Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **44** (2024) no. 1, 53–83. doi:10.1285/i15900932v44n1p53

Uniform Sobolev, interpolation and geometric Calderón–Zygmund inequalities for graph hypersurfaces

Serena Della Corte[i]
Delft Institute of Applied Mathematics (DIAM), Delft University of Technology, The Netherlands
s.dellacorte@tudelft.nl

Antonia Diana[ii]
Scuola Superiore Meridionale, Napoli, Italy
antonia.diana@unina.it

Carlo Mantegazza[iii]
Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Università di Napoli Federico II & Scuola Superiore Meridionale, Napoli, Italy
carlo.mantegazza@unina.it

Received: 16.07.2023; accepted: 07.06.2024.

Abstract. In this note, our aim is to show that families of smooth hypersurfaces of $\mathbb{R}^{n+1}$ which are all "C^1–close" enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for mathematical analysis, like Sobolev, Gagliardo–Nirenberg and "geometric" Calderón–Zygmund inequalities.

Keywords: Embedded hypersurface, Sobolev inequalities, interpolation inequalities, Calderón–Zygmund inequalities.

MSC 2020 classification: primary 53C42, secondary 35A23, 47J20.

1 Introduction and preliminaries

In this note, our aim is to show that families of smooth hypersurfaces of $\mathbb{R}^{n+1}$ which are all "C^1–close" enough to a fixed compact, embedded one, have uniformly bounded constants in some relevant inequalities for mathematical analysis, like Sobolev, Gagliardo–Nirenberg, "geometric" Calderón–Zygmund, trace and extension inequalities. These technical results are quite useful, in par-

[i]Supported by The Netherlands Organisation for Scientific Research (NWO), Grant Number 613.009.148

[ii]Member of the INDAM–GNAMPA research group

[iii]Member of the INDAM–GNAMPA research group and partially supported by PRIN Project 2022E9CF89 "GEPSO – Geometric Evolution Problems and Shape Optimization"

ticular, in the study of the geometric flows of hypersurfaces, when one studies the behavior of the hypersurfaces "close" (in some norm, for instance in C^1–norm) to critical ones (possibly "stable") or the asymptotic limits of flows existing for all times (see for instance [2, 3, 10, 13], where such controls on the constants are necessary).

We start by setting up some notation and recall some basic facts about hypersurfaces in Euclidean spaces that we need in the sequel, possible references are [6, 1, 15].

We will consider smooth, compact hypersurfaces M, embedded in $\mathbb{R}^{n+1}$, getting a Riemannian metric g by pull–back of the standard scalar product $\langle \cdot \,|\, \cdot \rangle$ of $\mathbb{R}^{n+1}$ via the embedding map $\varphi : M \to \mathbb{R}^{n+1}$, hence, turning it into a Riemannian manifold (M, g). Then, we use ∇ for the associated Levi–Civita covariant derivative and μ for the canonical measure induced by the metric g, which actually coincides with the n–dimensional Hausdorff measure $\mathcal{H}^n$ of $\mathbb{R}^{n+1}$ restricted to M. Then, the components of g in a local chart are

$$g_{ij} = \left\langle \frac{\partial \varphi}{\partial x_i} \,\Big|\, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

and the "canonical" measure μ, induced on M by the metric g is then locally described by $\mu = \sqrt{\det g_{ij}}\, \mathscr{L}^n$, where $\mathscr{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The inner product on M, extended to tensors, is given by

$$g(T, S) = g_{i_1 s_1} \dots g_{i_k s_k} g^{j_1 z_1} \dots g^{j_l z_l} T^{i_1 \dots i_k}_{j_1 \dots j_l} S^{s_1 \dots s_k}_{z_1 \dots z_l}$$

where g_{ij} is the matrix of the coefficients of the metric tensor in the local coordinates and g^{ij} is its inverse. Clearly, the norm of a tensor is then

$$|T| = \sqrt{g(T, T)}\,.$$

The induced Levi–Civita covariant derivative on (M, g) of a vector field X and of a 1–form ω are respectively given by

$$\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k\,, \qquad \nabla_j \omega_i = \frac{\partial \omega_i}{\partial x_j} - \Gamma^k_{ji} \omega_k\,,$$

where Γ^i_{jk} are the Christoffel symbols of the connection ∇, expressed by the formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \Big(\frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \Big)\,. \tag{1.1}$$

With $\nabla^m T$ we will mean the m–th iterated covariant derivative of a tensor T.

Being M embedded, we can assume it is a subset of $\mathbb{R}^{n+1}$ (hence the embedding map is the identity) and we denote with $\nu : M \to \mathbb{R}^{n+1}$ its global unit normal vector field, *pointing outward*. It is indeed well known (theorem of Jordan–Brouwer, see [6, Proposition 12.2], for instance) that any compact, embedded M "divides" $\mathbb{R}^{n+1}$ in two connected components, one of them bounded (called "the interior"), both having M as its smooth boundary, hence the hypersurface is orientable and such field ν exists.

Then, we define the *second fundamental form* B which is a symmetric 2–form given, in a local chart, by its components

$$\mathrm{B}_{ij} = -\left\langle \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \,\middle|\, \nu \right\rangle$$

and whose trace is the *mean curvature* $\mathrm{H} = g^{ij}\mathrm{B}_{ij}$ of the hypersurface (with these choices, the standard sphere of $\mathbb{R}^{n+1}$ has positive mean curvature).

