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# An affine proof of uniqueness for the smallest generalized quadrangles, including the determination of their automorphism groups

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**Abstract.** We show, by elementary means, that any two generalized quadrangles with three points per line and three lines per point are isomorphic. The proof uses affine quadrangles, it also yields that the doily is point-homogeneous, and that its full group of automorphisms is isomorphic to the group of all permutations of a set of order 6.

Extending the methods, we treat the case of generalized quadrangles with three points per line (and arbitrary line size) as well.

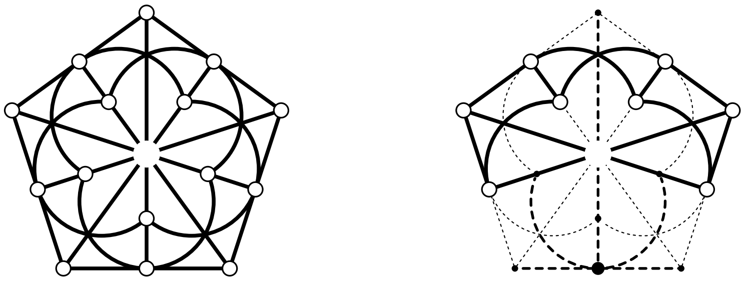
**Keywords:** generalized quadrangle, affine generalized quadrangle, affine derivation, cube, hypercube, automorphism, duality

**MSC 2000 classification:** primary 51E12, secondary 05B25 51A10 51A15

## Introduction

We use affine derivations of generalized quadrangles in the present paper. While affine planes have long been utilized for the investigation (and construction) of projective planes (i.e. generalized triangles), the interest in affine quadrangles seems to be rather new, cf. [8], [10], [9]. Generalized quadrangles may be considered as polar spaces of rank 2. See [3] and [5] for a discussion of affine parts of polar spaces of higher rank.

We concentrate on generalized quadrangles with 3 points per line. In fact, such structures exist with either 15 points, 15 lines, and 45 flags (i.e. incident point–line pairs), or 27 points, 45 lines, and 135 flags. Other small examples of generalized quadrangles come with 4 points per line and 4 lines per point, these have 40 points, 40 lines, and 160 flags. The structures that we treat in the sequel are those generalized quadrangles that minimize the number of flags (and the total number of points *and* lines).



(a) (b)

Figure 1. The doily (a), and an affine derivation (b).

# 1 The doily

The smallest generalized quadrangle has 15 points and 15 lines such that 3 points lie on each line and 3 lines pass through each point. It is well known that this geometry is unique (up to isomorphism), a popular graphic description is the "doily", see Fig. 1(a).

A non-experienced reader may explore the doily by checking that it satisfies the following:

**1 Axioms for Generalized Quadrangles.**

($) Every line has at least 3 points on it, every point has at least 3 lines through it, and not all the points are collinear.

($\perp$) For each point $p$ and each line $\ell$ not passing through $p$ there are exactly one line $m$ through $p$ and exactly one point $q$ on $\ell$ such that $q$ also lies on $m$.

We will denote $m$ by $\lambda(p, \ell)$ and $q$ by $\pi(p, \ell)$.

It is easy to derive from these axioms:

- Through any two points there is at most one line ("there are no digons").
- If three points are pairwise collinear then they are collinear ("there are no triangles").
- For any two points there is a point that is joined to both.
- For any two lines there is a line that meets both.

If two points $x, y$ are joined by a line then this (unique) line will be denoted by $xy$.

**2 Smallest Generalized Quadrangles.** For the sake of brevity, every generalized quadrangle with 3 points per line and 3 lines per point will be called a *smallest generalized quadrangle*. Clearly the doily is *one* example. We want to show, by elementary means, that every smallest generalized quadrangle is isomorphic to the doily. (Later on, we will determine all generalized quadrangles with 3 points per line.)

As a first step, we count the number of points and lines in a generalized quadrangle $Q$: start with a line $\ell$, count the set $Q_1(\ell)$ of points on $\ell$, and then count the set $Q_3(\ell)$ of points that are joined to points in $Q_1(\ell)$ but do not lie on $\ell$; cf. Fig. 2. The axioms ensure that this procedure takes care of each point, and that no point is counted twice.

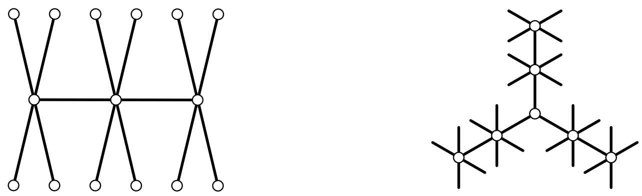


Figure 2. Counting points and lines in a generalized quadrangle.

