Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **44** (2024) no. 2, 127–142. doi:10.1285/i15900932v44n2p127

On the convergence of an implicit and relaxed Krasnoselskii-Mann type process for nonexpansive mappings

Ramzi May
Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia
rmay@kfu.edu.sa

Marouane Rabaoui
Laboratoire d'Analyse Mathématiques et Applications LR11ES11, Département de Mathématiques, Faculté des Sciences de Tunis, Université de Tunis El Manar, 2092 El Manar, Tunis, Tunisia
marouane.rabaoui@fst.utm.tn

Received: 28.09.24; accepted: 13.12.24.

Abstract. In this paper, we answer a question raised by Hong-Kun Xu in [14] about the weak convergence of a generalized semi-implicit relaxed version of the Krasnoselskii-Mann process for nonexpansive mappings. The stability of this process is then studied, and some numerical experiments are provided in order to measure the effect of the process parameters on the convergence speed.

Keywords: Hilbert space, nonexpansive mapping, Krasnoselskii-Mann process

MSC 2020 classification: 47H09, 47J05, 47J25

1 Introduction and main results

Let $\mathcal{H}$ be a real Hilbert space equipped with an inner product denoted by $\langle . \rangle$ and let $\|.\|$ denote the associated Euclidean norm. The study of several problems in Mathematics and Physics turns into the determination of a fixed point of an operator $T : C \to C$ where C is a given closed and nonempty subset of $\mathcal{H}$, see for instance [5] and the references therein. Therefore, many results are concerned with the construction of algorithms providing numerical approximations of fixed points for wide classes of operators. Among these results, it figures the classical well-known Banach's fixed point theorem (see [2]):

Theorem 1. *Let X be a closed and nonempty subset of a Banach space E and let $T : X \to X$ be a contraction mapping, i.e. there exists $\rho \in [0,1)$ such that, for every $x, y \in X$,*

$$\|T(x) - T(y)\| \leq \rho \|x - y\| .$$

Then, T has a unique fixed point $x^* \in X$. Moreover, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by the iterative process:

$$x_{n+1} = T(x_n), n = 0, 1, \dots \tag{1.1}$$

converges strongly in E to x^*. In addition, for every $n \in \mathbb{N}$,

$$\|T(x_n) - x_n\| \leq \rho^n \|T(x_0) - x_0\| , \quad \|x_n - x^*\| \leq \frac{\rho^n}{1-\rho} \|T(x_0) - x_0\| .$$

In many cases, the considered operator T does not satisfy the contraction condition, but it satisfies instead the weaker condition

$$\|T(x) - T(y)\| \leq \|x - y\| , \quad \forall x, y \in C.$$

Such an operator is called a nonexpansive operator. In this paper, we study such operators assuming the following three conditions:

(H1) C is a nonempty, closed and convex subset of $\mathcal{H}$.

(H2) $T : C \to C$ is a nonexpansive mapping.

(H3) The set $\mathrm{Fix}(T) = \{x \in C : T(x) = x\}$ of fixed points of T is nonempty.

Let us first notice that T being nonexpansive does not imply that it has necessarily a fixed point. One can consider for example the mapping defined on $\mathcal{H}$ by $T(x) = x + v$, where v is a non-zero, fixed element of $\mathcal{H}$. However, in [11], Kirk proved that any nonexpansive mapping from a nonempty, closed, convex and bounded subset of $\mathcal{H}$ to itself has at least one fixed point. Besides, unlike the case of a contraction mapping, the iterative process (1.1) does not converge in general under the assumptions (H1), (H2) , and (H3). As an example, one can consider the case $T = -I$, where I denotes the identity operator of $\mathcal{H}$. To overcome this, Krasnoselskii [12] and Mann [8] introduced separately the following relaxed version of process (1.1):

$$x_{n+1} = (1 - t_n)x_n + t_n T(x_n), \; n = 0, 1, 2, \dots \tag{1.2}$$

and they established the weak convergence of the sequence $\{x_n\}$ to a fixed point of T under the assumptions (H1),(H2) and (H3), together with a suitable assumption on the sequence $\{t_n\}$. Precisely, they proved the following result (see [13]):

Theorem 2. *Let $\{t_n\}$ be a sequence in $[0,1]$ satisfying the condition*

$$\sum_{n=0}^{\infty} t_n(1-t_n) = \infty. \tag{1.3}$$

Then, for every initial data $x_0 \in C$, the sequence $\{x_n\}$ generated by the process (1.2) converges weakly in $\mathcal{H}$ to some element x_∞ of $Fix(T)$. Moreover, the sequence $\{\|x_n - T(x_n)\|\}$ is non increasing and satisfies the following estimations:

$$\|x_n - T(x_n)\| \leq \frac{dist(x_0, Fix(T))}{\sqrt{\Gamma_n^0}}, \tag{1.4}$$

$$\lim_{n\to\infty} \sqrt{\Gamma_n^0}\, \|x_n - T(x_n)\| = 0, \tag{1.5}$$

where $\Gamma_n^0 := \sum_{k=0}^{n} t_k(1-t_k)$, and $dist(x_0, Fix(T))$ denotes the distance between the initial data x_0 and the set $Fix(T)$.

