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Hypergeometric series, lemniscate functions, q–extensions and Jacobi elliptic functions

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Abstract. In this paper, we provide a solution to an open problem posed by Campbell and Chu [12] concerning the explicit evaluation of a constant known as the 'lemniscate-like constant.' We demonstrate that by utilizing tools related to the primary Jacobi elliptic functions, we can derive a closed-form expression in terms of $q-$generalizations of Zeta and Polylogarithm functions and well-known mathematical constants. Lastly, we establish that our primary outcome establishes a non-obvious connection between various and disparate mathematical entities.

Keywords: Closed forms, hypergeometric functions, Jacobi elliptic functions, $q-$Zeta and $q-$Polylogarithm functions, lemniscate functions.

MSC 2020 classification: 33E05, 05A30, 33C20, 11S40

1 Introduction

The analysis of closed forms in terms of important mathematical constants and special functions for hypergeometric functions is a classical study. It is well known that this type of research is of interest, not only from a mathematical point of view (for example, for the study of Diophantine approximations, in order to prove the irrationality of certain constants) but also in other scientific fields and, sometimes it provides a possible link between seemingly distant topics; a very exhaustive illustration of these facts can be found in [8]. In the large family of hypergeometric functions, those that can be written as a series whose terms are powers of central binomial coefficients have been widely studied; a classical example is the work of Ramanujan about the series involving $1/\pi$ (for a survey of this topic see for example [6], and for formulas via hypergeometric transformations see [17]).

In this paper, we focus on a class of the so called *lemniscate-like constants*, introduced by Campbell and Chu in [12]. We recall that the lemniscate constants

$$L_1 := \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}},\ L_2 := \frac{\sqrt{2\pi^3}}{\Gamma^2\left(\frac{1}{4}\right)}$$

are important constants in the history of mathematics (see, e.g., [20, 29]). It is well known that

$$L_1 = \int_0^1 \frac{1}{\sqrt{1-t^4}}dt = \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{1}{4n+1};$$

$$L_1 = \int_0^1 \frac{t^2}{\sqrt{1-t^4}}dt = \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{1}{4n+3}$$

and so it is also known the connection with the classical Gauss' arc lemniscate sine function

$$\operatorname{arcsinlem}(w) := \int_0^w \frac{1}{\sqrt{1-t^4}}dt = \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{w^{4n+1}}{4n+1} \tag{1.1}$$

(see [4] for more details about the previous function) and its hyperbolic form

$$\operatorname{arcsinhlem}(w) := \int_0^w \frac{1}{\sqrt{1+t^4}}dt = \sum_{n\geq 0} \binom{2n}{n} \frac{(-1)^n}{4^n} \frac{w^{4n+1}}{4n+1}. \tag{1.2}$$

In consideration of the research that has been done over the years regarding the classical lemniscate constants, were introduced in [12] the lemniscate-like constants as series of the form

$$L_1^{f_n} := \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{f_n}{4n+1}, \; L_2^{f_n} := \sum_{n\geq 0} \binom{2n}{n} \frac{1}{4^n} \frac{f_n}{4n+3}$$

where $f_n : \mathbb{N}_0 \to \mathbb{C}$ are suitable arithmetic functions, that is, sequences that do not contain $\frac{1}{4n+1}$ and $\frac{1}{4n+3}$, respectively, or a factor that cancels $\binom{2n}{n}$ or $1/4^n$. In particular, the authors studied the cases $f_n := O_n$, O_{2n} where

$$O_n := \sum_{k=1}^{n} \frac{1}{2k-1}$$

is the odd harmonic number of order n. As pointed out in [12], the case $f_n := H_n$, where H_n is the n-th harmonic number, cannot be treated with the technique proposed by Campbell and Chu. Furthermore, known methods also appear to be insufficient in seeking a closed form for these particular series. In this paper, we show a new way to deal with $L_1^{H_n}$ and we prove that this choice of f_n is very interesting since the problem of finding a closed form for this particular lemniscate-like constant can be interpreted in multiple ways.

This paper is structured as follows: In Section 2, we introduce the notations, definitions, and useful properties that we will employ in subsequent sections. In Section 3, we demonstrate that the challenging evaluation of the proposed lemniscate-like constant can be accomplished by employing tools from the theory of elliptic functions and certain $q-$extensions of the Riemann Zeta function and Polylogarithm functions. In Section 4, we establish that $L_1^{H_n}$ can be interpreted from different perspectives, and consequently, the closed-form expression presented in Theorem 2 establishes a compelling connection between various topics.

