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Unitals without O'Nan configurations are classical if they admit all translations

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Abstract. We prove the statement in the title: if a (finite) unital admits all translations and contains no O'Nan configurations then the unital is classical, i.e., isomorphic to the Hermitian unital of the same order.

Keywords: unital, O'Nan configuration, translation, Hermitian unital

MSC 2020 classification: MSC:51E05 MSC:51A45 MSC:05B05

Introduction

We attempt a brief overview of the background for the present note. See Section 1 for precise definitions.

In [13], Wilbrink has characterized the finite classical unitals (i.e., those defined by a Hermitian form, see 1.3 below) among all finite unitals by three elementary conditions (see 2.2 below). We show that these conditions are satisfied if the unital in question admits all translations, and contains no O'Nan configurations. Then Wilbrink's characterization gives the result announced in the title.

Unitals admitting all translations have been studied in [4]. Using the classification of finite simple groups, it is proved in that paper that such unitals are classical. Our present treatment is much more elementary, but imposes the fairly strong hypothesis that no O'Nan configurations appear in the unital. Using translations (assumed to exist by our second strong hypothesis) we then check Wilbrink's conditions (see 2.2), and obtain the result.

It has been conjectured (see [9, p. 102], [1, p. 87]) that finite classical unitals are characterized by the absence of O'Nan configurations. To the author's knowledge, this conjecture has not been proved yet. In many of the known non-classical unitals, O'Nan configurations have been found.

1 Unitals, translations

Definitions 1.1. Throughout the present paper, let $\mathcal{U} = (P, \mathcal{B}, \in)$ be a finite *unital* of order q; i.e., the set P of points has size $q^3 + 1$ with $q > 1$, the set $\mathcal{B}$ of blocks consists of subsets of size $q+1$ in P, and each 2-element subset $\{x, y\} \subset P$ is contained in exactly one block (called the *joining* block, and denoted by $x \vee y$). In other words, $\mathcal{U}$ is a 2-$(q^3 + 1, q + 1, 1)$-design.

An *O'Nan configuration* is a set of 4 mutually intersecting blocks together with 6 points, such that each one of the 6 points lies on exactly two of these blocks. See Figure 1. (In axiomatic projective geometry, that configuration is called Veblen-Young configuration.)

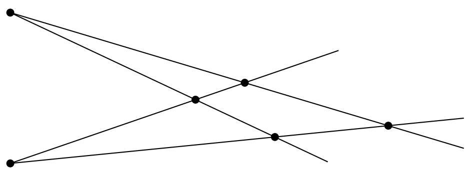

Figure 1: An O'Nan configuration.

A *translation* (with center c) of the unital $\mathcal{U}$ is an automorphism of $\mathcal{U}$ that fixes a point c and every block through c.

Lemma 1.2 ([4, Thm. 1.3])**.** *For each point c of $\mathcal{U}$, the set $T_{[c]}$ of all translations with center c forms a subgroup of* $\mathrm{Aut}(\mathcal{U})$ *that acts semi-regularly on $P \smallsetminus \{c\}$.*

In particular, the group $T_{[c]}$ induces a semi-regular group on $B \smallsetminus \{c\}$, for each block B containing c. We say that $\mathcal{U}$ *admits all translations with center c* if $T_{[c]}$ is transitive on $B \smallsetminus \{c\}$. (This is equivalent to $|T_{[c]}| = q$.) If $\mathcal{U}$ admits all translations with center c, for each $c \in P$, we say that $\mathcal{U}$ *admits all translations*. In [4], it has been proved that each unital admitting all translations is isomorphic to the classical unital of the same order (see 1.3 below). That proof uses the classification of finite simple groups.

There do exist examples of unitals admitting all translations for many (but not all) centers (see [5], [7]), and unitals admitting all translations for just one single center (see [5, Sec. 5]). At least some of those unitals do contain O'Nan configurations (e.g., see [6, 6.7, 6.10]).

Examples 1.3. Let $E|F$ be a separable quadratic field extension, and let $\sigma \colon s \mapsto \bar{s}$ denote the generator of the Galois group. There is an (essentially unique) isotropic non-degenerate σ-Hermitian form h on E^3. The classical unital $\mathcal{H}_{E|F}$ (cp. [2, p. 104], [1, 2.1, 2.2, see also p. 29]) has the set $P_{E|F}$ of all one-dimensional isotropic subspaces of E^3 as point set, the blocks are the intersections of $P_{E|F}$

with lines that meet that set in more than one point. Up to a choice of basis, the Hermitian form h maps $(x,y) \in E^3 \times E^3$ with $x = (x_0, x_1, x_2)$, $y = (y_0, y_1, y_2)$ to $x_0\overline{y_2} - x_1\overline{y_1} + x_2\overline{y_0}$.

We note that the classical unital $\mathcal{U}_{E|F}$ does not contain any O'Nan configurations. (See [8, Proposition, p. 507] for the finite case, and [3, 2.2]).