Remark 1. If the hypersurface M is locally the graph of a function $f : U \to \mathbb{R}$ with U an open subset of $\mathbb{R}^n$, that is, $M = \{(x, f(x)) \;:\; x \in U\}$, then we have

$$g_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\,, \qquad \nu = -\frac{(\nabla^{\mathbb{R}^n} f, -1)}{\sqrt{1 + |\nabla^{\mathbb{R}^n} f|^2}}\,, \tag{1.2}$$

$$\mathrm{B}_{ij} = -\frac{\mathrm{Hess}^{\mathbb{R}^n}_{ij} f}{\sqrt{1 + |\nabla^{\mathbb{R}^n} f|^2}}\,, \tag{1.3}$$

$$\mathrm{H} = -\frac{\Delta^{\mathbb{R}^n} f}{\sqrt{1 + |\nabla^{\mathbb{R}^n} f|^2}} + \frac{\mathrm{Hess}^{\mathbb{R}^n} f(\nabla^{\mathbb{R}^n} f, \nabla^{\mathbb{R}^n} f)}{\left(\sqrt{1 + |\nabla^{\mathbb{R}^n} f|^2}\right)^3} = -\,\mathrm{div}^{\mathbb{R}^n}\left(\frac{\nabla^{\mathbb{R}^n} f}{\sqrt{1 + |\nabla^{\mathbb{R}^n} f|^2}}\right) \tag{1.4}$$

where $\mathrm{Hess}^{\mathbb{R}^n} f$ is the (standard) Hessian of the function f.

Then, the following *Gauss–Weingarten relations* hold,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k} - \mathrm{B}_{ij}\nu \qquad \frac{\partial \nu}{\partial x_j} = \mathrm{B}_{jl} g^{ls}\frac{\partial \varphi}{\partial x_s}\,, \tag{1.5}$$

which easily imply

$$\nabla^2 \varphi = -\mathrm{B}\nu \qquad \text{and} \qquad \Delta\varphi = -\mathrm{H}\nu\,.$$

The symmetry properties of the covariant derivative of B are given by the following Codazzi equations,

$$\nabla_i \mathrm{B}_{jk} = \nabla_j \mathrm{B}_{ik} = \nabla_k \mathrm{B}_{ij}$$

which imply the following *Simons' identity* (see [23]),

$$\Delta \mathrm{B}_{ij} = \nabla_i\nabla_j \mathrm{H} + \mathrm{H}\,\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}\,. \tag{1.6}$$

Finally, the Riemann tensor can be expressed as (*Gauss equations*),

$$\mathrm{R}_{ijkl} \;=\; \mathrm{B}_{ik}\mathrm{B}_{jl} - \mathrm{B}_{il}\mathrm{B}_{jk}\,. \tag{1.7}$$

If now we choose a fixed smooth, compact, embedded hypersurface M_0 of $\mathbb{R}^{n+1}$, it is well known (by its compactness and smoothness) that, for $\varepsilon > 0$ small enough, M_0 has a *tubular neighborhood*

$$N_\varepsilon = \big\{x \in \mathbb{R}^{n+1} \,:\, d(x, M_0) < \varepsilon\big\}$$

(where d is the Euclidean distance on $\mathbb{R}^{n+1}$) such that the *orthogonal projection map* $\pi : N_\varepsilon \to M_0$ giving the (unique) closest point on M_0, is well defined and smooth. Then, if E is "the interior" of M_0, the *signed distance function* $d_E : N_\varepsilon \to \mathbb{R}$ from M_0

$$d_E(x) = \begin{cases} d(x, M_0) & \text{if } x \notin E \\ -d(x, M_0) & \text{if } x \in E \end{cases}$$

is smooth in N_ε and $\nu(x) = \nabla^{\mathbb{R}^{n+1}} d_E(x)$, for every $x \in M_0$. Moreover, for every $x \in N_\varepsilon$, the projection map π is given explicitly by

$$\pi_E(x) = x - \nabla^{\mathbb{R}^{n+1}} d^2_E(x)/2 = x - d_E(x)\nabla^{\mathbb{R}^{n+1}} d_E(x)$$

(indeed, actually $\nabla^{\mathbb{R}^{n+1}} d_E(x) = \nabla^{\mathbb{R}^{n+1}} d_E(\pi_E(x)) = \nu(\pi_E(x))$ for every $x \in N_\varepsilon$).

From now on, we will consider smooth hypersurfaces contained in N_ε that can be written (possibly after reparametrization) as graph over M_0, that is,

$$M = \big\{x + \psi(x)\nu(x) \,:\, x \in M_0\big\},$$

for a smooth "height function" $\psi : M_0 \to \mathbb{R}$ with $|\psi(x)| < \varepsilon$, for every $x \in M_0$.

We define the following families (clearly all containing M_0),

$$\mathfrak{C}^1_\delta(M_0) = \Big\{M = \big\{x + \psi(x)\nu(x) \,:\, x \in M_0\big\} \\ \text{for a smooth } \psi : M_0 \to \mathbb{R} \text{ with } \|\psi\|_{C^1(M_0)} < \delta\Big\}$$

where $\delta \in (0, \varepsilon)$ and we are considering on M_0 the induced metric from $\mathbb{R}^{n+1}$ (in order to define $|d\psi|$). Sometimes, we will use the expression "C^1–close to M_0", meaning that the above constant δ is small. Moreover, since we will use it, we also define the subfamily $\mathfrak{C}^{1,\alpha}_\delta(M_0)$ of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ such that the "height function" satisfies $\|\psi\|_{C^{1,\alpha}(M_0)} < \delta$.