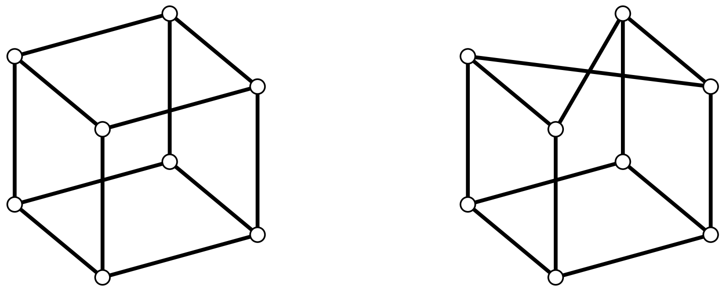
For any *smallest* generalized quadrangle we obtain 15 points. Dually (that is, by interchanging the roles of points and lines) we see that every smallest generalized quadrangle has 15 lines as well.

Later on, we will also be interested in generalized quadrangles with three points per line, but $m + 1 > 3$ lines per point. We note that then there are $3 + 3 \cdot m \cdot 2 = 3 \cdot (1 + 2m)$ points.

## 2 Affine derivations

In any generalized quadrangle $Q$, we may pick any point $p$ and form the affine derivation $Q^p$ by deleting all lines through $p$ together with all the points on them. Points and lines in $Q^p$ will be termed *affine*, while the deleted ones will be called points and lines *at infinity*.

An example (where $p$ is taken to be the point in the middle of the bottom line) is shown in Fig. 1(b), a more suggestive picture of that structure is given in Fig. 3(a). A structure with the same number of points and lines, and also not containing any triangles, is shown in Fig. 3(b). In fact, one can show that these two graphs are the only ones with eight vertices and three edges per vertex that do not contain triangles, see Prop. 32 in the appendix. This would be one way to prove the characterization Thm. 11 of smallest generalized quadrangles. We choose another way that uses more information about affine derivations, and generalizes to larger generalized quadrangles.



(a) (b)

Figure 3. A familiar picture of the affine quadrangle (a), and another graph on 8 vertices (b).

**3 Definitions.** Let $Q$ be a generalized quadrangle, fix a point $p$ in $Q$, and consider two lines $\ell, m$ of $Q$ that do not pass through $p$. In other words, we consider lines of the affine derivation $Q^p$. We say that $\ell$ and $m$ are *weakly parallel* if $\lambda(p,\ell) = \lambda(p,m)$. Clearly this is an equivalence relation on the set of lines of $Q^p$. For every point $a$ in $Q^p$ there is exactly one line $\ell_a$ through $a$ that meets $\lambda(p,\ell)$: we call $\ell_a$ the *weak parallel* to $\ell$ through $a$.

The relation of weak parallelism is used to reconstruct the lines through $p$. Aiming at the reconstruction of the points on these, we call lines $\ell$, $m$ of $Q^p$ *strongly parallel* if $\pi(p,\ell) = \pi(p,m)$. Clearly this is another equivalence relation, its classes correspond to the points on lines through $p$ (but different from $p$). In particular, strongly parallel lines are also weakly parallel.

**4 Remarks.** The notions of weak and strong parallelism are taken from [9]. In [10] these relations appear in the following way: for two lines $g, h$ in an affine quadrangle the relation $g \diamond h$ holds if, and only if, the lines have distance 0 or 6 in the incidence graph. This is the relation of strong parallelism. On the set of equivalence classes (i.e. the points at infinity) a second relation is established, this corresponds to the coarser equivalence relation of weak parallelism. The classes of the latter relation describe lines at infinity.

**5 Lemma.** *Let $\ell, m$ be two lines of $Q^p$ that are weakly parallel, and assume that in $Q^p$ there exists a line that meets both. Then we have*

($\#$) *for every affine point $a$ on $\ell$ the point $\pi(a,m)$ belongs to $Q^p$.*

*Conversely, if two affine lines satisfy* ($\#$) *then they are weakly parallel.*

Proof. Let $k$ be the line in $Q^p$ that meets both $\ell$ and $m$. Non-existence of triangles then yields that $\ell$ and $m$ have no point in common. Axiom ($\perp$) says that $\lambda(a,m)$ and $\pi(a,m)$ exist in $Q$. It just remains to check that $\pi(a,m)$ is an affine point: this follows from the fact that the unique line through $\pi(p,\ell)$ meeting $m$ does not pass through an affine point, but through $\pi(p,m)$.

Conversely, assume that $\ell$ and $m$ satisfy ($\#$), and consider the point $q := \pi(p,m)$. If $f := \pi(q,\ell)$ were an affine point, the point $\pi(f,m) = q$ would be affine, too. Thus $f$, $q$ and $p$ are pairwise collinear points, and nonexistence of triangles implies that they lie on a common line. This line is $\lambda(p,\ell) = \lambda(p,m)$, and the lines $\ell$ and $m$ are weakly parallel. $\boxed{QED}$

## 3 Extending embeddings

**6 Definition.** Let $X$ and $Y$ be point–line geometries. A *full embedding* of $X$ into $Y$ is an injective mapping of points such that for each line $\ell$ of $X$ the set of all points of $\ell$ is mapped *onto* the set of all points of some line in $Y$.

