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Almost starshaped CMC hypersurfaces in space forms are geodesic spheres

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Abstract. We prove that a constant mean curvature hypersurface of a simply connected real space form which is almost starshaped is a geodesic sphere.

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1 Introduction

Over the last decades, many characterizations of geodesic spheres have been obtained. The most famous one is the Alexandrov Theorem [2] which says that a closed embedded hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with constant mean curvature must be a geodesic sphere. The hypothesis, that the hypersurface is be embedded, cannot be removed as proved by many counter-examples constructed by Wente [22], Kapouleas [10], Hsiang [6] or Hsiang-Teng-Yu [7] for instance. Further, this result has been extended by Ros to scalar curvature [16] and then higher order mean curvatures [17] and more generally for any concave function of the principal curvatures by Korevaar [14]. In the same note, Korevaar also explain that the proof of Alexandrov is also valid in the hyperbolic space and the half-sphere. Another proof was given by Montiel and Ros [15]. Note that for higher order mean curvatures, the necessity of the embedding is still an open question.

Many other characterizations of geodesic spheres have been proved where the embeddedness has been replaced by another assumption in addition of the constancy of the mean curvature or more generally of one of the higher order mean curvature. For instance, Bivens proved in [3] that if two consecutive higher order mean curvatures are constant, then the hypersurface is a geodesic sphere. This have been extend by [11] to the constancy of the ratio of two consecutive higher order mean curvatures and finally by Koh and Lee to the ratio any two higher order mean curvatures [12, 13]. As a consequence, if two higher order mean curvatures are constant, then the hypersurface is a geodesic sphere. In [18] and [19], we were able to relax the hypothesis of the result of Bivens for the first two mean curvature. Precisely, if the mean curvature is constant and the second mean curvature H_2 is almost constant, the conclusion also holds. We first prove it in [18] for pointwise almost constancy of H_2 and then in [19] if H_2 is close to a constant for the L^p-norm. All these results hold in the three space forms.

We introduce the following notations before stating the main result of this note. $\mathbb{M}^{n+1}(\delta)$ denotes the simply connected $(n+1)$-dimensional space form of constant curvature δ, that is $\mathbb{M}^{n+1}(\delta)$ is the Euclidean space $\mathbb{R}^{n+1}$ if $\delta = 0$, the hyperbolic space $\mathbb{H}^{n+1}(\delta)$ if $\delta < 0$ and the upper half-sphere $\mathbb{S}^{n+1}_+(\delta)$ if $\delta > 0$. Let $r(\cdot) = d(p_0, \cdot)$ be the distance function to a base point p_0. We define the position vector Z as $Z = s_\delta(r)\overline{\nabla}r$, where $\overline{\nabla}r$ is the gradient of r in $\mathbb{M}^{n+1}(\delta)$ and the functions c_δ and s_δ are defined by

$$c_\delta(t) = \begin{cases} \cos(\sqrt{\delta}t) & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \\ \cosh(\sqrt{-\delta}t) & \text{if } \delta < 0 \end{cases} \quad \text{and} \quad s_\delta(t) = \begin{cases} \frac{1}{\sqrt{\delta}}\sin(\sqrt{\delta}t) & \text{if } \delta > 0 \\ t & \text{if } \delta = 0 \\ \frac{1}{\sqrt{-\delta}}\sinh(\sqrt{-\delta}t) & \text{if } \delta < 0. \end{cases}$$

In the present note, we are interested in the following characterization due to Hisung [8] for hypersurfaces of Euclidean spaces, but easily extended to spheres and hyperbolic spaces. Consider (M, g) a closed, connected and oriented Riemanniann manifold isometrically immersed into $\mathbb{M}^{n+1}(\delta)$. By the orientability of M, there exists on M a globally defined normal unit vector field ν. We assume that M is starshaped as a hypersurface, that is, the support function $\langle Z, \nu\rangle$ never vanishes on M, and so has a fixed sign. If, in addition, the mean curvature H is constant, then M is a geodesic sphere. Hsiung also showed the same result for any higher order mean curvature H_k, $1 \leqslant k \leqslant n$. Our goal is to relax the starshapedness assumption and obtain a new characterization of geodesic spheres with a weaker assumption that we will call *almost starshaped.* First, we will precise what we understand by almost starshaped. Let ρ_0 be a positive integer, we assume that there exists a smooth positive function ε such that at any point of M, we have

$$\langle Z, \nu\rangle \leqslant -\rho_0(1 - \varepsilon). \tag{1}$$

In the sequel, ε will be a function with small L_1-norm such that the hypersurface M is not necessarily starshaped since we allow the support function $\langle Z, \nu\rangle$ to be positive at some points but its positive part is small. Now, we can state the main result of this note.