We are going to see that the constants in Sobolev, Gagliardo–Nirenberg, some geometric Calderón–Zygmund inequalities, trace and extension inequalities are uniformly bounded, depending only on M_0 and δ.

Before starting discussing that, we introduce another technical construction. We notice that, possibly choosing a smaller $\varepsilon > 0$, the tubular neighborhood N_ε of M_0 defined above, can be covered by a finite number of open hypercubes $Q_1, \dots, Q_k \subseteq \mathbb{R}^{n+1}$ respectively centered at some points $p_1, \dots, p_k \in M_0$, such that, for every $i \in \{1, \dots, k\}$ and every $M \in \mathfrak{C}^1_\delta(M_0)$, with $\delta \in (0, \varepsilon)$, the "pieces" of hypersurfaces $M \cap Q_i$ can be written as *orthogonal* graphs on the affine hyperplanes $\Pi_{p_i} M_0 = p_i + T_{p_i} M_0$, parallel to the tangent hyperplanes to M_0 at the points $p_i \in M_0$ and passing through them, as in the following figure.

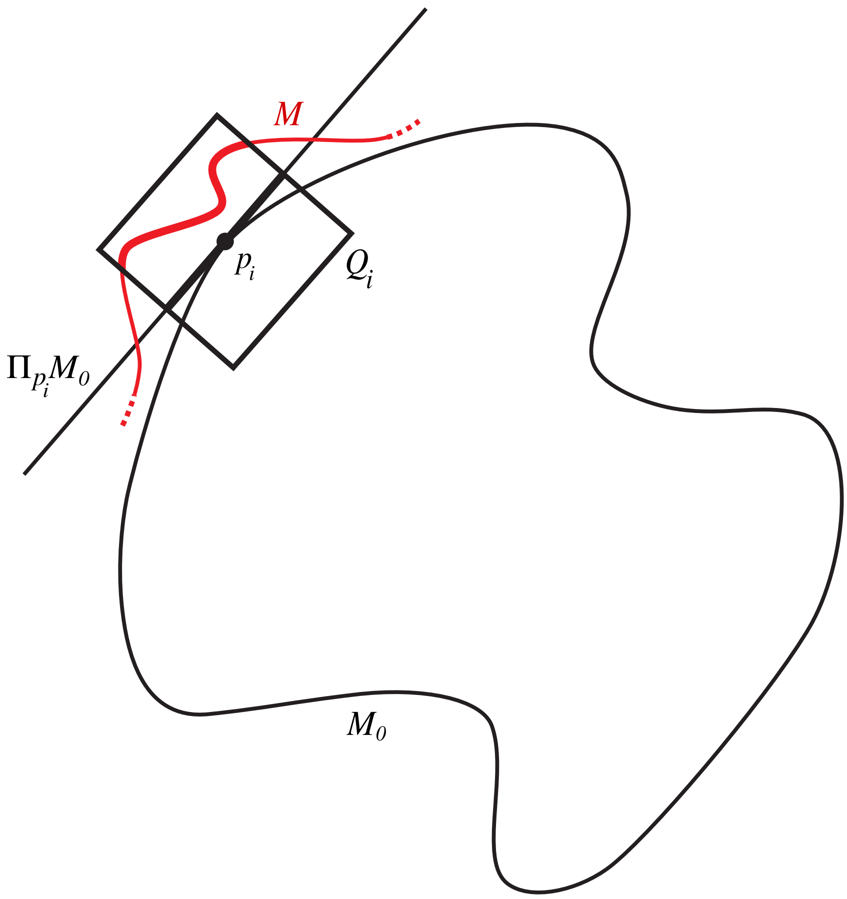

Figure 1

Then, we let $\rho_i : \mathbb{R}^{n+1} \to [0,1]$ a smooth partition of unity (with compact support) for N_ε, associated to the open covering Q_i, hence, if $M \in \mathfrak{C}^1_\delta(M_0)$ and $u : M \to \mathbb{R}$, there holds

$$u(y) = \sum_{i=1}^{k} u(y)\rho_i(y)$$

with the compact support of $u\rho_i : M \to \mathbb{R}$ contained in the piece $M \cap Q_i$ of the hypersurface M, which is described as the graph of a smooth function $\theta_i : \Pi_{p_i} M_0 \to \mathbb{R}$, that is, $M \cap Q_i$ is the image of the map $x \mapsto \Theta(x) = x + \theta_i(x)\nu(p_i)$ on $\Pi_{p_i} M_0 \cap Q_i$. Moreover, it is easy to see that, possibly choosing an even smaller $\varepsilon > 0$, we have $\|\theta_i\|_{C^1(\Pi_{p_i} M_0)} \leqslant 2\delta$, for every $i \in \{1, \dots, k\}$, since also M_0 can be locally written as an orthogonal graph on $\Pi_{p_i} M_0$.

We notice and underline that the family (and the number) of the hypercubes Q_i, as well as the width $\varepsilon > 0$ of the tubular neighborhood N_ε that we considered for this construction, only depend on M_0, precisely on its local and global geometry (in particular, on its second fundamental form B_0 – see [9] for more details).

We highlight to the reader that in the following, we will often denote with C a constant which may vary from a line to another.

