, whose elements are the subtori/subspaces G such that the corresponding partition π_G has exactly one non-singleton block. In this case, if there is no ambiguity, we will use the symbol π_G also to denote the only non-singleton block of the partition.

Proposition 2. $\mathcal{G}$ *is a building set.*

Proof. Let $H \notin \mathcal{G}$ and let $B_1, \ldots, B_\ell$, $\ell \geq 2$, be the non-singleton blocks of the associated partition π_H. We claim that the minimal elements (w.r.t. the inclusion of subspaces) of $\mathcal{G}_H := \{G \in \mathcal{G} \mid H \subset G\}$ are $G_1, \ldots, G_\ell$, where G_i is such that $\pi_{G_i} = B_i$ for $i = 1, \ldots, \ell$. This is clear from the fact that at partition level we are looking for the coarsest partitions with only one non-singleton block that are finer than π_H.

From this it follows easily that $H = G_1 \cap \cdots \cap G_\ell$ and this intersection is transversal. $\boxed{QED}$

Remark 5. As shown in [25], in the case of $\Gamma = K_n$ this building set corresponds to the minimal one, i.e. the building set of irreducibles. However, this is not true in general. For example, suppose that the graph Γ contains the edges $(1, 2)$ and $(2, 3)$ but not $(1, 3)$: in this case the element $G \in \mathcal{G}$ whose block is $\pi_G = \{1, 2, 3\}$ is not irreducible as a subspace.

When we consider the cone $\widehat{\Gamma}$, the corresponding building set $\widehat{\mathcal{G}}$, defined as in Definition 8, admits a useful description in terms of the building set $\mathcal{G}$ associated with Γ.

Proposition 3. *Let*

$$\iota: \quad \begin{aligned} \Pi_n(\Gamma) &\longrightarrow \Pi_{n+1}(\widehat{\Gamma}) \\ \{\pi_1 \ldots, \pi_k\} &\longmapsto \{\{0\}, \pi_1, \ldots, \pi_k\} \end{aligned}$$

where we consider the elements of $\Pi_{n+1}(\widehat{\Gamma})$ as partitions of $\{0, 1, \ldots, n\}$. Then each element of $\widehat{\mathcal{G}} \subseteq \Pi_{n+1}(\widehat{\Gamma})$ is associated with a partition of one of the following forms:

(1) $\iota(G)$ for some $G \in \mathcal{G}$;

(2) $\{\{0\} \cup S, \{i_1\}, \dots, \{i_k\}\}$ for some $S \in \mathcal{P}([n]) \setminus \{\emptyset\}$ where $[n] \setminus S = \{i_1, \dots, i_k\}$.

Proof. In fact, the partitions of type 1. and 2. above are exactly the ones with a single non-singleton block that induces a connected subgraph of $\widehat{\Gamma}$. $\boxed{QED}$

We now have all the ingredients to introduce projective wonderful models associated with a graphic hyperplane/toric arrangement.

Let us begin with the toric case; as we have seen in Section 2, the starting point is a good toric variety. As noticed in [9], for any graphic toric arrangement whose graph has n vertices there is a canonical choice of such a variety: the one associated with the fan Δ induced by the Weyl chambers of the root system of type A_{n-1}. In the end, following the construction recalled in Section 2, we obtain the projective wonderful model $\mathcal{Y}^T(\Gamma, \mathcal{G}) := \mathcal{Y}^T(\mathcal{A}^T(\Gamma), \mathcal{G})$.

In the hyperplane setting we do not consider the arrangement $\mathcal{A}^H(\Gamma)$ and instead we study $\mathcal{A}^H(\widehat{\Gamma})$, the arrangement associated with the cone $\widehat{\Gamma}$, and the building set $\widehat{\mathcal{G}}$ defined above, obtaining the model $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}}) := \mathcal{Y}^H(\mathcal{A}^H(\widehat{\Gamma}), \widehat{\mathcal{G}})$. In the next section we investigate the relationship between $\mathcal{Y}^T(\Gamma, \mathcal{G})$ and $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}})$.

4 The isomorphism between the models

Let Γ be a graph on n vertices and let $\mathcal{G}$ be the building set defined in Definition 8. As spoiled by the title of this section, we want to prove the following result:

Theorem 2. *The two models $\mathcal{Y}^T(\Gamma, \mathcal{G})$ and $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}})$ are isomorphic.*

Proof. The first step of the proof consists in noticing that the complement $\mathcal{M}(\mathcal{A}^T(\Gamma))$ can be realized as the complement of a subspace arrangement in a suitable projective space.