Theorem 1. *Let $n \geqslant 2$ be an integer and ρ_0 a positive real number. Let us consider (M^n, g) a closed, connected and oriented Riemannian manifold of di-*

mension n isometrically immersed into $\mathbb{M}^{n+1}(\delta)$ with second fundamental form B and mean curvature H. If $\delta > 0$, we assume in addition that M is contained in a geodesic ball of radius $R < \frac{\pi}{2\sqrt{\delta}}$.
Let $h > 0$, then there exists a positive constant ε_0 depending on n, h, δ, $\|B\|_\infty$ and $\mathrm{Vol}(M)$, and also on R if $\delta > 0$, so that if M has constant mean curvature $H = h$ and is almost starshaped in the sense of (1) with $\|\varepsilon\|_1 \leqslant \varepsilon_0$, then M is a geodesic sphere.

2 Preliminaries

Let (M^n, g) be an n-dimensional closed, connected and oriented Riemannian manifold isometrically immersed into the $(n+1)$-dimensional simply connected real space form $\mathbb{M}^{n+1}(\delta)$ of constant curvature δ. For more convenience, we will denote in the sequel the metric g by $\langle \cdot, \cdot \rangle$. The second fundamental form $II : TM \times TM \longrightarrow NM$ of the immersion of M into $\mathbb{M}^{n+1}(\delta)$ is defined for any vector fields U and V tangent to M by

$$\overline{\nabla}_U V = \nabla_U V + II(U, V),$$

where ∇ and $\overline{\nabla}$ are the Riemannian connections on M and $\mathbb{M}^{n+1}(\delta)$ respectively. In other word, II is the normal part of $\overline{\nabla}$ over M. Since we consider an oriented hypersurface, there exists a globally defined normal unit vector field ν on M which allows us to view the second fundamental form as real-valued. Indeed, the real-valued second fundamental form B of the immersion is the bilinear symmetric form on TM defined for two vector fields U, V by

$$B(U, V) = \langle II(U, V), \nu \rangle = \langle \overline{\nabla}_U V, \nu \rangle = -\langle \overline{\nabla}_U \nu, V \rangle.$$

In the sequel, we will denote by B the second fundamental form. Moreover, we will denote by S the symmetric endomorphism associated with this the quadratic form B defined by $SU = -\overline{\nabla}_U \nu$. Note that S is called the shape operator or the Weingarten operator .

From B (or S), we can define the mean curvature, which is simply the normalized trace of S:

$$H = \frac{1}{n} \mathrm{tr}\,(S).$$

Now, we recall the Gauss formula. For $U, V, W, X \in \Gamma(TM)$,

$$R(U, V, W, X) = \overline{R}(U, V, W, X) + \langle SU, W \rangle \langle SV, X \rangle - \langle SV, W \rangle \langle SU, X \rangle \qquad (2)$$

where R and $\overline{R}$ are respectively the curvature tensor of M and $\mathbb{M}^{n+1}(\delta)$. By taking the trace on U and W and for $X = V$, we get

$$\mathrm{Ric}(V) = \overline{\mathrm{Ric}}(V) - \overline{R}(\nu, V, \nu, V) + nH \langle SV, V \rangle - \left\langle S^2 V, V \right\rangle. \qquad (3)$$

By taking the trace a second time and since $\mathbb{M}^{n+1}(\delta)$ has constant sectional curvature δ, we have

$$\mathrm{Scal} = n(n-1)\delta + n^2H^2 - \|S\|^2, \tag{4}$$

or equivalently

$$\mathrm{Scal} = n(n-1)\left(H^2+\delta\right) - \|\tau\|^2, \tag{5}$$

where $\tau = S - H\mathrm{Id}_{TM}$ is the traceless part of the shape operator, also called *umbilicity tensor*. If $\tau = 0$, M is said *totally umbilical*.