**7 Theorem.** *Let $Q$ and $R$ be generalized quadrangles, let $a$ be a point of $Q$, and let $b$ be a point of $R$. Then every full embedding of $Q^a$ into $R^b$ extends to a full embedding of $Q$ into $R$.*

Proof. Two lines of $Q^a$ are strongly parallel if, and only if, no line in $Q^a$ meets both. We claim that any full embedding $\eta$ maps strongly parallel pairs of lines of $Q^a$ to strongly parallel pairs in $R^b$. Assume, to the contrary, that $\ell$ and $m$ are two strongly parallel lines in $Q^a$ such that some line meets both the images under $\eta$, and let $u$ on $\ell$ and $v$ on $m$ be the pre-images of the intersection points (here we use that $\eta$ is a full embedding). Pick any line $k$ through $u$ different from $\ell$. Then $\pi(v,k)$ is different from $u$, and the images of $u$, $v$, and $\pi(v,k)$ form a triangle in $R^b$.

This observation shows that we may extend, without ambiguities, the embedding $\eta$ to the set of all points of $Q$ by stipulating that $a$ is mapped to $b$, and that the image of any other point $x$ at infinity is the point where the images of any two affine lines through $x$ meet.

It remains to discuss lines at infinity. Using Lemma 5 and the fact that $\eta$ is a full embedding we see that images of weakly parallel lines are weakly parallel, again. Thus we have to map a line at infinity to the line joining the image of any of its points with $b$.

It is straightforward to see that this extension of $\eta$ is a full embedding of $Q$ into $R$. $\boxed{QED}$

**8 Corollary.** *Every isomorphism between affine derivations of quadrangles extends to an isomorphism of generalized quadrangles.*

The condition that the embedding is full is indispensable in Thm. 7:

**9 Example.** For every commutative field $K$ the edge graph of the cube admits an embedding into any affine derivation of the symplectic quadrangle $\mathrm{W}(K)$. However, the doily is embeddable in $\mathrm{W}(K)$ only if $K$ has characteristic 2.

## 4 Cubes and hypercubes

In order to determine all affine quadrangles with two points per line, we construct *substructures* (consisting of a subset $S$ of the point set, together with *all* lines that join points of $S$) that are built up from cubes:

**10 Lemma.** *Let $Q$ be a generalized quadrangle with three points per line, and let $p$ be a point of $Q$. Then the affine derivation $Q^p$ contains a substructure isomorphic to the edge graph of the cube. Moreover, every finite part of the point set of $Q^p$ is contained in a disjoint union of point sets of such substructures.*

Proof. Pick an affine point $a$ and three different lines through $a$. The (unique) affine points on these lines apart from $a$ will be denoted by $b$, $c$, and $d$, respectively. We draw the line through $b$ that is weakly parallel to $ac$, and denote its second affine point by $c'$. Analogously, we define $d'$ as the second affine point on the line through $b$ that is weakly parallel to $ad$. According to Lemma 5, there is a line through $c$ that meets $bc'$ in an affine point: this point has to be $c'$. In the same way, we see that $d$ and $d'$ are collinear. Now draw weak parallels to $ad$ through $c$ and $c'$, and note that the second affine points $x$ and $x'$ on these lines are collinear with $d$ and $d'$, respectively. A final application of Lemma 5 shows that these last two points are collinear as well.

We remark that parallel edges in the graph correspond to weakly parallel lines in the affine derivation, cf. Lemma 5. Thus we obtain the same cube if we start from any of its points, and draw weak parallels to the three lines that we have chosen. Moreover, opposite points in such a cube are not collinear: to fix notation, assume that $c$ and $d'$ belong to a line $m$, then $m$ is weakly parallel to $ad$ by Lemma 5. Now $cx$ and $m$ are different weak parallels to $ad$ through $c$, contradicting Def. 3.

Now assume that there exists an affine point $w$ outside $\{ a, b, c, d, c', d', x, x' \}$. We construct the cube containing $w$, using weak parallels to the edges of the previous one. Then the cubes have no points in common. Repeating this procedure for any point not covered by the cubes that have been constructed so far, we obtain the last assertion of the lemma. $\boxed{QED}$

Affine derivations of smallest generalized quadrangles have just eight points, and Lemma 10 yields:

**11 Theorem.** *Every affine derivation of a smallest generalized quadrangle is isomorphic to the graph of the cube.*

**12 Example.** For the smallest examples, we describe the reconstruction procedure (Cor. 8) explicitly. Let $Q$ be a smallest generalized quadrangle, and let $p$ be one of its points. The affine derivation $Q^p$ is isomorphic to the edge graph of the cube, see Thm. 11. We can see the missing points and lines inside $Q^p$, as follows:

a. *Points outside $Q^p$ but different from $p$:* through each of these points there are two lines that do not pass through $p$, and leave a trace in $Q^p$. There cannot be a line in $Q^p$ that meets both of the traces because there are no triangles in $Q$. This means that the two traces in the cube form a set of two diagonally opposed parallel edges. Conversely, every such set stems from a point outside $Q^p$ but different from $p$.

b. *Lines outside $Q^p$:* these are the lines through $p$. Each of these lines has (apart from $p$) two points corresponding to sets of diagonally opposed parallel edges. Nonexistence of triangles implies that the two sets are parallel as well. Thus the three lines outside $Q^p$ correspond to the three classes of parallel edges in the cube.

c. *The point $p$* is just the single point that belongs to each of the three lines reconstructed above, but to no other line.