2 Sobolev, Poincaré and Gagliardo–Nirenberg interpolation inequalities

We start discussing the Sobolev constants $C_S(M, p)$ of any compact n–dimensional hypersurface M, for every $p \in [1, n)$, entering in the following inequalities (which are known to hold, see [5, Chapter 2], for instance),

$$\begin{aligned}
\|u\|_{L^{p^*}(M)} &= \Big(\int_M |u|^{p^*}\, d\mu\Big)^{1/p^*} \\
&\leqslant C_S(M,p)\Big(\int_M |\nabla u|^p + |u|^p\, d\mu\Big)^{1/p} \\
&= C_S(M,p)\|u\|_{W^{1,p}(M)}
\end{aligned}$$

for every C^1–function $u : M \to \mathbb{R}$ (or $u \in W^{1,p}(M)$), where $p^* = \frac{np}{n-p}$ is the *Sobolev conjugate exponent* of p. It is well known that a bound on $C_S(M, 1)$ implies a bound on $C_S(M, p)$, for every $p \in [1, n)$ (see [5, Chapter 2, Section 5], for instance), hence we concentrate on the case $p = 1$, where $1^* = \frac{n}{n-1}$.

We first want to argue localizing things by means of the construction of the previous section. We then have a finite family of hypercubes Q_i centered at

$p_i \in M_0$, the partition of unity ρ_i and a parametrization $x \mapsto \Theta(x) = x + \theta_i(x)\nu_i$ on $\Pi_{p_i}M_0 \cap Q_i$ of each piece $M \cap Q_i$ of any smooth hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$, where $\nu_i = \nu(p_i)$ and the functions $\theta_i : \Pi_{p_i}M_0 \to \mathbb{R}$ satisfy $\|\theta_i\|_{C^1(\Pi_{p_i}M_0)} \leqslant 2\delta$, for every $i \in \{1, \dots, k\}$. Moreover, in dealing with any piece $M \cap Q_i$, we will assume (without clearly losing generality) that $\Pi_{p_i}M_0 = \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ and we observe that in such parametrization, by formula (1.2), the Riemannian measure μ associated to the (induced) metric g on M is given by $\mu = J\Theta\,\mathscr{L}^n$, with $\mathscr{L}^n$ the Lebesgue measure on $\Pi_{p_i}M_0 = \mathbb{R}^n$ and $J\Theta = \sqrt{1 + |\nabla^{\mathbb{R}^n}\theta_i|^2}$, which clearly satisfies $1 \leqslant J\Theta \leqslant 1 + 2\delta$.

For every C^1–function $u : M \to \mathbb{R}$, we can write

$$\Big(\int_M |u|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} = \Big(\int_M \Big|\sum_{i=1}^k u\rho_i\Big|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leqslant \sum_{i=1}^k \Big(\int_{M\cap Q_i} |u\rho_i|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}}$$

as the compact support of $u\rho_i$ is contained in $M \cap Q_i$.

Then, for every C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, there holds

$$\begin{aligned}\Big(\int_{M\cap Q_i} |v(y)|^{\frac{n}{n-1}}\,d\mu(y)\Big)^{\frac{n-1}{n}} &= \Big(\int_{\mathbb{R}^n} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\,J\Theta(x)\,dx\Big)^{\frac{n-1}{n}}\\ &\leqslant C(\delta)\Big(\int_{\mathbb{R}^n} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\,dx\Big)^{\frac{n-1}{n}},\end{aligned}$$

as $J\Theta \leqslant 1+2\delta$ and applying the Sobolev inequality for functions with compact support in $\mathbb{R}^n$, we have

$$\begin{aligned}&\Big(\int_{\mathbb{R}^n} |v(x+\theta_i(x)\nu_i)|^{\frac{n}{n-1}}\,dx\Big)^{\frac{n-1}{n}}\\ &\leqslant C\int_{\mathbb{R}^n} |\nabla^{\mathbb{R}^n}[v(x+\theta_i(x)\nu_i)]|\,dx\\ &= C\int_{\mathbb{R}^n} \big|\nabla v(x+\theta_i(x)\nu_i)\circ\big(\mathrm{Id} + \nabla^{\mathbb{R}^n}\theta_i(x)\otimes\nu_i\big)\big|\,dx\\ &\leqslant C\int_{\mathbb{R}^n} |\nabla v(x+\theta_i(x)\nu_i)|\,\big|\mathrm{Id} + \nabla^{\mathbb{R}^n}\theta_i(x)\otimes\nu_i\big|\,dx\\ &= C\int_{\mathbb{R}^n} |\nabla v(x+\theta_i(x)\nu_i)|\,\sqrt{1+|\nabla^{\mathbb{R}^n}\theta_i|^2}\,dx\\ &= C\int_M |\nabla v(y)|\,d\mu(y)\,,\end{aligned} \tag{2.8}$$

as $\sqrt{1+|\nabla^{\mathbb{R}^n}\theta_i|^2} = J\Theta$. Hence,

$$\Big(\int_{M\cap Q_i} |v|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leqslant C(\delta)\int_M |\nabla v|\,d\mu$$

and setting $v_i = u\rho_i$, after summing on $i \in \{1, \dots, k\}$, we conclude

$$\begin{aligned}\Big(\int_M |u|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} &\leqslant \sum_{i=1}^k \Big(\int_{M\cap Q_i} |v_i|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}}\\ &\leqslant C(\delta)\sum_{i=1}^k \int_M |\nabla v_i|\,d\mu\\ &= C(\delta)\sum_{i=1}^k \int_M |\nabla u|\rho_i + |u|\,|\nabla\rho_i|\,d\mu\\ &\leqslant C(\delta)\int_M |\nabla u|\,d\mu + C(M_0,\delta)\int_M |u|\,d\mu\,, \end{aligned} \tag{2.9}$$

as $|\nabla\rho_i| \leqslant C(M_0,\delta)$, for every $i \in \{1,\dots,k\}$. This clearly gives a uniform bound on $C_S(M,1)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, depending only on M_0 (in particular, on its second fundamental form B_0, as we said in the previous section) and $\delta > 0$.