In order to improve the rate of the convergence and the stability properties of the process (1.2), Alghamdi, Shahzad, and Xu introduced in [1] the following implicit mid-point version of the algorithm (1.2),

$$x_{n+1} = (1-t_n)x_n + t_n T(\frac{x_n + x_{n+1}}{2}). \tag{1.6}$$

They proved the following convergence result:

Theorem 3. *Let $\{t_n\}$ be a sequence in $[0,1]$ satisfying the following conditions:*

(a) $\liminf t_n > 0$.

(b) *There exist $b > 0$ and $n_0 \in \mathbb{N}$ such that $t_{n+1}^2 \leq b\, t_n$, for every $n \geq n_0$.*

Then, for any initial data $x_0 \in C$, the sequence $\{x_n\}$ generated by the process (1.6) converges weakly in $\mathcal{H}$ to some element of $Fix(T)$.

We notice here that the second assumption in the previous theorem is superfluous. In fact, it is a direct consequence of the first assumption and the fact that $\{t_n\}$ is bounded. Later, H. K. Xu improved greatly, in [14], the previous result by establishing the weak convergence of the process (1.6) under the more natural condition (1.3). Precisely he proved the following theorem:

Theorem 4. *Let $\{t_n\}$ be a sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty$. Then, for any initial data $x_0 \in C$, the sequence $\{x_n\}$ generated by the process (1.6) converges weakly in $\mathcal{H}$ to some element of $Fix(T)$.*

At the end of his paper [14], H. K. Xu raised the question of the convergence of the general averaged semi-implicit process:

$$x_{n+1} = (1-t_n)x_n + t_n T((1-\lambda)x_n + \lambda x_{n+1}),\ n = 0, 1, 2, \dots \tag{1.7}$$

where $\lambda \in [0,1]$ is a fixed parameter. Our first main contribution in this paper is a positive answer to this question. In order to clearly state our result, let us first recall the following classical notations:

Notation 1. Let $\{u_n\}$ and $\{v_n\}$ be two real positive sequences. The notation $u_n = o(v_n)$ means $\lim_{n\to\infty} \dfrac{u_n}{v_n} = 0$, and $u_n \sim v_n$ means that $\lim_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

Theorem 5. *Let $\lambda \in [0,1]$ and $\{t_n\}$ be a sequence in $[0,1)$ satisfying the condition*

$$\sum_{n=0}^{\infty} \frac{t_n(1-t_n)}{(1-\lambda t_n)^2} = \infty. \tag{1.8}$$

Then, for any initial data $x_0 \in C$, the sequence $\{x_n\}$ generated by the process (1.7) converges weakly in $\mathcal{H}$ to some fixed point of T. Moreover, the residual sequence $\{\|x_n - T(x_n)\|\}$ is non increasing and satisfies the following estimations:

$$\|x_{n+1} - T(x_{n+1})\| \le \frac{dist(x_0, Fix(T))}{\sqrt{\Gamma_n^\lambda}}, \tag{1.9}$$

$$\|x_{n+1} - T(x_{n+1})\| = o\Big(\frac{1}{\sqrt{\Gamma_n^\lambda}}\Big), \quad \text{where } \Gamma_n^\lambda := \sum_{k=0}^{n} \frac{t_k(1-t_k)}{(1-\lambda t_k)^2}. \tag{1.10}$$

Remark 1. From the previous theorem, we deduce that,for $\lambda = 1$ and $t_n = 1 - \frac{1}{(n+1)^\theta}$ with $\theta > 0$, the convergence rate of the residual sequence $\{x_n - T(x_n)\}$ to 0 is of smaller order than $o(\frac{1}{n_\theta})$, where $n_\theta = n^{\frac{\theta+1}{2}}$. In fact,

$$\Gamma_n^1 \sim \sum_{k=0}^{n} (k+1)^\theta \sim \frac{n^{\theta+1}}{\theta+1} \quad \text{as } n \to \infty.$$

Remark 2. Let $\lambda \in [0,1]$ and $\{t_n\}$ in $[0,1)$, and assume that $x_n \in C$, the process (1.7), is constructed for some $n \in \mathbb{N}$. Then, the mapping $\Psi_{\lambda,n} : C \to C$ defined by

$$\Psi_{\lambda,n}(x) = (1-t_n)x_n + t_n T((1-\lambda)x_n + \lambda x), \tag{1.11}$$

is a contraction with coefficient $\rho_{n,\lambda} = t_n\lambda$. Hence $\Psi_{\lambda,n}$ has a unique fixed point $x_{n+1} \in C$. Therefore, given $x_0 \in C$, the process (1.7) is well-defined. But, in general, the construction of x_{n+1} as a fixed point of $\Psi_{\lambda,n}$ can only be done approximately. Infact, by using for instance Theorem 1, we construct an approximation value of x_{n+1} for which $\|x_{n+1} - \Psi_{\lambda,n}(x_{n+1})\|$ is small enough and not necessary equal to 0. It is then natural to ask if such lack of precision in the construction of the sequence $\{x_n\}$ has a significant effect on its asymptotic behavior.

Theorem 6. *Let $\{t_n\}$ be a real sequence satisfying the assumptions of Theorem 5 and let $\{e_n\}$ be a nonnegative real sequence such that*

$$\sum_{n=0}^{\infty} \frac{e_n}{1-\lambda t_n} < \infty. \tag{1.12}$$

Then, every $\{x_n\}$ in C satisfying

$$\|x_{n+1} - [(1-t_n)x_n + t_n T((1-\lambda)x_n + \lambda x_{n+1})]\| \le e_n,$$

converges weakly in $\mathcal{H}$ to some element of $Fix(T)$.