2 Definitions, settings and preliminary results

In this section we introduce the notations and the definitions that we will use in the whole paper. The symbol

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}},\ 0<k<1$$

will denote the complete elliptic integral of the first kind and $K'(k) := K(k')$, where $k' := \sqrt{1-k^2}$. We will always denote the Beta function with the symbol $B(a,b)$, where

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0$$

(see [27], sections 19.2(ii) and 5.12).

As we anticipated in the introduction, we need to define the $q-$analog of the Riemann Zeta function and the $q-$analog of the polylogarithm function.

Definition 1. We define the $q-\zeta$ function as

$$\zeta_q(s) := \sum_{n\geq 1} \frac{1}{n^{1-s}} \frac{q^n}{1-q^n},\ |q|<1,\ s\in\mathbb{C} \tag{2.3}$$

and the $q-$Polylogarithm of order s as

$$\mathrm{Li}_s(x;q) := \sum_{n\geq 1} \frac{1}{n^{1-s}} \frac{x^n q^n}{1-q^n},\ |xq|<1, |q|\neq 1,\ s\in\mathbb{C}. \tag{2.4}$$

Note that this functions, and their generalizations, are well known and studied (for the definition of $\zeta_q(s)$ see, for example, [32]; for $\mathrm{Li}_s(x;q)$ see [31]); some examples include studies about the irrationality of some special values (see, e.g., [2, 9, 28, 33]), linear independence over suitable fields (see, e.g., [10, 31]), etc. Also note that the classical definition of (2.3) and (2.4) have been constructed for values of s that belong to subsets of $\mathbb{C}$ (more precisely, taking s as natural number) but clearly there is no convergence issue if $s \in \mathbb{C}$, so we will work in this setting, even if we are interested only in particular integer values of s. We recall that there exist different definitions for $q-$analogs of the classical ζ function (see [16, 25]), however, it is possible to prove, defining ζ_q as in (2.3), that

$$\lim_{\substack{q\to 1\\ |q|<1}} (1-q)^s \zeta_q(s) = (s-1)!\zeta(s)$$

for $s = 2, 3, \ldots$ (see [28]), then we have a interesting connection with $\zeta(s)$.

Now, taking $q := e^{-\pi K'(k)/K(k)}$, we recall the main Jacobian elliptic functions

$$\operatorname{sn}(z,k) := \frac{\theta_3(0,q)\,\theta_1(\mathscr{Z},q)}{\theta_2(0,q)\,\theta_4(\mathscr{Z},q)};$$

$$\operatorname{cn}(z,k) := \frac{\theta_4(0,q)\,\theta_2(\mathscr{Z},q)}{\theta_2(0,q)\,\theta_4(\mathscr{Z},q)}$$

$$\operatorname{dn}(z,k) := \frac{\theta_4(0,q)\,\theta_3(\mathscr{Z},q)}{\theta_3(0,q)\,\theta_4(\mathscr{Z},q)}$$

(see [27], 22.1), where $\mathscr{Z} := \frac{\pi z}{2K(k)}$ and the theta functions are the Jacobi theta functions (see [27], 20.2(i)).

3 Evaluation of a lemniscate-like constant

In this section we prove that the lemniscate-like constant $L_1^{H_n}$ can be written in terms of the previous defined $q-$functions and well known mathematical constant. Indeed, we have the following theorem:

Theorem 2. *Using the notation of the previous section, we have that*

$$\begin{aligned}\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^n}\frac{H_n}{4n+1} &= \frac{\Gamma\left(\frac{1}{4}\right)^2(\pi-6\log(2))}{8\sqrt{2\pi}} - \frac{\Gamma\left(\frac{1}{4}\right)\pi^{3/2}}{4\Gamma\left(\frac{3}{4}\right)}\\ &+4\sqrt{2}K\left[\log\left(\frac{2K}{\pi}\right)+\frac{2C}{\pi}+2\zeta_{q^4}(0)-5\zeta_{q^2}(0)\right.\\ &+2\zeta_q(0)+\frac{2i}{\pi}\left(\mathrm{Li}_{-1}\left(-1;q^4\right)-4\mathrm{Li}_{-1}\left(i;q^2\right)\right)\\ &\left.+\frac{i}{\pi}\left(4\mathrm{Li}_{-1}(i;q)-\mathrm{Li}_{-1}\left(-1;q^2\right)\right)\right]\end{aligned}$$

where C is the Catalan's constant, $K := K\left(\frac{1}{\sqrt{2}}\right) = \frac{4\Gamma\left(\frac{5}{4}\right)^2}{\sqrt{\pi}}$*, i is the imaginary unit and $q = e^{-\pi}$.*