The reconstruction process that we have just described may remind the reader of the general process of "projective completion" for affine planes. In fact, it is possible to describe affine derivations of arbitrary generalized quadrangles (not only smallest ones) axiomatically, and to generalize the indicated construction, see [10]. We are mainly interested in extensions of embeddings, cf. Thm. 7.

What we have achieved here, in an elementary way, is the following:

**13 Theorem.** *Let $Q$ be a smallest generalized quadrangle, and let $p$ be a point of $Q$. Then $Q^p$ is isomorphic to the edge graph of the cube, and $Q$ is isomorphic to the following structure:*

- *Points are the vertices of the cube, plus the "points at infinity" obtained as sets of diagonally opposed parallel edges, plus one extra point.*
- *Lines are the edges (each augmented by the point at infinity that it belongs to) plus the three classes of parallel edges (each augmented by the extra point).*

*In particular, every smallest generalized quadrangle is isomorphic to the doily.*

The *dual* of a generalized quadrangle is obtained by interchanging the roles of points and lines. Since the axioms treat points and lines in a symmetric way, the dual is a generalized quadrangle, again. Only in very special cases, however, a generalized quadrangle is *self-dual*, i.e., isomorphic to its own dual. See [11] for a discussion (involving affine quadrangles as well) of dualities in a class of generalized quadrangles that includes the smallest ones (they occur as symplectic quadrangle W(2), cf. Construction 25 below).

Because the dual of a smallest generalized quadrangle is another smallest generalized quadrangle, we have:

**14 Theorem.** *Every smallest generalized quadrangle is self-dual.*

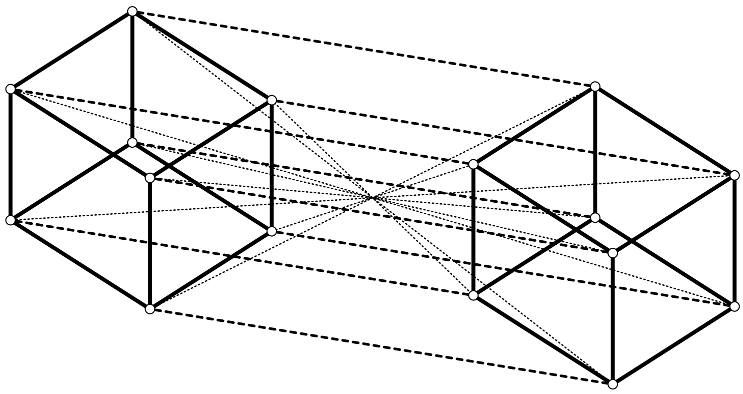


Figure 4. The hypercube, with added diagonals.

**15 Theorem.** *Let $Q$ be a generalized quadrangle with three points per line and more than three lines per point. Then every affine derivation is isomorphic to the edge graph of a hypercube of dimension* 4*, with eight extra edges joining pairs of opposite vertices, see Fig. 4.*

Proof. We know from Lemma 10 that every finite subset of the point set of the affine derivation is contained in a disjoint union of cubes. It remains to show that there are at most two cubes, and to determine the lines that join the cubes.

We resume the notation introduced in the proof of Lemma 10 for a first cube. Starting with any affine point $w \notin \{ a, b, c, d, c', d', x, x' \}$, we construct a second cube. By our construction, the edges of the second cube are induced by weak parallels to the lines $\ell := ab$, $ac$, and $ad$. Four parallel edges of any of our cubes are induced by lines belonging to a single weak parallel class, and thus to at most two strong parallel classes because the corresponding points at infinity are collinear. On the other hand, parallel edges on the same side of a cube cannot be strongly parallel because there are no triangles. Thus the four parallel edges form two strong parallel classes, consisting of diagonally opposed edges. (cf. the situation described in Example 12.)

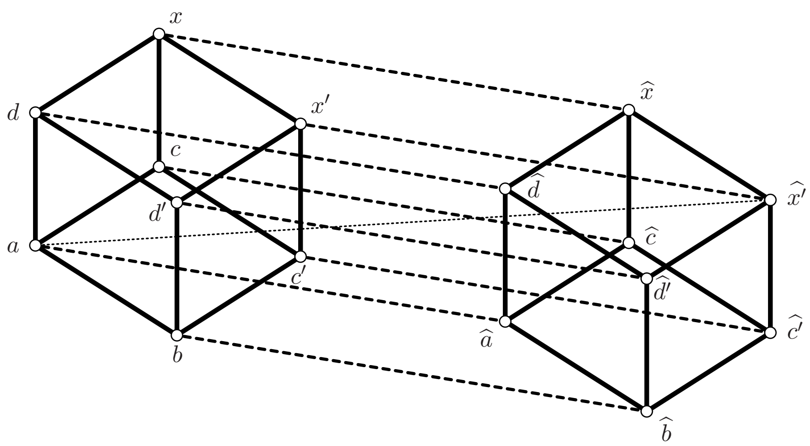



Figure 5. Labeling of the hypercube.