By combining Theorem 1, Remark 2, and Theorem 6, one can deduce that if λ, $\{t_n\}$ and $\{e_n\}$ satisfy respectively (1.8) and (1.12) then, the following algorithm provides, in a finite number of iterations, an approximate fixed point x_ε of T for any predefined tolerance number $\varepsilon > 0$ such that $\|x_\varepsilon - T(x_\varepsilon)\| \le \varepsilon$.

Algorithm 1.1

Input: Initial data $x_0 \in C$, parameter $\lambda \in [0,1]$, positive sequences $\{t_n\}$ and $\{e_n\}$, tolerance number $\varepsilon > 0$.
Step 1: Initialize $n \leftarrow 0$ and $N \leftarrow 0$.
Step 2: If $\|x_n - T(x_n)\| \le \varepsilon$ then $N \leftarrow N+1$ and Stop. Else, go to Step 3.
Step 3: Initialize $k \leftarrow 0$ and $u_0 \leftarrow x_n$.
Step 4: Calculate $u_{k+1} = \Psi_{\lambda,n}(u_k)$ and update $N \leftarrow N+1$.
Step 5: If $\|u_{k+1} - u_k\| \le e_n$, then $x_{n+1} \leftarrow u_k, n \leftarrow n+1$ and go to Step 2. Else, $k \leftarrow k+1$ and go to Step 4.
Output: $x_n = x_\varepsilon$ is an approximate fixed point of T with tolerance ε and N, called the number of iterations, is the number of times the operator T is applied in order to get x_ε.

Since 2015, many papers have been devoted to the study of the strong convergence of different algorithms that combine the viscosity approximation method, due essentially to A. Moudafi, and the semi-implicit process of type (1.6) (see [15, 10, 7]). In particular, Xu and al., in[15], considered the iterative process

$$x_{n+1} = t_n f(x_n) + (1-t_n)T(\frac{x_n + x_{n+1}}{2}), \tag{1.13}$$

where $f : C \to C$ is a given contraction mapping. They proved that if the sequence $\{t_n\}$ belongs to $(0,1]$ and satisfies the following conditions:

(C$_1$) $t_n \to 0$ as $n \to \infty$,

(C$_2$) $\sum_{n=1}^{\infty} t_n = \infty$,

(C$_3$) $\frac{|t_{n+1} - t_n|}{t_n} \to 0$ as $n \to \infty$ or $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$, then any sequence generated by (1.13) converges strongly in $\mathcal{H}$ to the unique fixed point $q^\star$ of the contaction mapping $P_{\mathrm{Fix}(T)} \circ f : \mathrm{Fix}(T) \to \mathrm{Fix}(T)$, where $P_{\mathrm{Fix}(T)}$ is the metric projection from $\mathcal{H}$ onto $\mathrm{Fix}(T)$. Besides, by adapting the method of the proof of [9, Theorem 4.1], one can verify that the previous strong convergence result remains true if the equality (1.13) is replaced by the perturbed estimation

$$\left\| x_{n+1} - [t_n f(x_n) + (1-t_n)T(\frac{x_n + x_{n+1}}{2})] \right\| \leq \mu_n,$$

provided that the nonnegative error sequence $\{\mu_n\}$ satisfies $\sum_{n=1}^{\infty} \mu_n < \infty$. Hence, if the sequences $\{t_n\}$ and $\{\mu_n\}$ satisfy the required conditions, the following algorithm gives an approximation $q_\varepsilon^\star$ of $q^\star$ in a finite number of iterations M.

Algorithm 1.2

Input: Contraction mapping $f : C \to C$, initial data $x_0 \in C$, positive real sequences $\{t_n\}$ and $\{\mu_n\}$, and a tolerance number $\varepsilon > 0$.
Step 1: Initialize $n = 0$ and $M = 0$.
Step 2: If $\|x_n - P_{\mathrm{Fix}(T)}(f(x_n))\| \leq \varepsilon$ then $M \leftarrow M + 1$. Else, go to Step 3.
Step 3: Initialize $k = 0$ and $u_0 = x_n$.
Step 4: Compute $u_{k+1} = t_n f(x_n) + (1-t_n)T(\frac{x_n+u_k}{2})$ and update $M \leftarrow M + 1$.
Step 5: If $\|u_{k+1} - u_k\| \leq \mu_n$, set $x_{n+1} \leftarrow u_k, n \leftarrow n + 1$ and go to Step 2. Else, set update $k \leftarrow k + 1$ and go to Step 4.
Output: $x_\varepsilon = x_n$ is an approximation of $q^\star$ and M is the number of times the operator T and/or the mapping f are applied in order to get x_ε.

Remark 3. It is clear that in Algorithm 1.2 the stopping criteria is difficult to check. For this reason, in practice, one can replace it by the weaker stopping criteria $\|x_n - T(x_n)\| \leq \varepsilon$. In fact, if the number of iterations M is reasonably large, the obtained output $x_\varepsilon = x_n$ is still an approximation of $q^\star$.