As in the beginning of Section 3, let $v = (1, \dots, 1)$ and let $\mathcal{E} \subseteq [n] \times [n]$ be the set of edges of Γ. We identify $(\mathbb{C}^*)^{n-1} = (\mathbb{C}^*)^n / (\mathbb{C}^* v)$ with

$$\mathbb{P}(\mathbb{C}^n) \setminus \bigcup_{i=1}^{n} \{t_i = 0\}$$

where $t_1, \dots, t_n$ are the projective coordinates on $\mathbb{P}(\mathbb{C}^n)$. The map is given by

$$\begin{aligned} (\mathbb{C}^*)^n / (\mathbb{C}^* v) &\longrightarrow \mathbb{P}(\mathbb{C}^n) \setminus \bigcup_{i=1}^{n} \{t_i = 0\} \\ [(t_1, \dots, t_n)] &\longmapsto [t_1, \dots, t_n] \end{aligned}$$

where on the left the notation with square brackets denotes the class of $(t_1, \ldots, t_n)$ in the quotient. Under this identification, the complement $\mathcal{M}(\mathcal{A}^T(\Gamma))$ is the same as the complement of an arrangement $\mathcal{A}'$ in $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ whose projective hyperplanes are

(i) $\{t_i = 0\}$, for $i = 1, \ldots, n$;

(ii) $\{t_i = t_j\}$, for $(i, j) \in \mathcal{E}$.

For convenience, we recall again the construction of the model $\mathcal{Y}^T(\Gamma, \mathcal{G})$ provided by [9]:

(1) first, we embed the torus $(\mathbb{C}^*)^n / (\mathbb{C}^* v)$ in the toric variety X_Δ associated with the Coxeter fan Δ of type A_{n-1};

(2) then, we blow up all the closures of the layers of $\mathcal{G}$ in an order that refines inclusion.

As a consequence of the identification of $\mathcal{A}^T(\Gamma)$ with a projective arrangement in $\mathbb{P}^{n-1}$, the construction just outlined is equivalent to the following:

(1) first, we obtain from $\mathbb{P}^{n-1}$ the toric variety X_Δ by blowing up the intersections of the subspaces of $\mathcal{A}'$ of type (i) in any order that refines inclusion (see for example [3, Section 2], [32, Section 3]);

(2) then we blow up the (proper transforms of the) elements of $\mathcal{G}$, obtained as intersections of the subspaces of type (ii), in any order that refines inclusion.

Now we focus on the construction of the model $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}})$ for the hyperplane arrangement $\mathcal{A}^H(\widehat{\Gamma})$. At first we build its projectivization in an opportune way. Recall that $\mathcal{A}^H(\widehat{\Gamma})$ is an arrangement in an n-dimensional space: if we choose to view it in the space

$$N := \{(0, x_1, \ldots, x_n) \mid x_i \in \mathbb{C}\} \subseteq \mathbb{C}^{n+1},$$

then its projectivization in $\mathbb{P}(N)$, with coordinates $[x_1, \ldots, x_n]$ omitting the leading zero, has the following projective hyperplanes:

• $\{x_i = 0\}$, for $i = 1, \ldots, n$;

• $\{x_i = x_j\}$, for $(i, j) \in \mathcal{E}$.

In other words, we obtain the same projective arrangement $\mathcal{A}'$ as the toric case.

Let us characterise the elements of $\widehat{\mathcal{G}}$ in this setting. Let π be a partition in $\Pi_{n+1}(\widehat{\Gamma})$ with only one non-singleton block. If 0 does *not* belong to this block, we will denote the corresponding subspace in $\widehat{\mathcal{G}}$ by H_π; otherwise we will denote the subspace by M_π. Notice that these two cases correspond to the subspaces of type 1. and type 2. respectively in Proposition 3.