Now, we define the higher order mean curvatures, for $k \in \{1, \cdots, n\}$, by

$$H_k = \frac{1}{\binom{n}{k}}\sigma(S) = \frac{1}{\binom{n}{k}}\sigma_k(\kappa_1, \cdots, \kappa_n),$$

where σ_k is the k-th elementary symmetric polynomial and $\kappa_1, \cdots, \kappa_n$ are the eigenvalues of S, that is the principal curvatures of the immersion. By convention, we set $H_0 = 1$ and from the definition, it is obvious that H_1 is the mean curvature H. We also remark from the twice traced Gauss formula (5) that

$$H_2 = \frac{1}{n(n-1)}\mathrm{Scal} - \delta. \tag{6}$$

Hence, the equation (5) becomes $H^2 - H_2 = \dfrac{1}{n(n-1)}\|\tau\|^2$ and thus $H^2 \geqslant H_2$. We also recall the very useful Hsiung-Minkowski formula

$$\int_M \Big(H_{k+1}\langle Z, \nu\rangle + c_\delta(r)H_k\Big)dv_g = 0, \tag{7}$$

for any $k \in \{0, \cdots, n-1\}$ and where Z defined above is the position vector of M. In particular, we will use in the proof of the theorem the first two formulas, that is,

$$\int_M \Big(H\langle Z, \nu\rangle + c_\delta(r)\Big)dv_g = 0, \tag{8}$$

and

$$\int_M \Big(H_2\langle Z, \nu\rangle + c_\delta(r)H\Big)dv_g = 0. \tag{9}$$

We can see for instance [1] or [5] for a proof. Note also that often in the litterature, these formula are written $\displaystyle\int_M H_{k+1}\langle Z, \nu\rangle\, dv_g = \int_M c_\delta(r)H_k dv_g$, that is with a different sign. This is due to our sign convention for the second fundamental form, namely $B(U, V) = -g\left(\overline{\nabla}_U\nu, V\right)$ which implies that geodesic spheres have positive constant mean curvature with respect to the inner normal vector field. Note that there is no need that the hypersurface bounds a domain to get Hsiung-Minkowski formulas and so there is no canonical choice of the unit normal vector field. However, if the mean curvature is positive and the hypersurface is suppose to be starshaped, then the first Hsiung-Minkowski formula implies that the support function $\langle Z, \nu \rangle$ is negative. That's why we wrote the almost starshapedness assumption as (1).

We finish this preliminaries section by recalling the following result that we proved with Grosjean in [4]. It is very classical fact that totally umbilical closed hypersurfaces of space forms are geodesic spheres. In [4], we proved a result about the stability of this characterization, that if the hypersurface has a sufficiently small umbilicity tensor, then the hypersurface is close to a sphere. Namely, we proved

Theorem 2 (Grosjean-Roth [4]/ Hu-Xu [9])**.** *Let (M^n, g) be an n-dimensional closed, connected and oriented Riemannian manifold isometrically immersed into the $(n+1)$-dimensional simply connected real space form $\mathbb{M}^{n+1}(\delta)$. Let us assume also that M lies in a ball of radius $R < \frac{\pi}{2\sqrt{\delta}}$ if $\delta > 0$. Let $\varepsilon < 1$, $r, q > n$. Then there exist positive constants:*

- *C and ε_1 depending on n, q, δ, $\mathrm{Vol}(M)$, $\|B\|_q$ and $\|H\|_\infty$ and also on R if $\delta > 0$,*
- *α depending on n and q,*

such that if $\varepsilon \leqslant \varepsilon_1$ and

(i) $\|\tau\|_r \leqslant \|H\|_r \varepsilon$.

(ii) $\left\| H^2 - \|H\|_\infty^2 \right\|_{r/2} \leqslant \|H\|_r^2 \varepsilon$,

then M is diffeomorphic and $C\varepsilon^\alpha$-quasi-isometric to $S\left(p_0, s_\delta^{-1}\left(\frac{1}{\sqrt{\|H\|_\infty^2 + \delta^2}}\right)\right)$, where p_0 is the center of mass of M. Moreover, M is embbeded.

By $C\varepsilon^\alpha$-quasi-isometric, we mean that the diffeomorphism from M into the desired sphere given by the theorem satisfies

$$\left| |dF_x(u)|^2 - 1 \right| \leqslant C\varepsilon^\alpha$$

for any $x \in M$, $u \in T_x M$ and $|u| = 1$.