In the next section, we prove our first main result Theorem 5. The third section is devoted to the proof of the second main result Theorem 6. In the last section, we apply the result of Theorem 6 to the numerical resolution of a nonnegative least square problem and we perform numerical experiments in order to study the relation between the convergence speed of the algorithm, the parameter λ and the sequence $\{t_n\}$. Moreover, we provide a numerical study of the speed of Algorithm 1.2 in the particular case of $f = 0$.

2 Proof of the first main result

In this section, we give a detailed proof of Theorem 5.

PROOF. Let q be a an element of $\mathrm{Fix}(T)$. By using the classical equality

$$\|(1-\theta)x+\theta y\|^2 = (1-\theta)\|x\|^2 + \theta\|y\|^2 - (1-\theta)\theta\|x-y\|^2 \tag{2.14}$$

which is valid for every x, y in $\mathcal{H}$ and every $\theta \in [0,1]$, one obtains

$$\|x_{n+1}-q\|^2 = (1-t_n)\|x_n-q\|^2 + t_n\|T(y_n)-q\|^2 - t_n(1-t_n)\|x_n-T(y_n)\|^2, \tag{2.15}$$

where $y_n := (1-\lambda)x_n + \lambda x_{n+1}$. Since T is nonexpansive and $T(q)=q$, one gets, thanks to the convexity of the function $x \mapsto \|x\|^2$,

$$\|T(y_n)-q\|^2 \leq (1-\lambda)\|x_n-q\|^2 + \lambda\|x_{n+1}-q\|^2.$$

Inserting this last estimate into (2.15) yields

$$\|x_{n+1}-q\|^2 \leq (1-\lambda t_n)\|x_n-q\|^2 + \lambda t_n\|x_{n+1}-q\|^2 - t_n(1-t_n)\|x_n-T(y_n)\|^2,$$

which, in turn, implies the key estimate

$$\|x_{n+1}-q\|^2 \leq \|x_n-q\|^2 - \frac{t_n(1-t_n)}{1-\lambda t_n}\|x_n-T(y_n)\|^2. \tag{2.16}$$

Therefore, the real sequence $\{\|x_n-q\|\}$ is non increasing and

$$\sum_{n=0}^{\infty}\frac{t_n(1-t_n)}{1-\lambda t_n}\|x_n-T(y_n)\|^2 \leq \|x_0-q\|^2. \tag{2.17}$$

On the other hand,

$$\begin{aligned}
\|x_{n+1}-T(x_{n+1})\| &\leq \|x_{n+1}-T(y_n)\| + \|T(y_n)-T(x_{n+1})\| \\
&\leq (1-t_n)\|x_n-T(y_n)\| + \|y_n-x_{n+1}\| \\
&= (1-t_n)\|x_n-T(y_n)\| + (1-\lambda)\|x_{n+1}-x_n\| \\
&= (1-t_n)\|x_n-T(y_n)\| + t_n(1-\lambda)\|x_n-T(y_n)\| \\
&= (1-\lambda t_n)\|x_n-T(y_n)\|.
\end{aligned}$$

Combining this last estimate with (2.17), one gets

$$\sum_{n=0}^{\infty}\frac{t_n(1-t_n)}{(1-\lambda t_n)^2}\|x_{n+1}-T(x_{n+1})\|^2 \leq \|x_0-q\|^2. \tag{2.18}$$

Now, let us prove that the sequence $\{\|x_n - T(x_n)\|\}$ is non increasing. For every $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - T(x_{n+1})\| &= \|(1-t_n)(x_n - T(x_{n+1}) + t_n(T(y_n) - T(x_{n+1}))\| \\ &\leq (1-t_n)\|x_n - T(x_{n+1})\| + t_n\|y_n - x_{n+1}\| \\ &\leq (1-t_n)\|x_n - T(x_n)\| + (1-t_n)\|T(x_{n+1}) - T(x_n)\| + t_n\|y_n - x_{n+1}\| \\ &\leq (1-t_n)\|x_n - T(x_n)\| + (1-t_n)\|x_{n+1} - x_n\| + t_n\|y_n - x_{n+1}\| \\ &= (1-t_n)\|x_n - T(x_n)\| + (1-t_n)\|x_{n+1} - x_n\| + t_n(1-\lambda)\|x_{n+1} - x_n\| \\ &= (1-t_n)\|x_n - T(x_n)\| + (1-\lambda t_n)\|x_{n+1} - x_n\|. \end{aligned} \tag{2.19}$$

Also, by the definition of x_{n+1},

$$\begin{aligned}\|x_{n+1} - x_n\| &= t_n\|x_n - T(y_n)\| \\ &\leq t_n\|x_n - T(x_n)\| + t_n\|T(x_n) - T(y_n)\| \\ &\leq t_n\|x_n - T(x_n)\| + t_n\|x_n - y_n\| \\ &= t_n\|x_n - T(x_n)\| + t_n\lambda\|x_{n+1} - x_n\|,\end{aligned}$$

which implies that

$$(1-\lambda t_n)\|x_{n+1} - x_n\| \leq t_n\|x_n - T(x_n)\|. \tag{2.20}$$

Hence, by combining the estimations (2.19) and (2.20), it follows that

$$\|x_{n+1} - T(x_{n+1})\| \leq \|x_n - T(x_n)\|.$$

Therefore, by invoking (2.18),

$$\|x_{n+1} - T(x_{n+1})\|^2 \leq \frac{\|x_0 - q)\|^2}{\Gamma_n^\lambda}.$$

The estimation (1.9) is then obtained by taking the minimum over all $q \in \mathrm{Fix}(T)$. Now, let us prove the estimation (1.10). Let ε be an arbitrary positive real number. From (2.18), it follows that there exists $n_0 \in \mathbb{N}$ such that

$$r_{n_0} := \sum_{k=n_0}^{\infty} \frac{t_k(1-t_k)}{(1-\lambda t_k)^2}\|x_{k+1} - T(x_{k+1})\|^2 < \varepsilon.$$