Proof. Following [12] we have

$$\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^n}\frac{H_n}{4n+1} = \frac{\Gamma\left(\frac{1}{4}\right)^2(\pi - 6\log(2))}{8\sqrt{2\pi}} + \int_0^1 \frac{\log(1+u)}{(1-u^2)^{3/4}}du$$

and so using the Taylor series of $\log(1+u)$, exchanging series and integral and computing the integral in terms of the Euler Beta function, we get

$$\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^n}\frac{H_n}{4n+1} = \frac{\Gamma\left(\frac{1}{4}\right)^2(\pi - 6\log(2))}{8\sqrt{2\pi}} + \sum_{n\geq 1}\frac{(-1)^{n+1}}{2n}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{4}\right)} \tag{3.5}$$

and since the series in the right side of (3.5) is absolutely convergent, we obtain

$$\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^n}\frac{H_n}{4n+1} = \frac{\Gamma\left(\frac{1}{4}\right)^2(\pi - 6\log(2))}{8\sqrt{2\pi}} - \frac{\Gamma\left(\frac{1}{4}\right)}{4}\sum_{n\geq 1}\frac{1}{n}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{4}\right)} + \frac{1}{2}\sum_{n\geq 0}\frac{1}{2n+1}B\left(n+1,\frac{1}{4}\right)$$

where $B(a,b)$ is the Beta function. Clearly

$$\sum_{n\geq 1}\frac{1}{n}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{4}\right)} = \frac{\pi^{3/2}}{\Gamma\left(\frac{3}{4}\right)}$$

since it is a known form of hypergeoemtric function $3F2(1)$ or, similarly, since it can be written in terms of a simple integral replacing the ratio of Gamma function as a Beta function. Hence, in the second series, rewriting $B\left(n+1,\frac{1}{4}\right)$ as an integral, using the change of variables $x \mapsto \frac{1-x}{1+x}$, making simple manipulations and exploiting the Taylor series of $(1+x^4)^{-1/2}$, we obtain

$$\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^n}\frac{H_n}{4n+1} = \frac{\Gamma\left(\frac{1}{4}\right)^2(\pi - 6\log(2))}{8\sqrt{2\pi}} - \frac{\Gamma\left(\frac{1}{4}\right)\pi^{3/2}}{4\Gamma\left(\frac{3}{4}\right)} + 4\sqrt{2}\sum_{n\geq 0}\left(\frac{1}{4}\right)^n\binom{2n}{n}\frac{(-1)^n}{(4n+1)^2}. \tag{3.6}$$

So, with this technique, we "removed" the H_n term keeping the "structure" of hypergeometric series. Hence, the problem boils down to find a closed form for the last series in (3.6). Now, by (1.2), we have

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^n\binom{2n}{n}\frac{(-1)^n}{(4n+1)^2} = -\int_0^1 \frac{\log(z)}{\sqrt{1+z^4}}dz = -\int_0^{\frac{2\Gamma\left(\frac{5}{4}\right)^2}{\sqrt{\pi}}} \log\left(\operatorname{sinhlem}(z)\right)dz \tag{3.7}$$

and since

$$\operatorname{sinhlem}(z) = \frac{1-\operatorname{cn}\left(2z;\frac{1}{\sqrt{2}}\right)}{\operatorname{sn}\left(2z;\frac{1}{\sqrt{2}}\right)} = \frac{\operatorname{sn}\left(z;\frac{1}{\sqrt{2}}\right)\operatorname{dn}\left(z;\frac{1}{\sqrt{2}}\right)}{\operatorname{cn}\left(z;\frac{1}{\sqrt{2}}\right)},\ |z|\leq K\left(\frac{1}{\sqrt{2}}\right)$$

(for the first equality, see [26], equation (2.10); for the second equality, just apply the sum of squares and double argument formulas, see, for example, [27], sections 22.6(i) and 22.6(ii)) and observing that

$$\frac{2\Gamma\left(\frac{5}{4}\right)^2}{\sqrt{\pi}} = \frac{K\left(\frac{1}{\sqrt{2}}\right)}{2} =: \frac{K}{2},$$

then we have to understand if the following integrals

$$\int_0^{\frac{K}{2}} \log\left(\operatorname{sn}\left(z;\frac{1}{\sqrt{2}}\right)\right)dz,\quad \int_0^{\frac{K}{2}} \log\left(\operatorname{dn}\left(z;\frac{1}{\sqrt{2}}\right)\right)dz,$$
$$\int_0^{\frac{K}{2}} \log\left(\operatorname{cn}\left(z;\frac{1}{\sqrt{2}}\right)\right)dz. \tag{3.8}$$

admit a closed form.

It is interesting to note that similar integrals have been studied by Glaisher [21]. However, the identities, relations, and strategies developed in the cited work cannot be used for our problem. The main issue, in a few words, is that in our case, we integrate from 0 to $K/2$, and this fact implies that we lose some crucial symmetries and properties linked to the periodicity of the Jacobi elliptic functions.