Let now see an alternate line, based on the "global" graph representation of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0.

For every C^1 function $u : M \to \mathbb{R}$, we have

$$\Big(\int_M |u(y)|^{\frac{n}{n-1}}\,d\mu(y)\Big)^{\frac{n-1}{n}} = \Big(\int_{M_0} |u(x+\psi(x)\nu(x))|^{\frac{n}{n-1}}\,J\Psi(x)\,d\mu_0(x)\Big)^{\frac{n-1}{n}}$$

where $J\Psi$ is the Jacobian of the map $\Psi : M_0 \to M$ and it is an easy check that, at every point $x \in M_0$, there holds

$$\frac{1}{C(\mathrm{B}_0,\delta)} \leqslant J\Psi \leqslant C(\mathrm{B}_0,\delta)\,, \tag{2.10}$$

for some constant $C(\mathrm{B}_0,\delta) > 0$, where B_0 is the second fundamental form of M_0. Moreover, $C(\mathrm{B}_0,\delta)$ goes to 1 as $\delta \to 0$. Notice that the fact that B_0 appears here can be seen from the expression of $d\Psi$, that is

$$d\Psi_x = \mathrm{Id}_{T_xM_0} + d\psi_x \otimes \nu(x) + \psi(x)d\nu_x\,,$$

as, by the Gauss–Weingarten relations (1.5), $d\nu_x$ is related to $\mathrm{B}_0(x)$.

Then, by applying the Sobolev inequality holding for M_0, we have

$$\begin{aligned}
&\Big(\int_{M_0} |u(x+\psi(x)\nu(x))|^{\frac{n}{n-1}}\,d\mu_0(x)\Big)^{\frac{n-1}{n}}\\
&\leqslant C_S(M_0,1)\int_{M_0}\big|\nabla^0[u(x+\psi(x)\nu(x))]\big|\,d\mu_0(x)\\
&\quad + C_S(M_0,1)\int_{M_0}|u(x+\psi(x)\nu(x))|\,d\mu_0(x)\\
&\leqslant C_S(M_0,1)\int_{M_0}|\nabla u(x+\psi(x)\nu(x))|\,|d\Psi(x)|\,d\mu_0(x)\\
&\quad + C_S(M_0,1)\int_{M_0}|u(x+\psi(x)\nu(x))|\,d\mu_0(x)\\
&\leqslant C(M_0,\delta)\int_M |\nabla u(y)|\,J\Psi^{-1}(y)\,d\mu(y)\\
&\quad + C(M_0,\delta)\int_M |u(y)|\,J\Psi^{-1}(y)\,d\mu(y)\\
&\leqslant C(M_0,\delta)\Big(\int_M |\nabla u(y)|\,d\mu(y) + \int_M |u(y)|\,d\mu(y)\Big)\,.
\end{aligned}$$

Hence,

$$\Big(\int_M |u|^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leqslant C(M_0,\delta)\Big(\int_M |\nabla u|\,d\mu + \int_M |u|\,d\mu\Big)\,.$$

As before, this means that the constant $C(M_0,\delta)$ uniformly bounds $C_S(M,1)$ for all the hypersurfaces in $\mathfrak{C}^1_\delta(M_0)$, moreover, since $C(M_0,\delta)\to 1$, as $\delta\to 0$, it also shows the continuous dependence of $C_S(M,1)$ under the C^1–convergence of the hypersurfaces.

Theorem 1. *Let $M_0\subseteq\mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$. Then, there exist uniform bounds, depending only on M_0 and δ (more precisely, on the "C^1– structure" of the immersion of M_0 in $\mathbb{R}^{n+1}$, its dimension and its second fundamental form), for all the hypersurfaces $M\in\mathfrak{C}^1_\delta(M_0)$ on:*

(i) *the volume of M from above and below away from zero,*

(ii) *the Sobolev constants for $p\in[1,n)$ of the embeddings $W^{1,p}(M)\hookrightarrow L^{p^*}(M)$,*

(iii) *the Sobolev constants for $p\in(n,+\infty]$ of the embeddings $W^{1,p}(M)\hookrightarrow C^{0,1-n/p}(M)$,*

(iv) *the constants in the Poincaré–Wirtinger inequalities on M for $p\in[1,+\infty]$,*

(v) *the constants in the embeddings of the fractional Sobolev spaces $W^{s,p}(M)$,*

(vi) *the constants in the Gagliardo–Nirenberg interpolation inequalities on M.*

Moreover, all these bounds go to the corresponding constants for M_0, as $\delta \to 0$.

Proof.
(i) This is trivial due to the C^1–closedness of M to M_0.

(ii) As explained at the beginning of the section, we can estimate the constant in the Sobolev inequality for $p \in [1, n)$, by means of $C_S(M, 1)$, which is uniformly bounded for all the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, by the above discussion.