With this notation, we have that the elements of type M_π correspond to the intersections of the subspaces of type (i) in the toric setting, and the elements of type H_π correspond to the elements of $\mathcal{G}$. With this in mind, we can conclude that the toric and hyperplane models are isomorphic if we can build the model $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}})$ by blowing up in $\mathbb{P}(N) \simeq \mathbb{P}^{n-1}$ the subspaces of $\widehat{\mathcal{G}}$ in the same order as the one described in the construction of the toric model $\mathcal{Y}^T(\Gamma, \mathcal{G})$, i.e. if we can reorder the elements of $\widehat{\mathcal{G}} = \{G_1, \ldots, G_m\}$ in the following way:

(1) first, we put all the subspaces of type M_π in any order that refines inclusion (say that these are $G_1, \ldots, G_{t_0}$);

(2) then, we put all the subspaces of type H_π in any order that refines inclusion (say that these are $G_{t_0+1}, \ldots, G_m$); notice that by construction the segment $\widehat{\mathcal{G}}_{t_0+1}^t := \{G_{t_0+1}, \ldots, G_t\}$ is building for every $t = t_0 + 1, \ldots, m$.

According to [28, Theorem 1.3], this is an admissible ordering of the elements of $\widehat{\mathcal{G}}$ if the initial segment $\widehat{\mathcal{G}}_t = \{G_1, \ldots, G_t\}$ is building for every $t = 1, \ldots, m$. Let us prove this, distinguishing two cases.

$\boxed{\text{Case 1.}}$ If all the subspaces in $\widehat{\mathcal{G}}_t$ are of type M_π, then any intersection of elements of $\widehat{\mathcal{G}}_t$ is still of type M_π and belongs to $\widehat{\mathcal{G}}_t$ since the ordering refines the inclusion. This implies that $\widehat{\mathcal{G}}_t$ is building.

$\boxed{\text{Case 2.}}$ If at least one subspace in $\widehat{\mathcal{G}}_t$ is of type H_π (notice that in this case all the subspaces of type M_π are in $\widehat{\mathcal{G}}_t$), then let L be the intersection of some elements of $\widehat{\mathcal{G}}_t$; after having removed the non-minimal subspaces, it has the form

$$L = L_1 \cap \cdots \cap L_k \tag{4}$$

where $k \geq 1$, L_1 is either of type M_{π_1} or of type H_{π_1}, all the other L_i's are of type H_{π_i} and the non-singleton blocks of $\pi_1, \ldots, \pi_k$ are pairwise disjoint. We notice that all the L_i's belong to $\widehat{\mathcal{G}}_t$:

- we already know that it is true for the subspaces of type M_π;

- for the subspaces of type H_π, this is true because each L_i of type H_{π_i} appears as a $\widehat{\mathcal{G}}_{t_0+1}^t$-factor of the intersection of some elements of $\widehat{\mathcal{G}}_{t_0+1}^t$, and $\widehat{\mathcal{G}}_{t_0+1}^t$ is building.

Moreover the partition corresponding to L is given by the disjoint union of the non-singleton blocks of $\pi_1, \ldots, \pi_k$ completed with the remaining singletons, so (4) is a transversal intersection. This ends the proof that $\widehat{\mathcal{G}}_t$ is building.

In conclusion, we have proven that the constructions in the toric case and in the hyperplane case are the same, therefore Theorem 1 can be applied and this shows that, up to isomorphism, $\mathcal{Y}^T(\Gamma, \mathcal{G})$ and $\mathcal{Y}^H(\widehat{\Gamma}, \widehat{\mathcal{G}})$ are the same. $\boxed{QED}$

5 Applications

In this section we present two examples of noteworthy graphic arrangements and the models associated with them. Both the examples lead to a geometric proof of an equality involving Eulerian polynomials.

5.1 A known example: The complete graph

Let K_n be the complete graph on n vertices. As already mentioned, both in the hyperplane case and in the toric one the graphic arrangement associated with it is the arrangement of Coxeter type A_{n-1}, which is well-known (see for example [24; 36] for the hyperplane case and [17; 32; 34; 35] for the toric one).

In this case it is trivial to notice that $\widehat{K}_n = K_{n+1}$, so a straightforward application of Theorem 2 gives the following corollary, which has already been announced (without a proper proof) in [25, Section 5].

Corollary 1. *The model $\mathcal{Y}^T(K_n, \mathcal{G})$ associated with the toric arrangement of type A_{n-1} is isomorphic to the model $\mathcal{Y}^H(K_{n+1}, \widehat{\mathcal{G}})$ associated with the hyperplane arrangement of type A_n.*

5.2 Disjoint union of pairs of complete graphs

Let $K_{n,m}$ be the graph obtained as the disjoint union of K_n and K_m, namely the graph on $n+m$ vertices $\{1, \ldots, n+m\}$ such that (i, j) is an edge if and only if either $i, j \leq n$ or $i, j > n$ (see Figure 1 for an example).