Now choose an edge $\widehat{\ell}$ of the second cube that is weakly but not strongly parallel to $\ell$. From Lemma 5 we know that $\widehat{a} := \pi(a, \widehat{\ell})$ and $\widehat{b} := \pi(b, \widehat{\ell})$ are affine points. Using Lemma 5 again, we see that the two cubes together with the weak parallels to $\lambda(a, \widehat{\ell})$ through the vertices form a hypercube. We label the vertices of the second cube with the elements of $\{\,\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}, \widehat{c'}, \widehat{d'}, \widehat{x}, \widehat{x'}\,\}$ in such a way that $u\widehat{u}$ and $\lambda(a, \widehat{\ell})$ are weakly parallel, see Fig. 5. Now $\widehat{x}\widehat{x'}$ is weakly, but not strongly parallel to $xx'$ and to $\ell$, and $a$ is joined to one of the points $\widehat{x}, \widehat{x'}$. Because $ac$ and $\widehat{a}\widehat{c}$ are weakly but not strongly parallel, the lines $ac$ and $\widehat{d}\widehat{x}$ are strongly parallel. This implies $\pi(a, \widehat{x}\widehat{x'}) = \widehat{x'}$. Thus $\lambda(a, \widehat{x}\widehat{x'})$ is one of the diagonals of the hypercube. The weak parallels to $\lambda(a, \widehat{x}\widehat{x'})$ form the eight diagonals of the hypercube.

Finally, we note that the edges through opposite vertices in any one of the cubes form three strongly parallel pairs, while the weak parallel classes are those represented by $ab$, $ac$, and $ad$, respectively. There are $2^3 = 8$ choices of three strong parallel classes such that the corresponding weak parallel classes are those that occur with the edges of our two cubes. Four of these choices are needed for pairs of opposite vertices of the first cube. Avoiding triangles, we have to use the remaining four choices in the second cube. This shows that a third cube is impossible: all the possible choices are ruled out by the first two cubes. $\boxed{QED}$

We obtain a new proof for a known result (cf. [2, Thm. 5.3]):

**16 Corollary.** *Every generalized quadrangle with three points per line either has three or five lines per point. In particular, there are no infinite generalized quadrangles with three points per line.*

In the light of Thm. 15 and Cor. 8 we obtain:

**17 Theorem.** *Any two generalized quadrangles with three points per line and five lines per point are isomorphic.*

## 5 Automorphisms of the doily

An *automorphism* of the doily is a permutation of the set of points such that lines are mapped to lines. The collection $A$ of all automorphisms forms a group (with respect to composition) called the *automorphism group* of the doily.

Some of the automorphisms are easy to see: for instance a rotation by 72 degrees maps Fig. 1(a) onto itself. We also see five reflections. However, there are more automorphisms of the doily than those that stem from symmetries of the pentagon. In fact, affine derivations at different points of the doily are isomorphic to each other by Thm. 11, and Cor. 8 yields:

**18 Corollary.** *The group $A$ acts transitively on the set of points of the doily, that is: for any two points $p, q$ there is at least one automorphism mapping $p$ to $q$.*

Using the fact (Thm. 14) that the doily is isomorphic to its dual, we see:

**19 Corollary.** *The group $A$ acts transitively on the set of lines of the doily.*

The following arguments require some background from elementary group theory. We write $\mathrm{Sym}_n$ for the group of all permutations of a set with $n$ elements.

**20 Theorem.** *The automorphism group of the edge graph of the cube has $2 \cdot 4! = 2^4 \cdot 3 = 48$ elements. In fact, this group is isomorphic to a direct product of $\mathrm{Sym}_4$ with a (cyclic) group of order 2.*

Proof. Let $W$ denote the automorphism group in question. The four diagonals are characterized in the edge graph by the fact that they join vertices at distance 3. The action of $W$ on the four diagonals yields a homomorphism from $W$ to $\mathrm{Sym}_4$. The group $R$ of rotations of the cube induces a subgroup of $W$ that acts faithfully on the set of diagonals, and transitively on the set of faces. The stabilizer of a face in $R$ is generated by a rotation of order 4. Thus $R$ has 24 elements, and is mapped bijectively onto $\mathrm{Sym}_4$.

It remains to determine the kernel $K$ of the action on the set of diagonals. Clearly, the reflection $\rho$ at the barycenter of the cube belongs to this kernel. If an element of $K$ fixes a vertex, it is easy to see that it fixes all the vertices. Thus $K$ consists of $\rho$ and the identity.

Now that we know that $W = RK$ consists of euclidian symmetries, we obtain that $R$ is a normal subgroup, and $W$ is the direct product of the two normal subgroups $R$ and $K$. $\boxed{QED}$

The order formula is well known for the group of euclidian symmetries of the cube, cf. [4, §3.5] for a collection of proofs for the fact that the group of rotations has order 24 (this is the factor isomorphic to $\mathrm{Sym}_4$, as one sees from the action on the 4 diagonals). See also [1, 12.5.4.2].