To deal with these integrals, we recall the following Fourier expansions:

$$\log\left(\operatorname{sn}(z;k)\right) = \log\left(\frac{2K}{\pi}\right) + \log\left(\sin\left(\frac{\pi z}{2K}\right)\right) - 4\sum_{n\geq 1}\frac{1}{n}\frac{q^n}{1+q^n}\sin^2\left(\frac{n\pi z}{2K}\right) \tag{3.9}$$

$$\log\left(\operatorname{cn}\left(z;k\right)\right)=\log\left(\cos\left(\frac{\pi z}{2K}\right)\right)-4\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+\left(-1\right)^{n}q^{n}}\sin^{2}\left(\frac{n\pi z}{2K}\right) \tag{3.10}$$

$$\log\left(\operatorname{dn}\left(z;k\right)\right)=-8\sum_{n\geq1}\frac{1}{2n-1}\frac{q^{2n-1}}{1-q^{4n-2}}\sin^{2}\left(\frac{\left(2n-1\right)\pi z}{2K}\right) \tag{3.11}$$

where $q:=e^{-\pi K'/K}=e^{-\pi}$, since $K'\left(\frac{1}{\sqrt{2}}\right)=K\left(\sqrt{1-\frac{1}{2}}\right)=K\left(\frac{1}{\sqrt{2}}\right)$ (see, e.g., [23], section 8.146, equations $20,21$ and 22). Let us now consider the first integral of (3.8). From (3.9), we firstly focus on

$$\int_{0}^{\frac{K}{2}}\log\left(\sin\left(\frac{\pi z}{2K}\right)\right)dz=\frac{2K}{\pi}\int_{0}^{\frac{\pi}{4}}\log\left(\sin\left(y\right)\right)dy=-\frac{K}{2\pi}\left(2C+\pi\log\left(2\right)\right)$$

where C is the Catalan's constant. Now, by the absolute convergence of

$$\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}},$$

we can consider

$$-4\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}}\int_{0}^{\frac{K}{2}}\sin^{2}\left(\frac{n\pi z}{2K}\right)dz=-K\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}}\left(1-\frac{2\sin\left(\frac{\pi n}{2}\right)}{\pi n}\right)$$

$$=-K\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}}+\frac{2K}{\pi}\sum_{n\geq1}\frac{1}{n^{2}}\frac{q^{n}\sin\left(\frac{\pi n}{2}\right)}{1+q^{n}}.$$

Now we recall al classical properties of the Lambert series, that is, if $f(n)$ is an arithmetical function, then

$$\sum_{n\geq1}\frac{f\left(n\right)q^{n}}{1+q^{n}}=\sum_{n\geq1}\frac{f\left(n\right)q^{n}}{1-q^{n}}-2\sum_{n\geq1}\frac{f\left(n\right)q^{2n}}{1-q^{2n}} \tag{3.12}$$

which can be proved with simple manipulations. Then, we get

$$\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}}=\zeta_{q}\left(0\right)-2\zeta_{q^{2}}\left(0\right)$$

and

$$\begin{aligned}\sum_{n\geq1}\frac{1}{n^{2}}\frac{q^{n}\sin\left(\frac{\pi n}{2}\right)}{1+q^{n}} &= \sum_{n\geq1}\frac{1}{\left(2n-1\right)^{2}}\frac{q^{2n-1}\left(-1\right)^{n+1}}{1+q^{2n-1}}\\ &= -i\sum_{n\geq1}\frac{1}{\left(2n-1\right)^{2}}\frac{q^{2n-1}i^{2n-1}}{1+q^{2n-1}}\\ &= -i\sum_{n\geq1}\frac{1}{n^{2}}\frac{q^{n}i^{n}}{1+q^{n}}+i\sum_{n\geq1}\frac{1}{\left(2n\right)^{2}}\frac{q^{2n}\left(-1\right)^{n}}{1+q^{2n}}\\ &= -i\left(\mathrm{Li}_{-1}\left(i;q\right)-2\mathrm{Li}_{-1}\left(i;q^{2}\right)\right.\\ &\left.-\frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^{2}\right)+\frac{1}{2}\mathrm{Li}_{-1}\left(-1;q^{4}\right)\right).\end{aligned}$$

Note that $\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1+q^{n}}$ can be written in also in terms of the Euler $\Phi(q)$ function

$$\Phi(q):=\prod_{n\geq1}\left(1-q^{n}\right)$$

or in terms of the $q-$pochhammer symbol

$$\left(a;q\right)_{\infty}:=\prod_{n\geq0}\left(1-aq^{n}\right)$$

(see [27], 17.2.4) since

$$\log\left(\Phi(q)\right)=\log\left(\left(q;q\right)_{\infty}\right)=-\sum_{n\geq1}\frac{1}{n}\frac{q^{n}}{1-q^{n}}=-\zeta_{q}\left(0\right)$$