By Theorem 2, the toric model associated with $K_{n,m}$ is isomorphic to the projective hyperplane model associated with $\widehat{K}_{n,m}$, and in particular

$$H^*(\mathcal{Y}^T(K_{n,m}, \mathcal{G}), \mathbb{Z}) \cong H^*(\mathcal{Y}^H(\widehat{K}_{n,m}, \widehat{\mathcal{G}}), \mathbb{Z}).$$

We will count the elements of two monomial bases of the two rings, obtained using the description of [24; 36] in the hyperplane case and [25] in the toric case. To do so, we briefly recall the main tools needed for these constructions.

Definition 9. Let Λ be a (simple) arrangement of subvarieties and let $\mathcal{G}$ be a building set for Λ. A subset $\mathcal{S} \subseteq \mathcal{G}$ is called ($\mathcal{G}$-)*nested* if for any set of pairwise
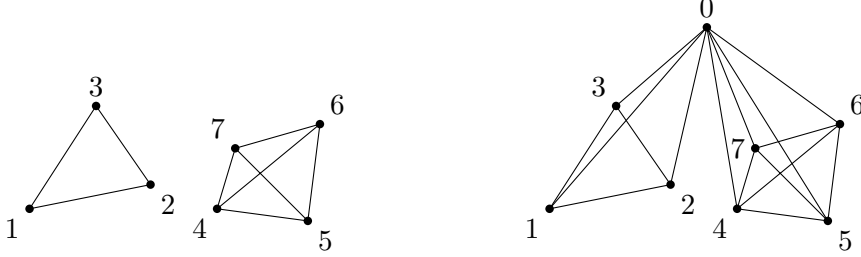
Figure 1. The graph $K_{3,4}$ and its cone $\widehat{K}_{3,4}$.

non-comparable (w.r.t. the inclusion) elements $\{A_1, \ldots, A_k\} \subseteq \mathcal{S}$, with $k \geq 2$, there is an element in Λ of which $A_1, \ldots, A_k$ are the $\mathcal{G}$-factors.

In the case of graphic arrangements, considering the building set $\mathcal{G}$ of Definition 8, the nested sets are $\{H_{\pi_1}, \ldots, H_{\pi_j}\}$ such that the non-singleton blocks of $\{\pi_1, \ldots, \pi_j\}$ are pairwise either comparable or disjoint.

Definition 10. A function $f \colon \mathcal{G} \to \mathbb{N}$ is $(\mathcal{G})$-*admissible* if it has both the following properties:

(1) supp f is $\mathcal{G}$-nested;

(2) for every $A \in \text{supp}\, f$ we have $f(A) < \dim M_f(A) - \dim A$, where $M_f(A)$ is the (connected) intersection of the elements of supp f that properly contain A.

Notice that the function such that $f(A) = 0$ for every $A \in \mathcal{G}$ is admissible.

For each element $G \in \mathcal{G}$, let T_G be a polynomial variable. The *admissible monomial* associated with an admissible function f is

$$m_f = \prod_{G \in \mathcal{G}} T_G^{f(G)}.$$

Theorem 3 ([24, Theorem 2.1]; see also [36]). *The set*

$$\{m_f \mid f \ \mathcal{G}\text{-admissible}\}$$

is a monomial basis for $H^*(\mathcal{Y}^H(\mathcal{A}, \mathcal{G}), \mathbb{Z})$.

In the toric case, let X_Δ be a good toric variety for $\mathcal{A}$ and let $\mathcal{R}$ be the set of its rays; moreover, for each ray r, let C_r be a polynomial variable.

Theorem 4 (see [25, Theorem 4.7]). *The set*

$$\{b\, m_f \mid f \ \mathcal{G}\text{-admissible}, \ b \in \mathcal{B}_f\}$$

is a monomial basis for $H^*(\mathcal{Y}^T(\mathcal{A},\mathcal{G}),\mathbb{Z})$, where $\mathcal{B}_f \subseteq \mathbb{Z}[C_r \mid r \in \mathcal{R}]$ is a monomial basis of $H^*(X_{\Delta(f)},\mathbb{Z})$ (here $\Delta(f)$ is a suitable subfan of Δ defined by supp f).