We also fix the notation for the integral norms. For $p \geqslant 1$ and f a continous function over M, we define $\|f\|_p$ by

$$\|f\|_p = \left(\frac{1}{\mathrm{Vol}(M)} \int_M |f|^p dv_g \right)^{\frac{1}{p}}.$$

Moreover, in the statement of the result, $\|\tau\|_r$ is the L^r-norm of $\|\tau\|$, where $\|\tau\|$ is the pointwise norm of the tensor τ, that is,

$$\|\tau\|_r = \left(\frac{1}{\mathrm{Vol}(M)}\int_M \|\tau\|^r dv_g\right)^{\frac{1}{r}}.$$

This results has been proved first by the author with Grosjean but with the condition that M lies into a ball of radius $R < \frac{\pi}{8\sqrt{\delta}}$. This result is based on a pinching result for the first eigenvalue of the Laplacian. Later, in [9], Hu and Xu improved this pinching result for the first eigenvalue of the Laplacian for hypersurfaces of spheres with the condition that the hypersurface lies into a ball of radius $R < \frac{\pi}{2\sqrt{\delta}}$ which automatically improved this almost umbilicity result. In the case where δ is positive, ε_1 depends on R. This comes from the improvement of Hu and Xu. More precisely, ε_1 goes to zero as R goes to $\frac{\pi}{2\sqrt{\delta}}$. In [20], the author with Scheuer were able to remove the second condition that the mean curvature is close to its maximum (or some L^p-norm more generally). We will use this result in Section 4 and will state it there. In the same paper [20], by a conformal change of metric, they were also able to obtain a proximity to sphere for almost umbilical hypersurfaces but only for Hausdorff distance and so without embeddedness *a priori*. Nevertheless, since were are interesting here with constant mean curvature hypersurfaces, Theorem 2 is enough for our use since the second hypothesis of is trivially satisfied.

The embeddedness of M in the conclusion of Theorem 2 comes from the fact that the diffeomorphism is explicitely constructed as the radial projection onto the sphere $S\left(p_0, s_\delta^{-1}\left(\frac{1}{\sqrt{\|H\|_\infty^2+\delta^2}}\right)\right)$. A control on the second fundamental form is needed to prove that this projection is a diffeomorphism. However, with a control only of the mean curvature, we can just obtain proximity for Hausdorff distance.

We also want to note that the constant ε_0 of Theorem 2 depends in of the quantities $\mathrm{Vol}(M)$, $\|B\|_q$ and $\|H\|_\infty$ in an scaling invariant way since the dependence is in fact in $\mathrm{Vol}(M)\|B\|_q^n$ and $\mathrm{Vol}(M)\|H\|_\infty^n$. For the use we need here, one can also remove the parameter q and get only a dependence on $\mathrm{Vol}(M)$ and $\|B\|_\infty$ instead of $\mathrm{Vol}(M)$, $\|B\|_q$ and $\|H\|_\infty$, also with invariance by scaling.

Now, we have all the ingredients to prove Theorem 1.

3 Proof of Theorem 1

The idea of the proof is to apply Theorem 2 to get embeddedness. Since we assume that M has constant mean curvature, point (ii) is trivial and point (i)

resumes to $\|\tau\|_r \leqslant h\varepsilon$, so that we just need to show that $\|\tau\|_r$ is small for some $r > n$. We will show that for $r = n+1$.
First, we have

$$\begin{aligned}
\|\tau\|_{n+1}^{2(n+1)} &= \left(\frac{1}{\mathrm{Vol}(M)}\int_M \|\tau\|^{(n+1)} dv_g\right)^2 \\
&= \left(\frac{1}{\mathrm{Vol}(M)}\int_M \|\tau\|^{n}\cdot\|\tau\| dv_g\right)^2.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we get

$$\|\tau\|_{n+1}^{2(n+1)} \leqslant \frac{1}{\mathrm{Vol}(M)^2}\left(\int_M \|\tau\|^{2n} dv_g\right)\left(\int_M \|\tau\|^{2} dv_g\right)$$

From this, we deduce immediately that

$$\|\tau\|_{n+1}^{2(n+1)} \leqslant \frac{1}{\mathrm{Vol}(M)}\|B\|_\infty^{2n}\left(\int_M \|\tau\|^{2} dv_g\right). \tag{10}$$

On the other hand, we have

$$\int_M \|\tau\|^2 dv_g = \frac{\rho_0}{\rho_0}\int_M \|\tau\|^2 dv_g. \tag{11}$$