(see [3], section 14, for further details and the connections between $\Phi(q)$ and the partition function $p(n)$). Hence we can finally write

$$\begin{aligned}\int_{0}^{\frac{K}{2}}\log\left(\mathrm{sn}\left(z;\frac{1}{\sqrt{2}}\right)\right)dz &= \frac{K}{2}\log\left(\frac{2K}{\pi}\right)-\frac{K}{2\pi}\left(2C+\pi\log\left(2\right)\right)\\ &-K\left(\zeta_{q}\left(0\right)-2\zeta_{q^{2}}\left(0\right)\right)\\ &-\frac{2K}{\pi}i\left(\mathrm{Li}_{-1}\left(i;q\right)-2\mathrm{Li}_{-1}\left(i;q^{2}\right)\right.\\ &\left.-\frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^{2}\right)+\frac{1}{2}\mathrm{Li}_{-1}\left(-1;q^{4}\right)\right).\end{aligned}$$

Now we have to deal with (3.10) and so, arguing similarly to the previous part, the problem boils down to the evaluation of

$$\int_{0}^{\frac{K}{2}}\log\left(\cos\left(\frac{\pi z}{2K}\right)\right)dz=\frac{2K}{\pi}\int_{0}^{\frac{\pi}{4}}\log\left(\cos\left(y\right)\right)dy=\frac{K}{\pi}\left(C-\frac{\pi}{2}\log\left(2\right)\right)$$

and

$$-4\sum_{n\geq 1}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n}\int_0^{\frac{K}{2}}\sin^2\left(\frac{n\pi z}{2K}\right)dz$$
$$=-K\sum_{n\geq 1}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n}\left(1-\frac{2\sin\left(\frac{\pi n}{2}\right)}{\pi n}\right).$$

We observe, using (3.12) and the absolute convergence of the series, that

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n}\frac{q^n}{1+(-1)^n q^n} &= \sum_{n\geq 1}\frac{1}{2n}\frac{q^{2n}}{1+q^{2n}}+\sum_{n\geq 1}\frac{1}{2n-1}\frac{q^{2n-1}}{1-q^{2n-1}}\\ &=\sum_{n\geq 1}\frac{1}{2n}\frac{q^{2n}}{1-q^{2n}}-\sum_{n\geq 1}\frac{1}{n}\frac{q^{4n}}{1-q^{4n}}+\sum_{n\geq 1}\frac{1}{2n-1}\frac{q^{2n-1}}{1-q^{2n-1}}\\ &=\sum_{n\geq 1}\frac{1}{n}\frac{q^n}{1-q^n}-\sum_{n\geq 1}\frac{1}{n}\frac{q^{4n}}{1-q^{4n}}=\zeta_q(0)-\zeta_{q^4}(0)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\geq 1}\frac{1}{n^2}\frac{q^n\sin\left(\frac{\pi n}{2}\right)}{1+(-1)^n q^n} &= \sum_{n\geq 1}\frac{1}{(2n-1)^2}\frac{q^{2n-1}(-1)^{n+1}}{1-q^{2n-1}}\\ &=-i\sum_{n\geq 1}\frac{1}{(2n-1)^2}\frac{q^{2n-1}i^{2n-1}}{1-q^{2n-1}}\\ &=-i\left(\sum_{n\geq 1}\frac{1}{n^2}\frac{q^n i^n}{1-q^n}-\frac{1}{4}\sum_{n\geq 1}\frac{1}{n^2}\frac{q^{2n}i^{2n}}{1-q^{2n}}\right)\\ &=-i\left(\mathrm{Li}_{-1}(i;q)-\frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^2\right)\right)\end{aligned}$$

hence

$$\begin{aligned}\int_0^{\frac{K}{2}}\log\left(\mathrm{cn}\left(z;\frac{1}{\sqrt{2}}\right)\right)dz &= \frac{K}{\pi}\left(C-\frac{\pi}{2}\log(2)\right)-K\left(\zeta_q(0)-\zeta_{q^4}(0)\right)\\ &\quad-\frac{2K}{\pi}i\left(\mathrm{Li}_{-1}(i;q)-\frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^2\right)\right).\end{aligned}$$