On the other hand, since $\{(\|x_n - T(x_n)\|\}$ is non increasing, then for every $n \geq n_0$,

$$\|x_{n+1} - T(x_{n+1})\|^2 \sum_{k=n_0}^{n} \frac{t_k(1-t_k)}{(1-\lambda t_k)^3} \leq r_{n_0},$$

which implies that

$$\|x_{n+1} - T(x_{n+1})\|^2 \Gamma_n^\lambda \leq \frac{\varepsilon}{1 - \frac{\Gamma_{n_0-1}^\lambda}{\Gamma_n^\lambda}}.$$

Hence, using (1.8), one gets as $n \to \infty$,

$$\overline{\lim_{n\to\infty}} \|x_{n+1} - T(x_{n+1})\|^2 \Gamma_n^\lambda \leq \varepsilon,$$

which clearly implies (1.10). Finally, the fact that $\{x_n\}$ converges weakly to some fixed point of T immediately follows from the following two facts already proved: $\{\|x_n - q\|\}$ is decreasing, for every $q \in \mathrm{Fix}(T)$ and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$, together with the following two classical lemmas:

Lemma 1. *[3, Corollary 4.18] Let $\{z_n\}$ be a sequence in C. If $\{z_n\}$ converges weakly in $\mathcal{H}$ to some z and $\{z_n - T(z_n)\}$ converges strongly in $\mathcal{H}$ to 0, then $z \in Fix(T)$.*

Lemma 2. *[3, Lemma 2.39] Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and let F be a nonempty subset of $\mathcal{H}$. Assume that*

(a) *For every $q \in F$, the real sequence $\{\|x_n - q\|\}$ converges.*

(b) *Every weak sequential cluster point of the sequence $\{x_n\}$ belongs to F.*

Then $\{x_n\}$ converges weakly in $\mathcal{H}$ to some point in F.

$\boxed{QED}$

3 Proof of the second main result

This section is devoted to prove Theorem 6. The proof uses the following classical result, which is a direct consequence of Riesz Representation Theorem and Banach-Steinhaus Theorem.

Lemma 3. *Let $\{x_n\}$ be a sequence in $\mathcal{H}$. Then $\{x_n\}$ converges weakly in $\mathcal{H}$ if and only if, the real sequence $\{\langle x_n, v\rangle\}$ converges, for every v in the unit ball of $\mathcal{H}$.*

Now, let us prove our second main result.

Proof. *of Theorem 6* Let v be in the unit ball of $\mathcal{H}$ and let $\varepsilon > 0$. There exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n\geq n_0} \frac{e_n}{1 - \lambda t_n} \leq \varepsilon.$$

Let us define the sequence $\{y_n)_{n\geq n_0}$ by

$$\begin{cases} y_{n_0} = x_{n_0} \\ y_{n+1} = (1-t_n)y_n + t_n T((1-\lambda)y_n + \lambda y_{n+1}), \; n \geq n_0. \end{cases}$$

According to Theorem 5, the sequence $\{y_n\}$ converges weakly in $\mathcal{H}$ to some $y_\infty(\varepsilon) \in \mathrm{Fix}(T)$. On the other hand, for any integer $n \geq n_0 + 1$,

$$\|y_{n+1} - x_{n+1}\| \leq (1-t_n)\|y_n - x_n\| + t_n(1-\lambda)\|y_n - x_n\| + t_n\lambda\|y_{n+1} - x_{n+1}\| + e_n.$$

It follows that

$$\|y_{n+1} - x_{n+1}\| \leq \|y_n - x_n\| + \frac{e_n}{1-\lambda t_n}.$$

Hence, for every $n \geq n_0$,

$$\|y_n - x_n\| \leq \sum_{k=n_0}^{n} \frac{e_k}{1-\lambda t_k} \leq \varepsilon. \tag{3.21}$$

Therefore, for every $m > n \geq n_0$,

$$\begin{aligned} |\langle x_m, v\rangle - \langle x_n, v\rangle| &\leq \|y_m - x_m\|\,\|v\| + \|y_n - x_n\|\,\|v\| + |\langle y_m, v\rangle - \langle y_n, v\rangle| \\ &\leq 2\varepsilon\,\|v\| + |\langle y_m, v\rangle - \langle y_n, v\rangle| \\ &\leq 2\varepsilon + |\langle y_m, v\rangle - \langle y_n, v\rangle|. \end{aligned}$$