The automorphism group $\mathrm{Aut}(\mathcal{H}_{E|F})$ of the classical unital $\mathcal{H}_{E|F}$ is the group $\mathrm{P\Gamma U}(E^3,h)$ induced by the group $\mathrm{\Gamma U}(E^3,h)$ of all semi-similitudes of the form h. (See [8] for the finite case, and [12] for the general case; cp. [11, 6.1, 5.6].) If the form is given as above, the subgroup

$$\Xi := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, z \in E, z + \bar{z} = x\bar{x} \right\}$$

is contained in the stabilizer of the point $E(0,0,1) \in P_{E|F}$, and

$$T_{[E(0,0,1)]} = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, z \in E, z + \bar{z} = 0 \right\}$$

is the group of translations with center $E(0,0,1)$. The group Ξ, together with any element $\gamma \in \mathrm{PSU}(E^3,h)$ that moves $E(0,0,1)$, shows that $\mathrm{PSU}(E^3,h) \leq \mathrm{Aut}(\mathcal{H}_{E|F})$ acts doubly transitively on $P_{E|F}$. For instance, we could use the element induced by $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \in \mathrm{SU}(E^3,h)$.

In particular, if E is finite of square order q^2 then there is a unique subfield F of order q. The involution σ is the appropriate power of the Frobenius endomorphism, it maps $s \in E$ to $\bar{s} = s^q$. Then $\mathcal{H}_{E|F} \cong \mathcal{H}_{\mathbb{F}_{q^2}|\mathbb{F}_q}$ is a unital of order q; the groups considered above show that this unital admits all translations.

2 Wilbrink's conditions

Definitions 2.1. Consider $B \in \mathcal{B}$ and $x \in P \setminus B$. A block B' with $x \notin B'$ is called *x-parallel to B* if B' meets each block joining x with a point on B.

If $\mathcal{U}$ does not contain any O'Nan configurations then there exists at most one block through a given point $y \neq x$ that is x-parallel to a given block B.

If $\tau \in T_{[x]}$ is a translation with center x then the image B^τ of B under τ is x-parallel to B.

Theorem 2.2 ([13]). *Let $\mathcal{U} = (P, \mathcal{B}, \in)$ be a unital of order q satisfying the following conditions.*

(I) *There are no O'Nan configurations in $\mathcal{U}$.*

(II) *Consider $L \in \mathcal{B}$ and $x, y \in P$ such that $x \notin L$ and $(x \vee y) \cap B \neq \emptyset$. Then there exists an x-parallel block L through y.*

(III) *Consider three blocks M_0, M_1, M_2 through a common point x, and points $y_i, z_i \in M_i \setminus \{x\}$ for $i \in \{0,1,2\}$. If $z_0 \vee z_i$ is x-parallel to $y_0 \vee y_i$ for $i \in \{1,2\}$ then $z_1 \vee z_2$ is x-parallel to $y_1 \vee y_2$.*

Then $\mathcal{U}$ is isomorphic to the classical unital of order q.

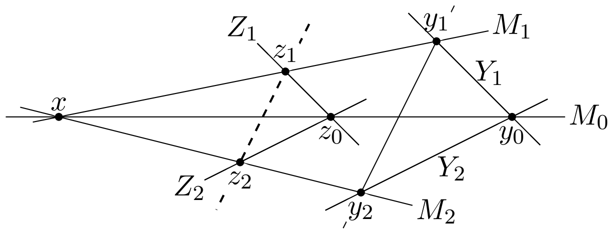

Figure 2: Wilbrink's condition (III)

Theorem 2.3. *Let $\mathcal{U} = (P, \mathcal{B}, \in)$ be a unital of order q admitting all translations, and with no O'Nan configurations. Then $\mathcal{U}$ is isomorphic to the classical unital of order q.*

Proof. It remains to show that $\mathcal{U}$ satisfies Wilbrink's conditions (II), (III), see 2.2.

Consider $B \in \mathcal{B}$ and $x \in P \setminus B$. If $x \vee y$ meets B in a point w, say, then our assumptions secure the existence of a translation $\tau \in T_{[x]}$ mapping w to y. Clearly the image B' of B under τ meets each block through x that also meets B. So B' is x-parallel to B, and condition (II) is verified.

Now consider three blocks M_0, M_1, M_2 through x, and points $y_i, z_i \in M_i \setminus \{x\}$ for $i \in \{0,1,2\}$ (see Figure 2). Let τ be the translation with center x that maps y_0 to z_0. For $i \in \{1,2\}$, the image Z_i of $Y_i := y_0 \vee y_i$ under τ contains z_0, and meets each block through x that meets Y_i. So Z_i is x-parallel to Y_i. By the absence of O'Nan configurations, we know that the x-parallel block to Y_i through z_0 is unique, and infer $Z_i = z_0 \vee z_i$. Thus z_i is the image of y_i under τ, and $z_1 \vee z_2$ is the image of $y_1 \vee y_2$ under τ. This yields that $z_1 \vee z_2$ is x-parallel to $y_1 \vee y_2$, and condition (III) is verified. QED

An application of the present theorem is given in [10, 6.4].

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