From the almost starshapedness condition (1), we get

$$\rho_0 \leqslant -\langle Z,\nu\rangle + \rho_0\varepsilon,$$

which give together with (11)

$$\begin{aligned}
\int_M \|\tau\|^2 dv_g &\leqslant \frac{1}{\rho_0}\int_M \|\tau\|^2\left(-\langle Z,\nu\rangle + \rho_0\varepsilon\right) dv_g \\
&\leqslant \frac{n(n-1)}{\rho_0}\int_M \left(H^2 - H_2\right)\left(-\langle Z,\nu\rangle + \rho_0\varepsilon\right) dv_g \\
&\leqslant -\frac{n(n-1)}{\rho_0}\int_M \left(H^2 - H_2\right)\langle Z,\nu\rangle dv_g + n(n-1)\times \\
&\quad \times \int_M \left(H^2 - H_2\right)\varepsilon dv_g
\end{aligned} \tag{12}$$

where we have used the fact that $-\langle Z,\nu\rangle + \rho_0\varepsilon > 0$ and $\|\tau\|^2 = n(n-1)\left(H^2 - H_2\right)$ to get the second line.
Now, we will estimate both terms of the right hand side. On one hand, since H is constant equal to h, we have

$$\int_M \left(H^2 - H_2\right)\langle Z,\nu\rangle dv_g = h\int_M H\langle Z,\nu\rangle dv_g - \int_M H_2\langle Z,\nu\rangle dv_g$$

so, by the Hsiung-Minkowski formulas, we have

$$\begin{aligned}\int_M \left(H^2 - H_2\right)\langle Z,\nu\rangle dv_g &= -h\int_M c_\delta(r)dv_g - \int_M H_2\langle Z,\nu\rangle dv_g \\ &= -h\int_M c_\delta(r)dv_g + \int_M c_\delta(r)H dv_g \\ &= -h\int_M c_\delta(r)dv_g + h\int_M c_\delta(r)dv_g \\ &= 0 \end{aligned} \tag{13}$$

On the other hand, we obviously have $H^2 - H_2 \leqslant 2\|B\|_\infty^2$.
Hence, (12) gives with this last estimate and (13)

$$\int_M \|\tau\|^2 dv_g \leqslant 2n(n-1)\|B\|_\infty^2 \int_M \varepsilon dv_g. \tag{14}$$

Hence, from (10) and (14), we get

$$\|\tau\|_{n+1}^{2(n+1)} \leqslant 2n(n-1)\|B\|_\infty^{2(n+1)}\|\varepsilon\|_1,$$

that is

$$\|\tau\|_{n+1} \leqslant (2n(n-1))^{\frac{1}{2(n+1)}}\|B\|_\infty\|\varepsilon\|_1^{\frac{1}{2(n+1)}}. \tag{15}$$

Now, we set $\varepsilon_0 = \dfrac{(h\varepsilon_1)^{2(n+1)}}{2n(n-1)\|B\|_\infty^{2(n+1)}}$, where ε_1 is the constant of Theorem 2.
Hence, if $\|\varepsilon\|_1 \leqslant \varepsilon_0$,we get from (15) that

$$\|\tau\|_{n+1} \leqslant h\varepsilon_1,$$

and we deduce from Theorem 2 that M is embedded. Finally, we conclude by using the Alexandrov Theorem, M is embedded and has constant mean curvature so M is a geodesic sphere.

4 A result with higher order mean curvatures in Euclidean spaces

As mentionned in the introduction, Hsiung showed that a closed connected and oriented starshaped hypersurface of the Euclidean space $\mathbb{R}^{n+1}$ with constant higher order mean curvature H_r, $r \in \{2,\dots,n\}$ is a geodesic sphere. This result is also true if the ambient space is a half-sphere or a hyperbloic space. In this last section, we will prove that the conclusion is still the same if we replace starshaped by almost starshaped in the case of Euclidean spaces only. Precisely, here is the result we can prove:

Theorem 3. *Let $n \geqslant 2$ and $r \in \{2, \cdots, n-1\}$ be two integers. Let ρ_0 be a positive real number. Let us consider (M^n, g) a closed, connected and oriented Riemannian manifold of dimension n isometrically immersed into the Euclidean space $\mathbb{R}^{n+1}$ so that the $(r+1)$-th mean curvature H_{r+1} is positive everywhere on M.*
Let $h > 0$, then there exists a positive constant ε_0 depending on n, r, h, $\|B\|_\infty$, $\mathrm{Vol}(M)$ and $\inf_M(H_{r+1;n,1})$ so that if M has constant r-th mean curvature $H_r = h$ and is almost starshaped in the sense of (1) with $\|\varepsilon\|_1 \leqslant \varepsilon_0$, then M is a geodesic sphere.