**21 Proposition.** *The automorphism group of the doily has order* $6! = 2^4 \cdot 3^2 \cdot 5 = 720$.

Proof. Let $p$ be a point of the doily, and let $A_p$ be its stabilizer (that is, the set of all elements of $A$ that fix $p$). Clearly, every element of $A_p$ induces an automorphism of the affine derivation at $p$, and thus an automorphism of the edge graph of the cube, cf. Thm. 11. Conversely, every automorphism of the affine derivation extends to a unique automorphism of the doily, see Thm. 13. This means that $A_p$ is isomorphic to the automorphism group of the cube, and has $2 \cdot 4!$ elements by Thm. 20. Now transitivity of $A$ on the set of 15 points shows that $A$ has $15 \cdot 2 \cdot 4! = 6!$ elements. $\boxed{QED}$

## 6 Ovoids, and the automorphism group of the doily

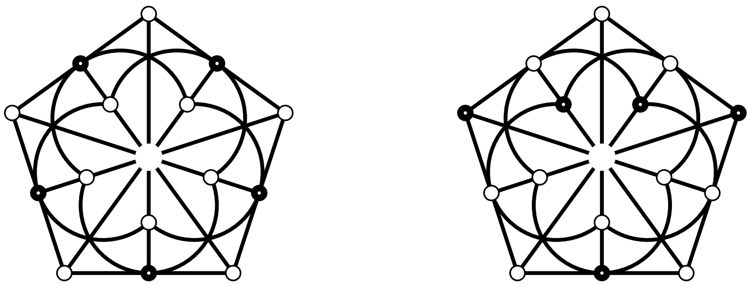
Inspired by the observation that $\mathrm{Sym}_6$ and $A$ have the same number of elements, we search for a set of 6 geometrically defined objects in the doily. This will imply that there is an isomorphism from $A$ onto $\mathrm{Sym}_6$. The next section is devoted to a proof of this fact. A different approach will be briefly described in Construction 27.

**22 Definition.** An *ovoid* in the doily is a set of 5 points such that no two of them are joined by a line.

See Fig. 6 for two examples of ovoids in the doily, marked by the dark points. Applying automorphisms (namely, rotations of the pentagon), we obtain five different ovoids from the one shown in Fig. 6(b), while the ovoid in Fig. 6(a) is invariant under these automorphisms (it is, however, moved by automorphisms that move any of its points to a point outside the ovoid).

We want to show that the six ovoids that we have found are in fact all the ovoids in the doily: this will be the set of six objects that we search for. We use affine derivations again:

Let $\mathcal{O}$ be an ovoid in the doily, and let $p$ be a point of $\mathcal{O}$. By the definition of ovoids, every point of $\mathcal{O} \setminus \{p\}$ belongs to the affine derivation $Q^p$. This derivation is (isomorphic to) the edge graph of a cube, and we search for 4 vertices that are pairwise not joined by edges. Picking any vertex of the cube,



(a) (b)

Figure 6. Two ovoids in the doily.

we have to avoid the three neighboring vertices. The opposite vertex may not be chosen because it would prohibit any other vertex. Thus $\mathcal{O}$ is (the vertex set of) one of the two tetrahedra inscribed in the cube in such a way that the edges are diagonals on the faces of the cube. As the automorphism group $A$ acts transitively on the points of the doily, we have thus proved:

**23 Theorem.** *Through every point of the doily there are exactly two ovoids.*

A simple counting argument shows that we have a total of $\frac{15\cdot 2}{5} = 6$ ovoids. In particular, the ovoids in Fig. 6 together with the four images of Fig. 6(b) under rotation cover all cases, and the ovoid from Fig. 6(a) may be moved to the one from Fig. 6(b). Thus the group $A$ acts transitively on the set of ovoids.

Fig. 6 shows that there are two ovoids that share a single point. Transitivity of $A$ on the set of points now yields that every point of the doily is the intersection of two suitable ovoids. These arguments show that the group $A$ acts faithfully on the set of ovoids: if an automorphism fixes each ovoid, it fixes each point of the doily, and is trivial. The faithful action on a set of size 6 yields an injective group homomorphism from $A$ to $\mathrm{Sym}_6$. Because $A$ and $\mathrm{Sym}_6$ have equal size (see Prop. 21), this homomorphism is an isomorphism, and we have proved:

**24 Theorem.** *The automorphism group of the doily is isomorphic to* $\mathrm{Sym}_6$.

## 7 Constructions

We indicate some constructions of smallest generalized quadrangles. Recall from Thm. 13 that all these quadrangles are mutually isomorphic.