Finally, we have to consider (3.11) and so

$$-\sum_{n\geq 1}\frac{8q^{2n-1}}{(2n-1)\left(1-q^{4n-2}\right)}\int_0^{\frac{K}{2}}\sin^2\left(\frac{(2n-1)\pi z}{2K}\right)dz$$

$$= -2K \sum_{n \geq 1} \frac{1}{2n-1} \frac{q^{2n-1}}{1-q^{4n-2}} - \frac{4K}{\pi} \sum_{n \geq 1} \frac{1}{(2n-1)^2} \frac{q^{2n-1}(-1)^n}{1-q^{4n-2}}$$

$$= -2K \left(\zeta_q(0) - \frac{1}{2}\zeta_{q^2}(0) \right) + 2K \left(\zeta_{q^2}(0) - \frac{1}{2}\zeta_{q^4}(0) \right)$$

$$-\frac{4K}{\pi} i \left(\mathrm{Li}_{-1}(i;q) - \frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^2\right) \right) + \frac{4K}{\pi} i \left(\mathrm{Li}_{-1}\left(i;q^2\right) - \frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^4\right) \right)$$

then

$$\int_0^{\frac{K}{2}} \log\left(\mathrm{dn}\left(z; \frac{1}{\sqrt{2}}\right)\right) dz = -2K \left(\zeta_q(0) - \frac{3}{2}\zeta_{q^2}(0) + \frac{1}{2}\zeta_{q^4}(0) \right)$$

$$-\frac{4K}{\pi} i \left(\mathrm{Li}_{-1}(i;q) - \frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^2\right) - \mathrm{Li}\left(i;q^2\right) + \frac{1}{4}\mathrm{Li}_{-1}\left(-1;q^4\right) \right).$$

So, making the necessary simplifications, we can conclude that

$$\begin{aligned}\sum_{n \geq 0} \left(\frac{1}{4}\right)^n \binom{2n}{n} \frac{(-1)^n}{(4n+1)^2} &= -\int_0^{\frac{K}{2}} \log\left(\frac{\mathrm{sn}\left(z; \frac{1}{\sqrt{2}}\right) \mathrm{dn}\left(z; \frac{1}{\sqrt{2}}\right)}{\mathrm{cn}\left(z; \frac{1}{\sqrt{2}}\right)} \right) dz \\ &= -\frac{K}{2} \log\left(\frac{2K}{\pi}\right) + \frac{2CK}{\pi} \\ &+ 2K\zeta_{q^4}(0) - 5K\zeta_{q^2}(0) + 2K\zeta_q(0) \\ &+ \frac{2K}{\pi} i \left(\mathrm{Li}_{-1}\left(-1;q^4\right) - 4\mathrm{Li}_{-1}\left(i;q^2\right) \right) \\ &+ \frac{K}{\pi} i \left(4\mathrm{Li}_{-1}(i;q) - \mathrm{Li}_{-1}\left(-1;q^2\right) \right)\end{aligned}$$

and this identity completes the proof. QED

Remark 3. It is natural to wonder if the same approach works with $L_2^{H_n}$. The answer is that it is not clear if we can use our technique for the evaluation of this series. Indeed, following the previous argument, we find the integral

$$-4\sqrt{2} \int_0^1 \frac{z^2 \log(z)}{(1+z^4)^{3/2}} dz = -\frac{4\sqrt{2}}{9} \int_0^1 \frac{\log(v)}{\left(1+v^{4/3}\right)^{3/2}} dv \tag{3.13}$$

and so it is not obvious how to treat (3.13). However, we do not exclude the possibility that there exists some transformation that allows us to reformulate the problem in terms of approachable definite integrals of Jacobian elliptic functions. In particular, we believe that some p, q generalizations of the Jacobian elliptic functions, for suitable p, q, could solve the problem. This approach would require extending some known properties to the case p, q. This idea will be the subject of future research.

4 Equivalent reformulations of the problem

In this section we will show that the particular choice $f_n = H_n$ is very interesting, as the main series $L_1^{H_n}$ can be interpreted in several ways. These observations could bring interesting new links between apparently distant topics.

4.1 Fourier-Legendre expansions

In a recent series of papers it has been shown that the Fourier-Legendre (FL) expansions are a very useful tool for finding the closed form for series whose coefficient are powers of the central binomial coefficients (see [11, 13, 14, 15, 24]). Even if the technique of FL expansions does not seem to provide a solution in our case, it is still interesting to note how it reformulates the problem. Indeed, from the power series representation

$$\sum_{n\geq 0}\binom{2n}{n}H_n x^n = \frac{2\log\left(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}\right)}{\sqrt{1-4x}},\ |x|<\frac{1}{4}$$