This result has to be compared to Theorem 1. First of all, we assume here that H_r is constant, but also that H_{r+1} is positive everywhere on M. Such an hypothesis is not require in Theorem 1, indeed, we do not assume that H_2 is positive. Second there is an additional dependence of the constant ε_0 here, namely $\inf_M(H_{r+1;n,1})$. This quantity $H_{r+1;n,1}$ is an extrinsic quantity defined form the second fundamental form and is positive if H_{r+1} is positive, which is the case here. Finally, Theorem 3 Is valid for the Euclidean space only for the moment. Our proof si based on the almost umbilicity result proved in [20] and which gives embeddedness only in the case of $\mathbb{R}^{n+1}$ as mentionned previously, in the other space forms, we deduce by a conformal change of metric proximity for Hausdorff distance only. However, it might be possible with other techniques to obtain embeddedness in half-spheres or hyperbolic spaces. Note also that Theorem 2 does not allow to conclude here since we have no control on the mean curvature so that we don't know if point (ii) is satisfied.

Before beginning the proof, we need to define the function $H_{r+1;n,1}$. Let $l \in \{2, \ldots, n\}$ and $i, j \in \{1, \ldots, n\}$ three integers with $i > j$. Then, we can define the extrinsic curvature term $H_{l;i,j}$ by

$$H_{l;i,j} = \frac{\partial H_l}{\partial \kappa_i \partial \kappa_j}$$

where H_l is the l-th mean curvature and κ_i, κ_j are the i-th and j-th principal curvatures (order by $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n$). The notation $\dfrac{\partial H_l}{\partial \kappa_i \partial \kappa_j}$ is to understand in the sense that H_l is defined form the principal curvature by

$$H_l = \frac{1}{\binom{n}{l}} \sigma_l(\kappa_1, \ldots, \kappa_n)$$

and thus

$$H_{l;i,j} = \frac{1}{\binom{n}{l}} \sigma_{l-2}(\kappa_1, \ldots, \kappa_{j-1}, \kappa_{j+1}, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_n).$$

In the sequel, we will only be interested in $H_{r+1;n,1} = \frac{1}{\binom{n}{r+1}}\sigma_{k-1}(\kappa_2, \dots, \kappa_{n-1})$. In [21], Scheuer showed that if H_r is positive then $H_{r+1;n,1}$ is also positive. Note that the proof of this fact is by contradiction so that we do not have an explicit inequality between H_r and $H_{r+1;n,1}$. Like for Theorem 1, we want to obtain almost umbilicity. For this, we begin by giving the following lemma

Lemma 1. *Let (M^n, g) be a closed, connected and oriented Riemannian manifold of dimension n isometrically immersed into the Euclidean space $\mathbb{R}^{n+1}$ so that the $(r+1)$-th mean curvature H_{r+1} is positive everywhere on M. Then, there exists a positive constant K depending on n, r,$\min_M(H_{r+1;n,1})$, $\min_M(H_r)$ and $\|B\|_\infty$ so that*

$$\|\tau\|^2 \leqslant K\big(HH_r - H_{r+1}\big).$$

Proof: First, we recall the classical following inequalities between higher order mean curvatures, for any $k \in \{1, \dots, n-1\}$,

$$H_k^2 - H_{k+1}H_{k-1} \geqslant 0.$$

Moreover, we have a more precise estimate of the positivity of this term. Namely,

$$H_k^2 - H_{k+1}H_{k-1} \geqslant c_n\|\tau\|^2 H_{k+1;n,1}^2 \tag{16}$$

where c_n is a constant depending only on n. One can find the proof in [21] for instance. We also recall the classical fact that since we assume that $H_{r+1} > 0$, then all the functions H_k are also positive for $k \in \{1, \cdots, n-1\}$ and we have in addition the so-called Maclaurin inequalities

$$H_{k+1}^{\frac{1}{k+1}} \leqslant H_k^{\frac{1}{k}} \leqslant \cdots \leqslant H_2^{\frac{1}{2}} \leqslant H.$$