Since, $\langle y_n, v\rangle \to \langle y_\infty, v\rangle$ as $n \to \infty$, we infer that there exists a positive integer $n_1 \geq n_0$ such that, for every $m > n \geq n_1$,

$$|\langle y_m, v\rangle - \langle y_n, v\rangle| \leq \varepsilon,$$

and so,

$$|\langle x_m, v\rangle - \langle x_n, v\rangle| \leq 3\varepsilon.$$

This means that $\{\langle x_n, v\rangle\}$ is a Cauchy real sequence and therefore it converges. Hence, by the previous lemma, $\{x_n\}$ converges weakly in $\mathcal{H}$ to some x_∞. Now, as n goes to ∞ in the estimation (3.21), one gets, thanks to the fact that the function $x \mapsto \|x\|^2$ is weakly lower semi-continuous (see [3, Lemma 2.35]),

$$\|x_\infty - y_\infty(\varepsilon)\| \leq \underline{\lim}_{n\to\infty} \|x_n - y_n\| \leq \varepsilon.$$

Now, as ε goes to 0, and using the facts that $\mathrm{Fix}(T)$ is a closed subset of $\mathcal{H}$ and $y_\infty(\varepsilon) \in \mathrm{Fix}(T)$, for every $\varepsilon > 0$, one concludes that $x_\infty \in \mathrm{Fix}(T)$. $\boxed{QED}$

4 Application to a nonnegative least square problem and numerical experiments

Let m and n be two positive integers. On the space $\mathbb{R}^m$ (or $\mathbb{R}^n$), we define the Euclidean norm $\|.\|$ by $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x^T x}$. Let

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$$

be the positive orthant, where the notation $x \geq 0$ means that all the components $x_1, \cdots, x_n$ are nonnegative. This section is devoted to the application of Theorem 6 to the numerical resolution of the constrained linear system

$$Ax = b, \ x \in \mathbb{R}^n_+, \tag{4.22}$$

where $A = (a_{ij}) \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$. The solutions of system (4.22) in the sense of the least square method are given by the following definition.

Definition 1. A nonnegative least square solution (NLS) of system (4.22) is a solution of the constrained convex optimization problem

$$\min_{x \in \mathbb{R}^n_+} \ f(x) = \frac{1}{2}\|Ax - b\|^2 \tag{4.23}$$

Let $\Omega(A, b)$ denotes the set of all nonnegative least square solutions of system (4.22). The following result proves that $\Omega(A, b)$ is always nonempty and provides a characterization of this set in term of fixed points of a special family of nonexpansive mappings.

Proposition 1. *The following assertions hold true:*

(a) *The set $\Omega(A, b)$ is a nonempty, closed and convex subset of $\mathbb{R}^n$.*

(b) *For every $\alpha > 0$, $\Omega(A, b)$ is equal to the set of all fixed points of the operator $T_{(\alpha)} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by*

$$T_{(\alpha)}(x) = \Pi_+(x - \alpha(A^T(Ax - b))),$$

where A^T is the transpose matrix of A, and $\Pi_+ : \mathbb{R}^n \to \mathbb{R}^n_+$ is the metric projection from $\mathbb{R}^n$ onto $\mathbb{R}^n_+$.

(c) *If $\alpha \in [0, \frac{2}{\lambda_1}]$, where λ_1 is the greatest eigenvalue of the real symmetric matrix $A^T A$, then the operator $T_{(\alpha)}$ is nonexpansive.*

Proof. (1) Since (4.23) is a constrained convex optimization problem and the objective function f is continuous then, the set of its solutions $\Omega(A,b)$ is a closed and convex subset of $\mathbb{R}^n$ (see [6, Corollary 21.1]). It remains to prove that $\Omega(A,b)$ is nonempty. Let us first notice that the optimal value $f^* = \min_{x\in\mathbb{R}^n_+} f(x)$ is given by

$$f^* = \frac{1}{2}\left(\inf_{v\in A(\mathbb{R}^n_+)} \|v-b\|\right)^2,$$

where $A(\mathbb{R}^n_+) := \{Ax : x \in \mathbb{R}^n_+\}$, which is a nonempty and convex subset of $\mathbb{R}^m$. Moreover, following the proof of ([4, Lemma 6.32]), one can establish that $A(\mathbb{R}^n_+)$ is also a closed subset of $\mathbb{R}^m$. Therefore, the metric projection $P_{A(\mathbb{R}^n_+)}$ from $\mathbb{R}^m$ onto $A(\mathbb{R}^n_+)$ is well-defined, $f^* = \frac{1}{2}\|b - P_{A(\mathbb{R}^n_+)}(b)\|^2$, and $\Omega(A,b) = \{x \in \mathbb{R}^n_+ : Ax = P_{A(\mathbb{R}^n_+)}(b)\}$. This implies, in particular, that $\Omega(A,b)$ is nonempty.
(2) From the variational characterization of the solutions of the constrained convex optimization problems (see [6, Lemma 21.2]), $\bar{x} \in \Omega(A,b)$ if and only if $\bar{x} \geq 0$ and $\langle \nabla f(\bar{x}), x - \bar{x}\rangle \geq 0$, for every $x \geq 0$. Hence, using the classical variational characterization of the metric projection, it follows that $\bar{x} \in \Omega(A,b)$ is equivalent to $\bar{x} = \Pi_+(\bar{x} - \alpha\nabla f(\bar{x}))$, for every $\alpha > 0$. Therefore, the required result follows from the fact that

$$\nabla f(\bar{x}) = A^T(A\bar{x} - b).$$

(3) Let $\alpha \in [0, \frac{2}{\lambda_1}]$, $x, y \in \mathbb{R}^n_+$ and put $w = x - y$. Since the operator Π_+ is nonexpansive,

$$\left\|T_{(\alpha)}(x) - T_{(\alpha)}(y)\right\|^2 \leq \left\|\omega - \alpha A^T A\omega\right\|^2. \tag{4.24}$$