Remark 6. Recall that for graphic toric arrangements we can choose the fan $\Delta = \Delta_{n-1}$ induced by the Weyl chambers of the root system of type A_{n-1}; in this case, for any $\mathcal{G}$-admissible function f we have that $\Delta(f)$ is isomorphic to a fan of the form Δ_{k-1} for some $k \leq n$, i.e. it is again a fan associated with a root system of type A_{k-1}. In particular we know that their Poincaré polynomials are given by Eulerian polynomials.

Finally, to count the elements of the monomial bases we just need to recall from [25] the definitions of *admissible trees* and *admissible forests*.

Definition 11. An *admissible tree* on m leaves is a labeled directed rooted tree such that

- it has m leaves, each labeled with a distinct non-zero natural number;

- each non-leaf vertex v has $k_v \geq 3$ outgoing edges, and it is labeled with the symbol q^i where $i \in \{1, \ldots, k_v - 2\}$.

By convention, the graph with one vertex and no edges is an admissible tree on one leaf (actually the only one). The *degree* of an admissible tree is the sum of the exponents of the labels of the non-leaf vertices. Denote by $\lambda(q,t)$ the generating function of the admissible trees, i.e. the series whose coefficient of $q^i t^k/k!$ counts the number of admissible trees of degree i on k leaves (see [24; 36]).

Definition 12. An *admissible forest* on n leaves is the disjoint union of admissible trees such that the sets of labels of their leaves form a partition of $\{1, \ldots, n\}$. The *degree* of an admissible forest is the sum of the degrees of its connected components.

Let us consider the graphic toric arrangement associated with $K_{n,m}$ and let $\mathcal{G}$ be building set of Definition 8. Let $\mathcal{A}_{n-1}^T$ and $\mathcal{A}_{m-1}^T$ be the two toric graphic arrangements associated with K_n and K_m respectively; it is easy to show that every $\mathcal{G}$-nested set $\mathcal{S}$ can be uniquely written as the disjoint union of a $\mathcal{F}_{n-1}$-nested set $\mathcal{S}_1$ and a $\mathcal{F}_{m-1}$-nested set $\mathcal{S}_2$, where $\mathcal{F}_{n-1}$ and $\mathcal{F}_{m-1}$ are the "building sets of irreducible elements" for $\mathcal{A}_{n-1}^T$ and $\mathcal{A}_{m-1}^T$ (see [25, Section 5.1]), and on the other hand every such union gives rise to a $\mathcal{G}$-nested set. Moreover, there is a bijection Ψ between the set of $\mathcal{G}$-admissible functions and the set

$$\{(f_1, f_2) \mid f_1 \text{ is } \mathcal{F}_{n-1}\text{-admissible}, f_2 \text{ is } \mathcal{F}_{m-1}\text{-admissible}\}$$

given by $\Psi(f) = (f|_{\mathcal{S}_1}, f|_{\mathcal{S}_2})$ if supp $f = \mathcal{S}_1 \sqcup \mathcal{S}_2$ as above. In other words, there is a grade-preserving bijection between the set of $\mathcal{G}$-admissible functions and

the set of pairs (F_1, F_2) of admissible forests on n and m leaves respectively (where $\deg(F_1, F_2) := \deg F_1 + \deg F_2$). Finally, it is not hard to prove that, given a $\mathcal{G}$-admissible function f corresponding to the pair (F_1, F_2), the fan $\Delta(f)$ is of the form $\Delta_{\ell-1}$, where ℓ is the total number of connected components of F_1 and F_2, therefore there is a bijection between $\mathcal{B}_f$ and the permutations in S_ℓ that is grade-preserving provided that we choose any Eulerian statistic on S_ℓ as the degree of a permutation. In conclusion, a basis for the $\mathbb{Z}$-cohomology of $\mathcal{Y}^T(K_{n,m}, \mathcal{G})$ is in one-to-one correspondence with the set of triples (F_1, F_2, σ) where F_1 is an admissible forest on n leaves, F_2 is an admissible forest on m leaves, and σ is a permutation in S_ℓ as above. This proves that, if we define the (exponential) generating function of the family

$$\Phi^T(q, x, y) := \sum_{n,m \geq 1} \mathrm{Poin}(\mathcal{Y}^T(K_{n,m}, \mathcal{G}), q) \frac{x^n}{n!} \frac{y^m}{m!},$$