Thus, dividing by H_kH_{k-1}, (16) becomes

$$\frac{H_k}{H_{k-1}} - \frac{H_{k+1}}{H_k} \geqslant c_n\|\tau\|^2 \frac{H_{k+1;n,1}^2}{H_kH_{k-1}}. \tag{17}$$

Thus, by summing equation (17) for k from 1 to r, we get

$$H - \frac{H_{r+1}}{H_r} \geqslant c_n\|\tau\|^2 \sum_{k=1}^{r} \frac{H_{k+1;n,1}^2}{H_kH_{k-1}}, \tag{18}$$

and so

$$HH_r - H_{r+1} \geqslant c_n\|\tau\|^2 \left(\sum_{k=1}^{r} \frac{H_{k+1;n,1}^2}{H_kH_{k-1}}\right) H_r. \tag{19}$$

Moreover, we have $H_kH_{k-1} \leqslant \|B\|_\infty^{2k-1}$. In addition, since H_{r+1} is positive, then all the function H_k are also positive and thus, as proved by Scheuer in [21], the functions $H_{k;n,1}$ are also positive. In addition, since they are the normalized symmetric polynomial evaluated for $\kappa_2, \ldots, \kappa_{n-1}$, they also satisfy the Maclaurin inequality, up to a normalization constant, that is

$$(H_{k;n,1})^{\frac{1}{k-2}} \geqslant a_{n,k} \left(H_{k+1;n,1}\right)^{\frac{1}{k-1}},$$

where $a_{n,k}$ is a positive constant depending only on n and k, and so

$$(H_{k;n,1})^{\frac{1}{k-2}} \geqslant b_{n,k,r} \left(H_{r+1;n,1}\right)^{\frac{1}{r-1}},$$

where $b_{n,k,r}$ is a positive constant depending only on n, k and r. Note that the exponents come from the fact that $H_{k;n,1}$ is the symmetric polynomial of degree $k-2$. Thus (19) gives

$$HH_r - H_{r+1} \geqslant c_n\|\tau\|^2 \left(\sum_{k=1}^{r} \frac{b_{n,k+1,r}^{2(k-1)} H_{r+1;n,1}^{\frac{2(k-1)}{r-1}}}{\|B\|_\infty^{2k-1}} \right) H_r \geqslant C\|\tau\|^2, \tag{20}$$

where $C = c_n \min\limits_{1\leqslant k\leqslant r} \left(b_{n,k+1,r}^{2(k-1)}\right) \dfrac{\min\limits_M(H_r)}{2\|B\|_\infty} \displaystyle\sum_{k=1}^{r} \left(\frac{\min\limits_M (H_{r+1;n,1})^{\frac{1}{r-1}}}{\|B\|_\infty} \right)^{2(k-1)}$. This concludes the proof of the lemma by setting $K = \dfrac{1}{C}$ which depends only on n, r, $\min\limits_M(H_{r+1;n,1})$, $\min\limits_M(H_r)$ and $\|B\|_\infty$. □

Now, we can prove Theorem3. The strategy is to show almost umbilicity with the help of Lemma 1 in order to apply the following result that we proved with Scheuer in [20].

Theorem 4 (Roth-Scheuer [20]). *Let M be a closed, connected, oriented hypersurface of $\mathbb{R}^{n+1}$. Let $p > n \geq 2$. Then there exist constants c and ε_2, depending on n, p, $\mathrm{Vol}(M)$, $\|B\|_p$ as well as a constant $\alpha = \alpha(n,p)$, such that whenever there holds*

$$\|\tau\|_p \leqslant \|H\|_p \varepsilon_2,$$

there also holds

$$d_H(\Sigma, S_\rho) \leqslant \frac{c^\alpha \rho}{\|H\|_p^\alpha} \|\tau\|_p^\alpha = \rho\varepsilon^\alpha,$$

and M is diffeomorphic and ε^α-quasi-isometric to a sphere S_ρ of approriate radius ρ. In addition, M is embedded.