On the other hand, since A^TA is symmetric and positive semi-definite, there exists an orthogonal matrix Q, i.e. $QQ^T = I_n$, and a diagonal matrix $D = diag(\lambda_1, \cdots, \lambda_n)$ such that $A^TA = QDQ^T$, where of $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues of A^TA. Let $\tilde{\omega} = Q^T\omega$, then

$$\begin{aligned}\left\|\omega - \alpha A^T A\omega\right\|^2 &= \left\|(I_n - \alpha A^TA)\omega\right\|^2 = \left\|Q(I_n - \alpha D)Q^T\omega\right\|^2 \\ &= \|(I_n - \alpha D)\tilde{\omega}\|^2 = \sum_{j=1}^n (1-\alpha\lambda_j)^2\tilde{\omega}_j^2\end{aligned}$$

Hence, combining this last estimation with the fact that $\|\tilde{\omega}\| = \|\omega\|$, one obtains

$$\left\|T_{(\alpha)}(x) - T_{(\alpha)}(y)\right\| \leq \kappa(\alpha)\,\|x - y\|,$$

where

$$\kappa(\alpha) = \sup_{1\leq j\leq n} |1 - \alpha\lambda_j|.$$

This ends the proof since the assumption on α implies that $\kappa(\alpha) \leq 1$. $\boxed{QED}$

Remark 4. The metric projection Π_+ from $\mathbb{R}^n$ onto the positive orthant $\mathbb{R}^n_+$ is explicitly defined by:

$$\Pi_+(x) = (\max\{x_1, 0\}, \ldots, \max\{x_n, 0\}),\ \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Remark 5. If $A \in \mathbb{R}^{m\times n}$ with $rank(A) = n$, then the matrix $A^T A$ is positive defined and therefore, all its eigenvalues are positive. Hence, for every $\alpha \in (0, \frac{2}{\lambda_1})$, the constant $\kappa(\alpha)$ belongs to $(0, 1)$. As a result, the operator $T_{(\alpha)}$ is a contraction. So, by the second assertion of the previous proposition, $\Omega(A, b)$ is reduced to a single element.

Remark 6. It is not necessary to calculate the greatest eigenvalue λ_1 of the matrix $A^T A$ in order to determine a range of α that makes the operator $T_{(\alpha)}$ nonexpansive. In fact, according to Gerschgorin's Theorem,

$$\lambda_1 \leq \mu(A) := \max_{i\in\{1,\ldots,n\}} |b_{ij}|\,,$$

where (b_{ij}) denotes the entries of $A^T A$. Hence, according to the last assertion of the previous proposition, for every $\alpha \in (0, \frac{2}{\mu(A)}]$, the operator $T_{(\alpha)}$ is nonexpansive.

According to the previous proposition and remarks, one can apply the two algorithms, stated at the end of the introduction section, to the nonexpansive mapping $T_{(\alpha)}$ in order to obtain numerical approximations of the nonnegative least square solutions of (4.22). This will be the subject of the next subsection.

4.1 Numerical experiments

In this subsection, we consider the particular case of system (4.22), where A and b are given by:

$$A = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 4 \\ 4 \end{pmatrix}.$$

Since

$$b = A\,(2, 1, 3, 1)^T,$$

$$N(A) = \text{span}\{w = (1, -2, 2, -1)^T\},$$

one can easily deduce that the solutions of the corresponding system (4.22) are

$$\Omega(A, b) = \{(2 + t, 1 - 2t, 3 + 2t, 1 - t)^T : -\frac{3}{2} \leq t \leq \frac{1}{2}\}.$$

In our case, $\mu(A) = 23$. Hence, the operator

$$T(x) = T_\alpha(x) = \Pi_+(x - \alpha A^T(Ax - b)),$$

for $\alpha = \frac{2}{23}$, is nonexpansive with $\mathrm{Fix}(T) = \Omega(A, b)$. In the first part of this subsection, we apply Algorithm 1.1 to this operator T, for the particular initial data $x_0 = (0, 0, 0, 0)^T$, the tolerance precision $\varepsilon = 0.001$, error sequence $\{e_n = \frac{1}{1+n^2}\}$ and different values of the parameter $\lambda \in [0, 1]$ and different examples of sequences $\{t_n\}$ in $[0, 1]$ satisfying the conditions of Theorem 3. We summarize the obtained results in the following table:

$\{t_n\}$	$\lambda = 0$	$\lambda = 0.2$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$	$\lambda = 1$
0.1	1469	1453	1427	1399	1367	1343
0.5	293	279	255	237	213	185
0.9	163	153	129	116	91	78
$\frac{1}{\sqrt{n+1}}$	2793	2793	2793	2749	2635	2537
$1 - \frac{1}{\sqrt{n+1}}$	181	165	145	125	103	85
$1 - \frac{1}{n+1}$	155	141	121	101	81	65