(iii) If $p > n$, we show that there exists a uniform constant $C(M_0, p, \delta)$ such that

$$\|u\|_{C^{0,\alpha}(M)} \leqslant C(M_0, p, \delta)\|u\|_{W^{1,p}(M)} \tag{2.11}$$

with $\alpha = 1 - n/p$ and

$$\|u\|_{C^{0,\alpha}} = \sup_{y \in M} |u(y)| + \sup_{y, y^* \in M,\ y \neq y^*} \frac{|u(y) - u(y^*)|}{|y - y^*|^\alpha},$$

for all $M \in \mathfrak{C}^1_\delta(M_0)$ and every C^1 function $u : M \to \mathbb{R}$.

In the same setting and notation at the beginning of this section, it is easy to see that we can choose a special family of hypercubes Q_i such that enlarging their edges of a small value $\sigma > 0$, we have hypercubes $\widetilde{Q}_i$ with the further property that $M \cap \widetilde{Q}_i$ can be still written as an orthogonal graph on $\Pi_{p_i} M_0 = \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$.

The following holds

$$\sup_{y \in M} |u(y)| \leqslant \sum_{i=1}^{k} \sup_{y \in M \cap Q_i} |u(y)\rho_i(y)|$$

and for every C^1 function $v : M \to \mathbb{R}$ with compact support in $M \cap Q_i$, by applying the Sobolev inequality for $p > n$ in $\mathbb{R}^n$ and arguing as in obtaining estimate (2.8), we have

$$\begin{aligned}
\sup_{y \in M \cap Q_i} |v(y)| &= \sup_{x \in \mathbb{R}^n} |v(x + \theta_i(x)\nu_i)| \\
&\leqslant C\Big(\int_{\mathbb{R}^n} \big|\nabla v(x + \theta_i(x)\nu_i) \circ \big(\mathrm{Id} + \nabla^{\mathbb{R}^n}\theta_i(x) \otimes \nu_i\big)\big|^p \, dx\Big)^{1/p} \\
&\leqslant C(\delta)\Big(\int_{\mathbb{R}^n} |\nabla v(x + \theta_i(x)\nu_i)|^p \, dx\Big)^{1/p} \\
&\leqslant C(\delta)\Big(\int_{\mathbb{R}^n} |\nabla v(x + \theta_i(x)\nu_i)|^p J\Theta \, dx\Big)^{1/p} \\
&= C(\delta)\Big(\int_M |\nabla v(y)|^p \, d\mu(y)\Big)^{1/p},
\end{aligned} \tag{2.12}$$

as $J\Theta \geqslant 1$. Setting $v_i = u\rho_i$ and estimating as in getting inequality (2.9), we conclude

$$\sup_M |u| \leqslant C(M_0, p, \delta)\Big(\int_M |\nabla u|^p + |u|^p\, d\mu\Big)^{1/p}. \tag{2.13}$$

Regarding the seminorm $[u]_{C^{0,\alpha}} = \sup_{y,y^*\in M,\ y\neq y^*} \frac{|u(y)-u(y^*)|}{|y-y^*|^\alpha}$, given two points $y, y^* \in M$, we have

$$|u(y) - u(y^*)| = \Big|\sum_{i=1}^k v_i(y) - v_i(y^*)\Big| \leqslant \sum_{i=1}^k |v_i(y) - v_i(y^*)|\,. \tag{2.14}$$

Then, for any C^1 function $v : M \to \mathbb{R}$ with compact support in $M\cap Q_i$, if y and y^* both belong to the intersection of M with the "enlarged" hypercube $\widetilde{Q}_i$, we can write $y = x + \theta_i(x)\nu_i$ and $y^* = x^* + \theta_i(x^*)\nu_i$ for some $x, x^* \in \widetilde{Q}_i \cap \Pi_{p_i} M_0$ (by our initial choice of the family Q_i) and there holds

$$\begin{aligned}
|v(y) - v(y^*)| &= |v(x + \theta_i(x)\nu_i) - v(x^* + \theta_i(x^*)\nu_i)|\\
&\leqslant C(M_0, p)\, |x - x^*|^\alpha\, \|\nabla^{\mathbb{R}^n}(v\circ\Theta)\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha\, \|\nabla^{\mathbb{R}^n}(v\circ\Theta)\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha\, \|\nabla v\|_{L^p(M)}\,,
\end{aligned}$$

where the first inequality follows as in the proof of Theorem 4 in Section 5.6.2 of [14], the second one holds since $|x-x^*| \leqslant |y-y^*|$ and the third one is obtained arguing like in estimate (2.12).

If both y^* and y do not belong to $M\cap\widetilde{Q}_i$ clearly $|v(y) - v(y^*)| = 0$, while if $y \in M\cap\widetilde{Q}_i$ with $v(y) \neq 0$ but $y^* \notin M\cap\widetilde{Q}_i$, then $y \in M\cap Q_i$, hence $|y - y^*| \geqslant \sigma$ and

$$\frac{|v(y) - v(y^*)|}{|y - y^*|^\alpha} \leqslant \frac{|v(y)|}{\sigma^\alpha} \leqslant C(M_0, p, \delta)\frac{\|\nabla v\|_{L^p(M)}}{\sigma^\alpha}\,,$$

by estimate (2.12).