we have that

$$\Phi^T(q, x, y) = \sum_{\ell_1, \ell_2 \geq 1} \mathrm{Poin}(X_{\Delta_{\ell_1 + \ell_2 - 1}}, q) \frac{\lambda^{\ell_1}(q, x)}{\ell_1!} \frac{\lambda^{\ell_2}(q, y)}{\ell_2!}$$

$$= \sum_{\ell_1, \ell_2 \geq 1} \frac{A_{\ell_1 + \ell_2}(q)}{q} \frac{\lambda^{\ell_1}(q, x)}{\ell_1!} \frac{\lambda^{\ell_2}(q, y)}{\ell_2!}, \tag{5}$$

where $A_\ell(q)$ is the ℓ-th Eulerian polynomial, defined as

$$A_\ell(q) := \begin{cases} \sum_{k=1}^{\ell} A(\ell, k) q^k, & \ell \geq 1, \\ 1, & \ell = 0 \end{cases}$$

(see [8]; $A(\ell, k)$ is the number of permutations in S_ℓ with $k - 1$ descents, for $\ell \geq 1$ and $1 \leq k \leq \ell$).

Remark 7. In the previous formula, as well as from now on, q is a polynomial variable with degree 2.

We now write a series similar to (5) in the hyperplane scenario. Let $\widehat{K}_{n,m}$ be the cone of $K_{n,m}$, and let $\widehat{\mathcal{G}}$ the building set as in Definition 8. As usual, we identify an element $G \in \widehat{\mathcal{G}}$ with a subset of $\{0, 1, \ldots, n+m\}$, i.e. the single non-singleton block of the partition π_G. In this case the blocks relative to the elements of $\widehat{\mathcal{G}}$ can only be of one of the following types:

(1) a subset of $\{1, \ldots, n\}$;

(2) a subset of $\{n+1, \ldots, n+m\}$;

(3) any subset containing 0 of cardinality at least 2.

In particular, in any $\widehat{\mathcal{G}}$-nested set $\mathcal{S}$ the elements of type (3) form a linear chain with respect to the inclusion, and the set $\mathcal{S} \setminus \{\text{elements of } \mathcal{S} \text{ of type (3)}\}$ can be written as a disjoint union of a $\mathcal{F}_{n-1}$- and a $\mathcal{F}_{m-1}$-nested sets.

Once again, the concept of admissible forests can be used to describe the $\widehat{\mathcal{G}}$-admissible functions, albeit with a slight modification.

Definition 13. A *special admissible forest* of type (n, m) is an admissible forest on $n + m + 1$ leaves, numbered from 0 to $n + m$, such that

- its connected components not containing the leaf 0 are admissible trees with leaf labels either contained in $\{1, \ldots, n\}$ or contained in $\{n+1, \ldots, n+m\}$;

- the connected component containing the leaf 0 has the following property: all the connected components of the graph obtained by removing the leaf 0 and all the nodes that have 0 among their descendants are admissible trees with leaf labels either contained in $\{1, \ldots, n\}$ or contained in $\{n+1, \ldots, n+m\}$.

In fact, a special admissible forest defines a $\widehat{\mathcal{G}}$-admissible function f in the following way: each internal node v represents an element $G_v \in \widehat{\mathcal{G}}$ (in particular, π_{G_v} is the set of the labels of the leaves descending from v); the set $\mathcal{S} = \{G_v \mid v \text{ internal node}\}$ is a $\widehat{\mathcal{G}}$-nested set which is the support set of f; if q^i is the label associated with the node v then $f(G_v) = i$.

In conclusion, a basis for the $\mathbb{Z}$-cohomology of $\mathcal{Y}^H(\widehat{K}_{n,m}, \widehat{\mathcal{G}})$ is in grade-preserving one-to-one correspondence with the set of special admissible forests of type (n, m). Therefore we can study the (exponential) generating function

$$\Phi^H(q, x, y) := \sum_{n,m \geq 1} \mathrm{Poin}(\mathcal{Y}^H(\widehat{K}_{n,m}, \widehat{\mathcal{G}}), q) \frac{x^n}{n!} \frac{y^m}{m!},$$

which is equal to

$$\sum_{n,m \geq 1} \sum_F q^{\deg F} \frac{x^n}{n!} \frac{y^m}{m!}$$

where F varies in the set of special admissible forests of type (n, m).