(see [7]) we have

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^n\binom{2n}{n}\frac{H_n}{4n+1} = 2\int_0^1 \frac{\log\left(\frac{1+\sqrt{1-4x^4}}{2\sqrt{1-4x^4}}\right)}{\sqrt{1-4x^4}}dx$$

so, combining the previous identity with the FL expansion

$$\frac{1}{2\sqrt{x}}\log\left(\frac{1+\sqrt{x}}{4\sqrt{x}}\right) = \sum_{n\geq 0}(-1)^n H_n P_n(2x-1)$$

where $x \in (0,1)$ and where $P_n(x)$ is the Legendre polynomial of order n (see [15], formula (18)) and the evaluation

$$\int_0^1 x^{2\mu-1}P_n\left(1-2x^2\right)dx = \frac{(-1)^n\Gamma(\mu)^2}{2\Gamma(\mu+n+1)\Gamma(\mu-n)},\ \mathrm{Re}(\mu)>0,$$

(see [23], equation 7.233) we get

$$\begin{aligned}\sum_{n\geq 0}\left(\frac{1}{4}\right)^n\binom{2n}{n}\frac{H_n}{4n+1} &= \frac{\sqrt{\pi}\log(2)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \frac{1}{4}\sum_{n\geq 0}H_n\int_0^1 x^{-3/4}P_n(2x-1)\,dx \\ &= \frac{\sqrt{\pi}\log(2)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{3}{4}\right)}\sum_{n\geq 0}(-1)^n H_n\frac{\Gamma\left(n+\frac{3}{4}\right)}{\Gamma\left(n+\frac{5}{4}\right)} \\ &= \frac{\sqrt{\pi}\log(2)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \frac{\Gamma\left(\frac{1}{4}\right)}{4\Gamma\left(\frac{5}{4}\right)}\,{}_2F_1^{(0,1,0,0)}\left(\frac{3}{4},1;\frac{5}{4};-1\right)\end{aligned}$$

hence Theorem 2 provides a also a closed form for ${}_2F_1^{(0,1,0,0)}\left(\frac{3}{4},1;\frac{5}{4};-1\right)$.

4.2 Hypergeometric function

We start from the relation (3.6). It is not difficult to prove that

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^n\binom{2n}{n}\frac{(-1)^n}{(4n+1)^2}={}_3F_2\left(\frac{1}{2},\frac{1}{4},\frac{1}{4};\frac{5}{4},\frac{5}{4};-1\right) \tag{4.14}$$

and, as far as we know, a closed form expression for this function is unknown. Note that this ${}_3F_2$ is quite interesting; indeed, it is a special case of a classical classical result of Whipple

$$\begin{aligned}{}_3F_2\left(a,b,c;1+a-b,1+a-c;-1\right)=&\frac{\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+a-b-c\right)}\\&\times{}_3F_2\left(\frac{1}{2},b,c;1+\frac{a}{2},\frac{1}{2}+\frac{a}{2};1\right)\end{aligned}$$

(see [30], equation 9.3) but, unfortunately, even this new ${}_3F_2$ seems to not possess a known closed form in terms of well known special functions or mathematical constants.

4.3 Generalized Lerch trascendent

We start again from (3.6). Recalling the Lerch trascendent function

$$\Phi\left(z,s,a\right):=\sum_{n\geq 0}\frac{z^n}{\left(n+a\right)^s}$$

with $a\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $s\in\mathbb{C}$ if $|z|<1$ and $\mathrm{Re}(s)>1$ if $|z|=1$ (see [27], section 25.14(i)) and the fractional derivative of a function

$$\mathcal{D}_z^{\mu}\left[f\left(z\right)\right]:=\begin{cases}\frac{1}{\Gamma(-\mu)}\int_0^z\left(z-t\right)^{-\mu-1}f\left(t\right)dt, & \mathrm{Re}(\mu)<0\\ \frac{d^m}{dz^m}\left(\mathcal{D}_z^{\mu-m}\left[f\left(z\right)\right]\right) & m-1\leq\mathrm{Re}(\mu)<m,\ m\in\mathbb{N}\end{cases}$$

(see, e.g., [18], p. 181 et seq.) we define the generalized Lerch trascendent function $\Phi_\mu^*\left(z,s,a\right)$ as

$$\Phi_\mu^*\left(z,s,a\right):=\frac{1}{\Gamma\left(\mu\right)}\mathcal{D}_z^{\mu-1}\left[z^{\mu-1}\Phi\left(z,s,a\right)\right]=\sum_{n\geq 0}\frac{(\mu)_n}{n!}\frac{z^n}{\left(n+a\right)^s}.$$

This function was introduced in [22] and studied in a series of papers. Hence, we can see that