It follows that, for every y and y^* in M, we have

$$\frac{|v(y) - v(y^*)|}{|y - y^*|^\alpha} \leqslant C(M_0, p, \delta)(1 + \sigma^{-\alpha})\|\nabla v\|_{L^p(M)}\,.$$

Then, putting together this and inequality (2.14), we conclude, for every y and y^* in M,

$$|u(y) - u(y^*)| \leqslant \sum_{i=1}^k |v_i(y) - v_i(y^*)| \leqslant C(M_0, p, \delta)\, |y - y^*|^\alpha\, \|\nabla u\|_{W^{1,p}(M)}$$

which, with inequality (2.13) gives the desired estimate (2.11).

(iv) In order to obtain the conclusion for the Poincaré–Wirtinger inequality, for any $p \in [1,+\infty]$ and all $M \in \mathfrak{C}^1_\delta(M_0)$,

$$\|u-\widetilde{u}\|_{L^p(M)} \leqslant C(M_0,p,\delta)\|\nabla u\|_{L^p(M)}\,,$$

where $\widetilde{u} = \fint_M u\,d\mu$, we argue by contradiction assuming this uniform estimate is false. Then, for each $k \in \mathbb{N}$, there would exist a graph hypersurface $M_k \in \mathfrak{C}^1_\delta(M_0)$ and a function $u_k \in W^{1,p}(M_k)$ such that

$$\|u_k-\widetilde{u}_k\|_{L^p(M_k)} \geqslant k\|\nabla u_k\|_{L^p(M_k)}.$$

where $\widetilde{u}_k = \fint_{M_k} u_k\,d\mu_k$. We renormalize these function as

$$v_k = \frac{u_k-\widetilde{u}_k}{\|u_k-\widetilde{u}_k\|_{L^p(M_k)}}\,,$$

then, $\int_{M_k} v_k\,d\mu_k = 0$, $\|v_k\|_{L^p(M_k)} = 1$ and $\|\nabla v_k\|_{L^p(M_k)} \leqslant 1/k$.

If we consider the functions $w_k = v_k \circ \Psi_k : M_0 \to \mathbb{R}$, where $\Psi_k : M_0 \to M_k$ is given by $\Psi_k(x) = x + \psi_k(x)\nu(x)$ (as in the second way to deal with $C_S(M,1)$, at the beginning of this section), we have

$$0 < C'(M_0,p,\delta) \leqslant \|w_k\|_{L^p(M_0)} \leqslant C(M_0,p,\delta) \tag{2.15}$$

and

$$\|\nabla w_k\|_{L^p(M_0)} \leqslant C(M_0,p,\delta)/k\,. \tag{2.16}$$

In particular, the functions w_k are equibounded in $W^{1,p}(M_0)$, hence by the Rellich–Kondrachov embedding theorem and the estimate (2.16), there exists a subsequence (not relabeled) converging in $L^p(M_0)$ to a constant function equal to some $\lambda \in \mathbb{R}$ which cannot be zero, by the estimate (2.15). Moreover, there holds

$$\int_{M_0} w_k(x)\,J\Psi_k(x)\,d\mu_0(x) = \int_{M_k} w_k \circ \Psi_k^{-1}(y)\,d\mu_k(y) = \int_{M_k} v_k(y)\,d\mu_k(y) = 0\,,$$

hence, since $J\Psi_k$ are equibounded (formula (2.10)) and assuming, possibly passing again to a subsequence, that $\mathrm{Vol}(M_k) \to V > 0$, by means of point (i), we conclude

$$0 = \int_{M_0} (w_k(x)-\lambda)\,J\Psi_k(x)\,d\mu_0(x) + \lambda\int_{M_0} J\Psi_k(x)\,d\mu_0(x) \to \lambda V\,,$$

as $k \to \infty$, being $\int_{M_0} J\Psi_k(x)\,d\mu_0(x) = \mathrm{Vol}(M_k)$. This is clearly a contradiction, as $\lambda, V \neq 0$ and we are done.

The case $p = +\infty$ is analogous.

(v) As for the "usual" (with integer order) Sobolev spaces, all the constants in the embeddings of the fractional Sobolev spaces are also uniform for the family $\mathfrak{C}^1_\delta(M_0)$. The proof is along the same line, localizing with a partition of unity and using the inequalities holding in $\mathbb{R}^n$ (see [21] and [22]).

(vi) Finally, we want to show that for any q, r real numbers $1 \leqslant q \leqslant +\infty, 1 \leqslant r \leqslant +\infty$ and j, m integers $0 \leqslant j < m$, there exists a constant C depending on j, m, r, q, θ, M_0 and δ such that the following interpolation inequalities hold

$$\|\nabla^j u\|_{L^p(M)} \leqslant C\big(\|\nabla^m u\|_{L^r(M)} + \|u\|_{L^r(M)}\big)^\theta \|u\|^{1-\theta}_{L^q(M)}, \tag{2.17}$$

for all $M \in \mathfrak{C}^1_\delta(M_0)$, where

$$\frac{1}{p} = \frac{j}{n} + \theta\Big(\frac{1}{r} - \frac{m}{n}\Big) + \frac{1-\theta}{q}$$

for every $\theta \in [j/m, 1]$ such that p is nonnegative, with the exception of the case $r = \frac{n}{m-j} \neq 1$ for which the inequality is not valid for $\theta = 1$.