Now notice that a special admissible forest of type (n, m) can be obtained from two (regular) admissible forests F_1 and F_2 on n and m leaves respectively (where we relabel the leaves of the second one with the numbers $n+1, \ldots, n+m$) by choosing which of their trees are "attached" to the leaf 0 and how. This information is given by a permutation in $S_{\ell_1 + \ell_2}$, where ℓ_1 and ℓ_2 are the number

of trees of F_1 and F_2. To show how, we recall some definitions taken mainly from [23; 26].

Given an ordered list of distinct numbers (not necessarily a permutation), say $\sigma = [\sigma_1, \ldots, \sigma_N]$, we denote by $\mathrm{inv}(\sigma)$ the set of inversions of σ:

$$\mathrm{inv}(\sigma) := \{(i,j) \mid 1 \leq i < j \leq N, \ \sigma_i > \sigma_j\}.$$

Definition 14. A *hook* is an ordered list of distinct non-zero natural numbers $\eta = [t_1, \ldots, t_h]$, with $h \geq 2$, such that $t_1 > t_2$ and $t_2 < t_3 < \cdots < t_h$ (this second condition applies only for $h \geq 3$).

Remark 8. (1) Given s numbers $1 \leq j_1 < \cdots < j_s \leq N$ and $i \in \{1, \ldots, s-1\}$ there is a unique way to sort $\{j_1, \ldots, j_s\}$ so that they form a hook with exactly i inversions, namely $[j_{i+1}, j_1, \ldots, j_i, j_{i+2}, \ldots, j_s]$.

(2) There exists a unique way to write a list of distinct numbers σ as a concatenation $\sigma = p\,\eta_1 \cdots \eta_k$ where each η_i is a hook and p is a list of increasing numbers. This is called the *hook factorization* of σ. For example, the hook factorization of $\sigma = [1, 2, 8, 4, 3, 7, 9, 6, 5] \in S_9$ is

$$[1, 2, 8][7, 3, 4, 9][6, 5].$$

Notice that it is possible to have $k = 0$, if σ is an increasing sequence; also it may happen that $p = \emptyset$ ($\sigma = [3, 1, 2]$ is an example with $k = 1$).

Definition 15. Let σ be a list of distinct numbers. The statistic *lec* is defined as

$$\mathrm{lec}(\sigma) = \sum_{i=1}^{k} \#\mathrm{inv}(\eta_i)$$

where $p\,\eta_1 \cdots \eta_k$ is the hook factorization of σ.

For example if $\sigma = [1, 2, 8, 4, 3, 7, 9, 6, 5] \in S_9$, we have $\mathrm{lec}(\sigma) = 2 + 1 = 3$.

Lemma 1. *There exists a one-to-one correspondence between the set of triples (F_1, F_2, σ) where*

- *F_1 is an admissible forest on n leaves and with ℓ_1 trees;*

- *F_2 is an admissible forest on m leaves and with ℓ_2 trees;*

- *σ is a permutation in $S_{\ell_1 + \ell_2}$;*

and the set of special admissible forests of type (n, m). This correspondence is grade-preserving provided that we define $\deg(F_1, F_2, \sigma) := \deg(F_1) + \deg(F_2) + \mathrm{lec}(\sigma)$.
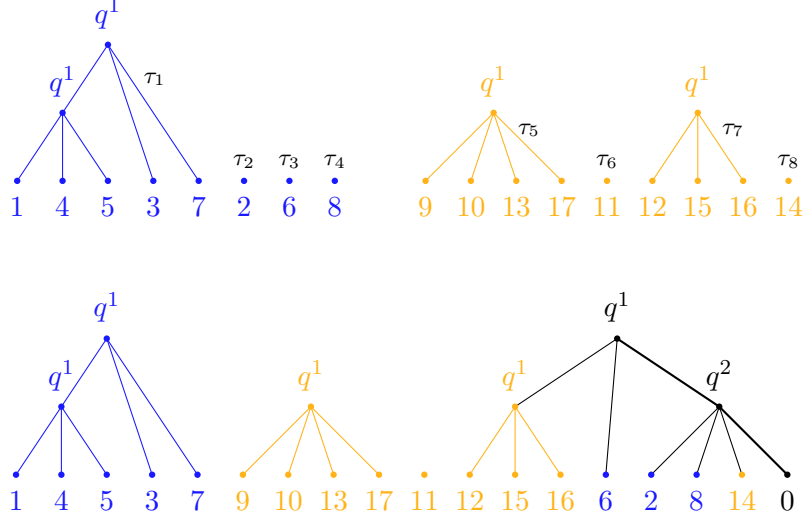
Figure 2. Top: two admissible forests on 8 and 9 leaves respectively, each with four connected components. Bottom: the special admissible forest of type $(8, 9)$ that can be obtained from the two forests above and the permutation $\sigma = [1, 5, 6, 7, 3, 8, 2, 4]$.