Table 1: $N(\{t_n\}, \lambda) := \min\{k : \|T(x_k) - x_k\| \leq 0.001\}$

Clearly, the obtained approximate solution of system (4.22) is given by $x_\varepsilon = (1.5190, 2.0000, 1.9863, 1.4916)^T$. It is approximately equal to $(1.5, 2, 2, 1.5)^T$, which is the point with minimal norm in $\mathrm{Fix}(T)$. In the next experiments, we study for the same operator T the effect of the regularization on the strong convergence speed of the process (1.13)to the particular fixed point $q^\star$ given by Theorem 6. To this end, we apply Algorithm 1.2 in the case where f is identically equal to the zero vector, $\lambda = \frac{1}{2}$, the initial data x_0 is the zero vector, the tolerance precisions are $\varepsilon = 0.1$ and $\varepsilon = 0.01$, the error sequence $\{\mu_n = \frac{1}{1+n^2}\}$, and with various examples of sequences $\{t_n\}$ in the form $\{t_n = \frac{1}{(n+1)^\theta}\}$ such that $0 < \theta < 1$. Let us notice that, in this case, the strong limit of the sequence $\{x_n\}$ is $q^\star = P_{\mathrm{Fix}(T)}(0) = (1.5, 2, 2, 1.5)^T$. We summarize the obtained numerical results in the next table. The notation ND (Not Defined) means that $N(\varepsilon, \theta) > 10^4$.

ε	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 0.9$
0.1	ND	ND	739	171	105
0.01	ND	ND	ND	3059	1357

Table 2: $N(\varepsilon, \theta) := \min\{k : \|T(x_k) - x_k\| \leq \varepsilon\}$

From these numerical experiments, we draw the following conclusions:

(a) The process convergence speed is better when the parameter λ is close to 1 and the sequence $\{t_n\}$ converges quickly to 1. This fact is theoretically expected as it was already mentioned in Remark 1.

(b) The weak limit of the process does not depend on the parameter λ and the sequence $\{t_n\}$. It seems that it only depends on the initial data x_0.

(c) The strong convergence of the process (1.13) is very slow. So, one can deduce that although the Tikhanov regularizing term $t_n f(x_n)$ forces the generated sequence $\{x_n\}$ to converge strongly to a particular fixed point of T, it has a clear negative effect on the the rate of the convergence of the sequence $\{x_n\}$.

5 Conclusion

In this paper we have studied the iterative process:

$$x_{n+1} = (1 - t_n)x_n + t_n T((1 - \lambda)x_n + \lambda x_{n+1}),\ n = 0, 1, 2, \dots \tag{5.25}$$

where T is a nonexpansive mapping and $\lambda \in [0, 1]$ is a fixed parameter and $\{t_n\} \in [0, 1]$ is a real sequence. We proved that, if the sequence $\{t_n\}$ satisfies a suitable condition, then for any initial data x_0, the sequence $\{x_n\}$ generated in this process converges weakly to a fixed point of T if any exists. Moreover we established a precise estimate on the rate of convergence of the residual sequence $r_n = T(x_n)-x_n$ to 0. We have also studied the stability of the considered process. In fact, we have established that a small perturbation on the definition of the sequence $\{x_n\}$ does not affect the weak convergence property of the process. In the last section of the paper, we studied numerically an optimization problem in order to have an idea about the effect of the parameter λ on the speed of the convergence of the process (5.25).

Acknowledgement

The authors acknowledge the Deanship of Scientific Research at King Faisal University for the financial support of this work under Annual Research Project Track (Grant No. 180082). The authors are also indebted to the anonymous referee for his/her comments and suggestions.

References

[1] M. A. Alghamdi, N. Shahzad, H. K. Xu, *The implicit midpoint rule for nonexpansive mappings*, Fixed Point Theory and Applications **96** (2014).

[2] C. Banach, *Sur les operateurs dans les ensembles abstraits et leur applications aux equations integrales*, Fundamenta Mathematicae **3**(1922), 133–181.

[3] H. H. Bauschke, P. L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*, CMS Books in Mathematics, Springer, New York, 2011.

[4] A. Beck, *Introduction to nonlinear optimization theory, algorithms, and applications with Matlab*, SIAM and Mathematical Optimization Sociaty, Philadelphia, 2014.

[5] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse problems **20** (2004), 103–120.

[6] E. K. P. Chong, S. H. Zak, *An introduction to optimization*, Wiley-Interscience Series in Discrete Mathematics and Optimization, 2001.

[7] G. Marino, R. Zaccono, *On strong convergence of some mid point type methods for nonexpansive mappings*, J. Nonlinear Var. Anal. **1(2)**(2017) 15–174.

[8] W. R. Mann, *Mean value method in iteration*, Pro. Amer. Math. Soc. **4**(1953) 506–510.

[9] R. May, Z. Binali, *On the strong convergence of a perturbed algorithm to the unique solution of a variational inequality problem*, J. Nonlinear Var. Anal. **7(2))**(2023) 209–221.

[10] Y. Ke, C. Ma, *The generalized viscosity implicit rules of nonexpansive mapping in Hilbert spaces*, Fixed Point Theory and Applications **190**(2015).

[11] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Mon. **72(9)**(1965), 1004–1006.

[12] M. A. Krasnosel'ki, *Two remarks on the method of successive approximations*, Uspehi Mat. Nauk **10**(1955) 123–127 (Russian).

[13] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67**(1979) 274–276.

[14] H. K. Xu, *A note on the implicit midpoint rule for nonxpansive mappings*, linear and nonlinear analysis **1**(2015) 113–119.

[15] H. K. Xu, M. A. Alghamdi, N. Shahzad, *The viscosity techniques for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory and Applications **41** (2015).