**25 Construction** (Symplectic quadrangles over GF(2))**.** Let $V$ be a vector space of dimension 4 over the field GF(2) with 2 elements, and let $\langle\cdot|\cdot\rangle$ be a

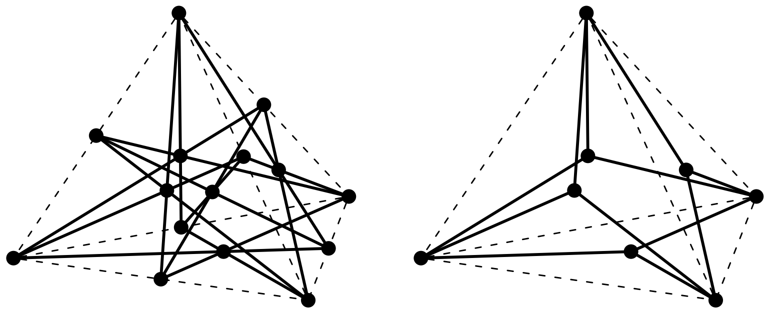


Figure 7. W(2) in PG$(3, 2)$ (left), and an affine derivation (right).

non-degenerate alternating bilinear form on $V$ (such a form is, up to a change of basis, unique). We write $t_n$ for the set of $n$-dimensional subspaces of $V$ on which the form vanishes. Using $t_1$ as set of points and $t_2$ as set of lines (and the natural relation of incidence) we obtain a generalized quadrangle, known as the symplectic quadrangle W(2) over GF(2). See [13, 2.3.17].

This construction embeds the smallest quadrangle in the 3-dimensional projective space PG$(3, 2)$ over GF(2). Fig. 7 is inspired by [6, p. 44 f].

**26 Construction** (Orthogonal quadrangles over GF(2))**.** We proceed as before but start with a non-degenerate quadratic form on a vector space of dimension 5 over GF(2). The sets of totally isotropic subspaces of dimensions 1 and 2 form the orthogonal quadrangle Q$(4, 2)$ which is a smallest generalized quadrangle. See [13, 2.3.12].

An explicit example[1] of such a quadratic form is $f(x_0, x_1, x_2, x_3, x_4) = x_0x_4 + x_1x_3 + x_2^2$, cf. [13, 2.3.4]. A second example is given by $g(x_0, x_1, x_2, x_3, x_4) = x_0^2 + x_0x_1 + x_1^2 + x_2x_3 + x_4^2$, a third one by $h(x_0, x_1, x_2, x_3, x_4) = x_0^2 + x_0x_1 + x_1^2 + x_2x_3 + x_4x_0 + x_4x_1 + x_4x_2 + x_4x_3 + x_4^2$. The latter two forms are restrictions of $q(x_0, x_1, x_2, x_3, x_4, x_5) = x_0^2 + x_0x_1 + x_1^2 + x_2x_3 + x_4x_5$ to the hyperplanes defined by the conditions $x_4 = x_5$ and $x_0 + x_1 + x_2 + x_3 + x_4 = x_5$.

This gives rise to two embeddings of the doily into the orthogonal quadrangle Q$(5, 2)$ such that the two subquadrangles share the point GF$(2)(1, 1, 1, 1, 0, 0)$. The traces of the two doilies in the affine derivation form a pair of cubes, as in Lemma 10.

**27 Construction** (Involutions in the symmetric group)**.** Let $J$ be the set of all transpositions in $\mathrm{Sym}_6$, and let $F$ be the set of involutions in $\mathrm{Sym}_6$ without fixed points (that is, products of three disjoint transpositions). Elements of $J$

[1] Note that the form $d(x_0, x_1, x_2, x_3, x_4) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2$ is a degenerate one over GF(2): the associated bilinear form vanishes.

are called points, elements of $F$ are called lines, and we say that a point and a line are incident if they commute in the group $\mathrm{Sym}_6$. This gives a smallest generalized quadrangle $(J, F)$. It is quite obvious that $\mathrm{Sym}_6$ acts (via conjugation) by automorphisms on $(J, F)$. This model is particularly interesting since it exhibits an automorphism of $\mathrm{Sym}_6$ that is not inner: the automorphism is induced by a duality (cf. Thm. 14) of the quadrangle, acting via conjugation on the full automorphism group $\mathrm{Sym}_6$, cf. Thm. 24. See [13, 1.4.2].

More constructions of smallest generalized quadrangles (and related structures) may be found in [7]. See also [12] where the doily is treated as a geometry for the group $\mathrm{Alt}_5$.

A graphical description of the generalized quadrangle with 3 points per line and 5 lines per point is given in [6, 4.4].

# 8 Automorphisms of hypercubes and quadrangles

Let $Q$ be a generalized quadrangle with 3 points per line and 5 lines per point. Then each affine derivation is isomorphic to a hypercube, with added diagonals, see Thm. 15. Applying Cor. 8, we obtain:

**28 Theorem.** *The automorphism group* $\mathrm{Aut}\,(Q)$ *of* $Q$ *acts transitively on the points of* $Q$*, and the stabilizer of any point is isomorphic to the automorphism group of the hypercube with added diagonals.*

In the proof of Lemma 10 we started with an arbitrary affine point $a$, and any choice of three lines through $a$. The hypercube in the proof of Thm. 15 was constructed from the resulting pair of cubes (with strongly parallel lines). This shows that the group of automorphisms of the affine derivation acts transitively on the set of affine points, and the stabilizer of the point $a$ acts at least threefold transitively on the set of lines through $a$. On the other hand, a suitable euclidian reflection shows that the stabilizer of $a$ contains an element that induces a transposition on the line pencil. Therefore, the stabilizer of $a$ induces on the pencil a subgroup of $\mathrm{Sym}_5$ with order at least $5 \cdot 4 \cdot 3 = 60$, and not contained in the alternating group: this has to be $\mathrm{Sym}_5$ itself. If an automorphism of the affine quadrangle fixes a point and each line through that point it fixes all the other points as well.