Proof. We only show the algorithm that associates a triple (F_1, F_2, σ) with a special admissible forest, which is similar to the one described in [25]. The reader can refer to Figure 2 to see an example.

Preliminary step. We define a total order on the set of trees of F_1 and F_2 in the following way: given two trees τ, τ', we say that $\tau < \tau'$ if

(1) τ belongs to F_1 and τ' belongs to F_2;

(2) if τ and τ' belong to the same forest, the minimum label of the leaves in τ is smaller than the minimum label of the leaves in τ'.

Let $\{\tau_1, \ldots, \tau_{\ell_1}\}$ and $\{\tau_{\ell_1+1}, \ldots, \tau_{\ell_1+\ell_2}\}$ be the trees of F_1 and F_2 ordered this way. Moreover, write σ as an ordered list $[\sigma(1), \ldots, \sigma(\ell_1 + \ell_2)]$ and let $p\, \eta_1 \cdots \eta_k$ be its hook factorization.

Step 1. Consider the last hook η_k and let $i_1 = \#\mathrm{inv}(\eta_k)$. Create a tree $\tau_0^{(1)}$ with a new internal vertex, labelled with q^{i_1}, to which the roots of the trees $\{\tau_j \mid j \in \eta_k\}$ are attached as well as the zero-labelled leaf.

Step 2. Consider the second-to-last hook η_{k-1} and let $i_2 = \#\mathrm{inv}(\eta_{k-1})$. Create a tree $\tau_0^{(2)}$ with a new internal vertex, labelled with q^{i_2}, to which the roots of the trees $\{\tau_j \mid j \in \eta_{k-1}\}$ are attached as well as the root of $\tau_0^{(1)}$.

Next steps. Continue with the other hooks, considering them from the last to the first. The part p of the hook factorization, if present, determines the trees that will not be attached to τ_0. $\boxed{QED}$

The key observation is that, if we define the bivariate exponential generating function of the lec statistic

$$\mathcal{L}(q, x, y) := \sum_{k_1, k_2 \geq 1} \left(\sum_{\sigma \in S_{k_1 + k_2}} q^{\mathrm{lec}(\sigma)} \right) \frac{x^{k_1}}{k_1!} \frac{y^{k_2}}{k_2!},$$

then the coefficient of $q^d x^{\ell_1} y^{\ell_2} / (\ell_1! \ell_2!)$ in $\mathcal{L}(q, \lambda(q, x), \lambda(q, y))$ counts exactly the triples (F_1, F_2, σ) as above and such that $\deg(F_1, F_2, \sigma) = d$. As a consequence, we have that

$$\Phi^H(q, x, y) = \sum_{\ell_1, \ell_2 \geq 1} \sum_{\sigma \in S_{\ell_1 + \ell_2}} q^{\mathrm{lec}(\sigma)} \frac{\lambda^{\ell_1}(q, x)}{\ell_1!} \frac{\lambda^{\ell_2}(q, y)}{\ell_2!}. \tag{6}$$

By Theorem 2 we know that the two series $\Phi^T(q, x, y)$ and $\Phi^H(q, x, y)$ are equal, and so are the two right-hand sides of (5) and (6). Now, $\lambda(q, t)$ is invertible with respect to the composition if viewed as a series in $\mathbb{Z}[q][[t]]$ (in fact $\lambda(q, t) = t + qt^3/3! + (q + q^2)t^4/4! + \cdots$). By composing by λ^{-1} twice both (5) and (6) we finally obtain

$$\frac{A_\ell(q)}{q} = \sum_{\sigma \in S_\ell} q^{\mathrm{lec}(\sigma)}.$$

In other words, the isomorphism between $\mathcal{Y}^T(K_{n,m}, \mathcal{G})$ and $\mathcal{Y}^H(\widehat{K}_{n,m}, \widehat{\mathcal{G}})$ provides yet another proof of the fact that the lec statistic is Eulerian.

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