As in the proof of Theorem 1, we will estimate the L^{n+1}-norm of the umbilicity tensor τ. We still start from(10), that is,

$$\|\tau\|_{n+1}^{2(n+1)} \leqslant \frac{1}{\mathrm{Vol}(M)}\|B\|_\infty^{2n}\left(\int_M \|\tau\|^2 dv_g\right).$$

The difference here is the way to estimate $\int_M \|\tau\|^2 dv_g$. First, from the almost starshapedness condition (1), we have like in the proof of Theorem 1,

$$\rho_0 \leqslant -\langle Z,\nu\rangle + \rho_0\varepsilon,$$

which give together with (11)

$$\int_M \|\tau\|^2 dv_g \quad\leqslant\quad \frac{1}{\rho_0}\int_M \|\tau\|^2\left(-\langle Z,\nu\rangle + \rho_0\varepsilon\right)dv_g \tag{21}$$

Now, we use Lemma 1 to obtain

$$\begin{aligned}\int_M \|\tau\|^2 dv_g &\leqslant \frac{K}{\rho_0}\int_M (HH_r - H_{r+1})\left(-\langle Z,\nu\rangle + \rho_0\varepsilon\right)dv_g\\ &\leqslant -\frac{K}{\rho_0}\int_M (HH_r - H_{r+1})\langle Z,\nu\rangle dv_g + K\int_M (HH_r - H_{r+1})\,\varepsilon dv_g\end{aligned} \tag{22}$$

On one hand, we have

$$\begin{aligned}\int_M (HH_r - H_{r+1})\langle Z,\nu\rangle dv_g &= h\int_M H\langle Z,\nu\rangle dv_g - \int_M H_{r+1}\langle Z,\nu\rangle dv_g\\ &= -h\mathrm{Vol}(M) + \int_M H_r dv_g\\ &= -h\mathrm{Vol}(M) + h\mathrm{Vol}(M)\\ &= 0\end{aligned} \tag{23}$$

where we have used the first are $(r+1)$-th Hsiung-Minkowksi formula and the fact that H_r is constant equal to h. Note that we are now in the Euclidean space so that in Hsiung-Minkowski formula, c_δ is just 1 here.
On the other hand, we have clearly $HH_r - H_{r+1} \leqslant 2\|B\|_\infty^{r+1}$, so that (22) gives

$$\int_M \|\tau\|^2 dv_g \quad\leqslant\quad 2K\|B\|_\infty^{r+1}\int_M \varepsilon dv_g. \tag{24}$$

Hence, reporting into (10), we obtain

$$\|\tau\|_{n+1}^{2(n+1)} \quad\leqslant\quad 2K\|B\|_\infty^{2n+r+1}\|\varepsilon\|_1 \tag{25}$$

that is

$$\|\tau\|_{n+1} \leqslant (2K)^{\frac{1}{2(n+1)}} \|B\|_{\infty}^{\frac{2n+r+1}{2(n+1)}} \|\varepsilon\|_{1}^{\frac{1}{2(n+1)}}. \quad (26)$$

we set $\varepsilon_0 = \dfrac{h^{\frac{2(n+1)}{r}} \varepsilon_2^{2(n+1)}}{2K\|B\|_{\infty}^{2n+r+1}}$ where ε_2 is the constant of Theorem 4. Hence, if $\|\varepsilon\|_1 \leqslant \varepsilon_0$, we get from (26) that

$$\begin{aligned} \|\tau\|_{n+1} &\leqslant h^{\frac{1}{r}} \varepsilon_2 \\ &= \|H_r^{\frac{1}{r}}\|_{n+1} \varepsilon_2 \\ &\leqslant \|H\|_{n+1} \varepsilon_2. \end{aligned}$$

We can apply Theorem 4 with $p = n+1$ and we deduce in particular that M is embedded. Since M has constant r-th mean curvature, by the Alexandrov theorem for H_r prove by Ros [17], we conclude that M is a geodesic sphere. Note that the constant ε_0 depends only on the quantities announced in the statement of the Theorem. Indeed, ε_0 depends on n, r, h and $\|B\|_\infty$ from its expression. It depends also on K and ε_2. First, ε_2 is obtained form Theorem 4 with $p = n+1$ and so depends on n, $\mathrm{Vol}(M)$ and $\|B\|_{n+1}$. The dependence on $\|B\|_{n+1}$ can clearly be replaced by $\|B\|_\infty$ when analysing the proof of Theorem 4. Finally, K comes from Lemma 1 and depends on n, r,$\min\limits_M(H_{r+1;n,1})$, $\min\limits_M(H_r)$ and $\|B\|_\infty$. But H_r is constant equal to h hence $\min\limits_M(H_r)$ is nothing else but h which concludes the proof.

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