A representation of the hypercube in the affine space over the field with 2 elements shows that there is a sharply transitive, elementary abelian group of automorphisms of the affine derivation of $Q$.

We have thus proved:

**29 Theorem.** *The stabilizer of an affine point in the automorphism group of any affine derivation of* $Q$ *is isomorphic to* $\mathrm{Sym}_5$*. The full group of automor-*

*phisms of such a derivation is a semidirect product of* $\mathrm{Sym}_5$ *with an elementary abelian group of order* 16*.*

**30 Corollary.** $\mathrm{Aut}\,(Q)$ *has order* $27 \cdot 120 \cdot 16 = 2^7 \cdot 3^4 \cdot 5$*.*

**31 Remarks.** See [13, 4.6.3] for the determination of the group of automorphisms of $\mathrm{Q}(5,2)$ (in fact, the automorphism group for the dual $\mathrm{H}(3,2^2)$ of $\mathrm{Q}(5,2)$ is determined there). The group $\mathrm{Aut}\,(\mathrm{Q}(5,2))$ contains a simple normal subgroup (isomorphic to $\mathrm{PSU}(4,2) \cong \mathrm{P\Omega}^-(6,2) \cong \mathrm{P\Omega}(5,3)$) of order $2^6 \cdot 3^4 \cdot 5$.

The group of euclidian isometries of a hypercube has order $2^7 \cdot 3$: it is a semidirect product of $\mathrm{Sym}_4$ with a (normal) elementary abelian subgroup of order $2^4$, cf. [1, 12.5.4.2]. This group occurs as the stabilizer of a weak parallel class (namely, the set of diagonals of the hypercube) in the affine derivation.

## Appendix: Graphs

For any smallest generalized quadrangle (with 3 points per line and 3 lines per point) each affine derivation has 2 points per line and 3 lines per point. Thus we may consider the derivation as a graph, where every vertex belongs to three edges. Because we have deleted $1 + 3 \cdot 2 = 7$ of the 15 points, there remain 8 vertices in the graph. Moreover, our graph does not contain triangles (because the generalized quadrangle does not contain quadrangles).

We give a more general, purely graph-theoretic characterization of the affine derivation of the doily (namely, the graph of the cube):

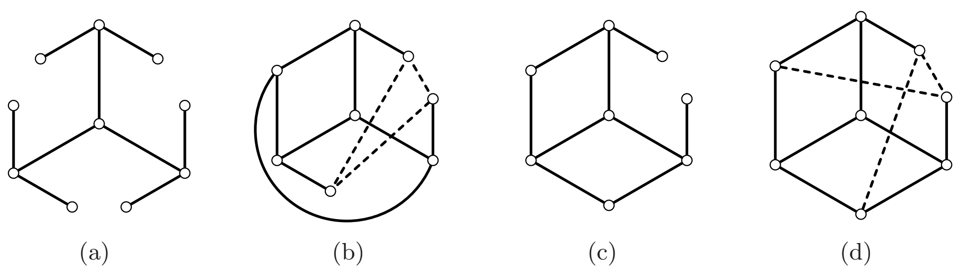



Figure 8. Constructing a graph: the wrong way (b), and the right way (d).

**32 Proposition.** *Let* $\Gamma$ *be a graph with eight vertices such that every vertex belongs to exactly three edges. If* $\Gamma$ *does not contain triangles then* $\Gamma$ *is isomorphic to one of the graphs shown in Fig. 3.*

Proof. Assume first that there is a vertex $q$ such that every other vertex has distance at most 2 from $q$ in the graph $\Gamma$. We have drawn a first approximation of this situation in Fig. 8(a). However, there are too many vertices: we have to identify some of them. Avoiding triangles, we may only identify some of the endpoints of $\Gamma$. If three vertices should be identified, these three would lie on three different branches starting from $q$, and we obtain Fig. 8(b): every vertex apart from the endpoints already has three edges, and introducing the missing edges for the endpoints we create a triangle. Therefore, we have to identify two pairs of endpoints, obtaining Fig. 8(c). Avoiding triangles, the only way of introducing the missing edges leads to Fig. 8(d): this is a graph isomorphic to Fig. 3(b).

Now assume that there are vertices $a$ and $z$ at distance 3 in $\Gamma$. We build our graph from Fig. 9(a) where the points at distance 1 from $a$ or $z$ are drawn. Every vertex $v$ at distance 1 from $a$ is joined to two vertices at distance 1 from $z$ (we have to avoid triangles!), and there is exactly one vertex $v'$ at distance 1 from $z$ such that $\{ v, v' \}$ is *not* an edge. We may arrange the vertices at distance 1 from $z$ in such a way that Fig. 9(b) is obtained: this is the graph of the cube, Fig. 3(a). $\boxed{QED}$

(a) (b)

Figure 9. Constructing the graph of the cube.

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