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^{n}\binom{2n}{n}\frac{(-1)^{n}}{(4n+1)^{2}}=\sum_{n\geq 0}\frac{\left(\frac{1}{2}\right)_{n}}{n!}\frac{(-1)^{n}}{(4n+1)^{2}}=\frac{1}{16}\Phi^{*}_{\frac{1}{2}}\left(-1,2,\frac{1}{4}\right)$$

so we can write

$$\sum_{n\geq 0}\binom{2n}{n}\frac{1}{4^{n}}\frac{H_{n}}{4n+1}=\frac{\Gamma\left(\frac{1}{4}\right)^{2}(\pi-6\log(2))}{8\sqrt{2\pi}}-\frac{\Gamma\left(\frac{1}{4}\right)\pi^{3/2}}{4\Gamma\left(\frac{3}{4}\right)}+\frac{\sqrt{2}}{4}\Phi^{*}_{\frac{1}{2}}\left(-1,2,\frac{1}{4}\right).$$

4.4 Definite integrals of elliptic functions

We have seen that

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^{n}\binom{2n}{n}\frac{(-1)^{n}}{(4n+1)^{2}}=-\int_{0}^{\frac{K}{2}}\log\left(\frac{\operatorname{sn}\left(z;\frac{1}{\sqrt{2}}\right)\operatorname{dn}\left(z;\frac{1}{\sqrt{2}}\right)}{\operatorname{cn}\left(z;\frac{1}{\sqrt{2}}\right)}\right)dz.$$

Now, from the functional relation

$$\frac{\operatorname{sn}(z;k)^{2}\operatorname{dn}(z;k)^{2}}{\operatorname{cn}(z;k)^{2}}=\frac{1-\operatorname{cn}(2z;k)}{1+\operatorname{cn}(2z;k)}$$

(see [23], section 8.155, equation 2) then

$$\sum_{n\geq 0}\left(\frac{1}{4}\right)^{n}\binom{2n}{n}\frac{(-1)^{n}}{(4n+1)^{2}}$$

$$=-\frac{1}{4}\int_{0}^{K}\left[\log\left(1-\operatorname{cn}\left(z;\frac{1}{\sqrt{2}}\right)\right)-\log\left(1+\operatorname{cn}\left(z;\frac{1}{\sqrt{2}}\right)\right)\right]dz$$

and this is one of the "left open case" in [21]. Indeed, Glaisher explicitly states that, with his method, we can only evaluate

$$\int_{0}^{K}\left[\log\left(1-\operatorname{cn}(z;k)\right)+\log\left(1+\operatorname{cn}(z;k)\right)\right]dz. \tag{4.15}$$

Furthermore, combining Glaisher 's results for (4.15) with Theorem 2, we also have a closed form for

$$\int_{0}^{K}\log\left(1-\operatorname{cn}(z;k)\right)dz,\ \int_{0}^{K}\log\left(1+\operatorname{cn}(z;k)\right)dz.$$

4.5 Lambert series

For the definition of $\zeta_q(s)$ and $\mathrm{Li}_s(x;q)$ we used Lambert series. Therefore, it is natural to ask if these series have interesting links or properties. In fact, it is well known that $\zeta q(s)$ belongs to a class of well-known Lambert series of the form:

$$\mathscr{L}_q(s,x) := \sum_{n\geq 1} \frac{n^s q^{nx}}{1-q^n},\ s\in\mathbb{C},\ 0\leq q<1,\ x>0$$

which have some interesting representations like

$$\mathscr{L}_q(-s,x) = \sum_{n\geq 0} \mathrm{Li}_s\left(q^{n+x}\right)$$

where

$$\mathrm{Li}_s(x) := \sum_{n\geq 1} \frac{x^n}{n^s}$$

is the Polylogarithm function and

$$\mathscr{L}_q(s,1) = \sum_{n\geq 1} \sigma_s(n) q^n$$

where $\sigma_s(n) := \sum_{d|n} d^s$ (see [1, 5]). Now, since we work with also $\mathrm{Li}_s(q;x)$, we define the following class of Lambert series

$$\mathscr{L}_q(y,s,x) := \sum_{n\geq 1} \frac{n^s y^n q^{nx}}{1-q^n},\ s\in\mathbb{C},\ 0\leq q<1,\ |y|\leq 1,\ x>0$$

and so, clearly,

$$\mathscr{L}_q(s,x) = \mathscr{L}_q(1,s,x).$$

Our aim is to mimic the proof of Lemma 2.1 of [5] for the series $\mathscr{L}_q(y,s,x)$ and so giving an explicit relation with $\mathrm{Li}_s(x)$. Indeed, if we define the operator

$$D := \frac{d}{dx}$$

and we consider

$$\frac{D}{e^D-1} := \sum_{n\geq 0} \frac{B_n}{n!} D^n$$

where B_n are the Bernoulli numbers, it is straightforward to see, following the proof of Lemma 2.1 of [5], that

$$\mathscr{L}_q(y,s,x) = \frac{D}{e^D-1} \frac{\mathrm{Li}_s(yq^x)}{\log(1/q)}.$$

Furthermore, it is not difficult to see that also an analogous of the Theorem 2.2 of [5] can be proved.

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