Moreover, if $u : M \to \mathbb{R}$ is a smooth function with $⨍_M u\, d\mu = 0$, inequality (2.17) simplifies to

$$\|\nabla^j u\|_{L^p(M)} \leqslant C\|\nabla^m u\|^\theta_{L^r(M)} \|u\|^{1-\theta}_{L^q(M)} . \tag{2.18}$$

We can obtain inequality (2.17) arguing as in Proposition 5.1 of [20], essentially following the line of the proof of Theorem 3.70 in [5], but substituting the *Sobolev–Poincarè inequality* (41) in the argument there with its version where the constant is uniform for all $M \in \mathfrak{C}^1_\delta(M_0)$. Indeed, the other "ingredients" in such proof are a bound on the volume (uniform, by point (i)) and some "universal" inequalities in which the constants do not depend on the hypersurfaces at all [5, Theorem 3.69].

Such Sobolev–Poincarè inequality (41) in Theorem 3.70 of [5] reads

$$\|u\|_{L^{p^*}(M)} \leqslant C_{SP}(M, p)\|\nabla u\|_{L^p(M)} , \tag{2.19}$$

for every C^1–function $u : M \to \mathbb{R}$ (or $u \in W^{1,p}(M)$) with $\int_M u\, d\mu = 0$, (here, as before, $p^* = \frac{np}{n-p}$ is the Sobolev conjugate exponent) and we actually need it with a uniform constant, in order to get inequality (2.18), by the very same proof of such theorem.

This inequality actually follows by points (ii) and (iv). Indeed, for every $u \in W^{1,p}(M)$, by Sobolev inequality, we have

$$\|u\|_{L^{p^*}(M)} \leqslant C(M_0, p, \delta)\big(\|\nabla u\|_{L^p(M)} + \|u\|_{L^p(M)}\big)$$

and, by Poincarè–Wirtinger inequality, as $\widetilde{u} = \int_M u\,d\mu = 0$,

$$\|u\|_{L^p(M)} \leqslant C(M_0, p, \delta)\|\nabla u\|_{L^p(M)}$$

hence, we obtain inequality (2.19) with $C_{SP}(M, p)$ bounded by a uniform constant $C(M_0, p, \delta)$, for every $M \in \mathfrak{C}^1_\delta(M_0)$. QED

Remark 2 (The fractional Sobolev spaces $W^{s,p}(M)$)**.** At point (v) of the theorem above we considered the fractional Sobolev space $W^{s,p}$ on the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, which are usually defined via local charts for M and partitions of unity, that is, getting back to the definition with the Gagliardo $W^{s,p}$–seminorms in $\mathbb{R}^n$ (we refer to [4, 12, 21, 22], for details). They can be also defined equivalently by considering directly on M the Gagliardo $W^{s,p}$–seminorm of a function $u \in L^p(M)$, for $s \in (0, 1)$, as follows:

$$[u]^p_{W^{s,p}(M)} = \int_M \int_M \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}}\, d\mu(x)\, d\mu(y)$$

and setting $\|u\|_{W^{s,p}(M)} = \|u\|_{L^p(M)} + [u]_{W^{s,p}(M)}$. Moreover, the constants giving the equivalence of the two norms obtained by localization or by this direct definition are uniform for all $M \in \mathfrak{C}^1_\delta(M_0)$. Indeed, the localization method of Section 1, is "uniform" for all $M \in \mathfrak{C}^1_\delta(M_0)$, meaning that the number of necessary local charts is fixed and the diffeomorphisms between $\mathbb{R}^n$ and "corresponding" (associated to correlated local charts, that is, being a graph on the same piece of M_0, as in our construction) local "pieces" of any different hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$, are uniformly "$C^1$–close" one to each other.

3 Geometric Calderón–Zygmund inequalities

Theorem 2. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $p \in (1, +\infty)$. Then, if $\delta > 0$ is small enough, there exists a constant $C(M_0, p, \delta)$ such that the following geometric Calderón–Zygmund inequality holds,*

$$\|\mathrm{B}\|_{L^p(M)} \leqslant C(M_0, p, \delta)\big(1 + \|\mathrm{H}\|_{L^p(M)}\big)$$

for every $M \in \mathfrak{C}^1_\delta(M_0)$.

Proof. We recall the local representation as graphs of the hypersurfaces $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0, as at the beginning of the previous section. We have a finite family of hypercubes Q_i centered at $p_i \in M_0$, the partition of unity ρ_i and a parametrization $x \mapsto \Theta(x) = x + \theta_i(x)\nu_i$ on $\Pi_{p_i} M_0 \cap Q_i$ of each piece $M \cap Q_i$ of any smooth hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$, where $\nu_i = \nu(p_i)$ and the functions

$\theta_i : \Pi_{p_i} M_0 \to \mathbb{R}$ satisfy $\|\theta_i\|_{C^1(\Pi_{p_i} M_0)} \leqslant 2\delta$, for every $i \in \{1, \dots, k\}$. Moreover, in dealing with any piece $M \cap Q_i$, we will assume (clearly without losing generality) that $\Pi_{p_i} M_0 = \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ and that $Q_i \cap \Pi_{p_i} M_0$ is the hypercube $Q_{2R} \subseteq \Pi_{p_i} M_0 = \mathbb{R}^n$ with edges of length $2R > 0$, centered at the origin. Finally, we can also ask that the family of hypercubes $Q_i' \subseteq \mathbb{R}^n$ with edges parallel to the ones of Q_i and of length R (half of the one of Q_i), centered at p_i, covers any hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$.

By formulas (1.3) and (1.4), in the parametrization of $M \cap Q_i$ given by Θ, the second fundamental form B and mean curvature H of M are then expressed by

$$\mathrm{B} \circ \Theta = -\frac{\mathrm{Hess}^{\mathbb{R}^n} \theta_i}{\sqrt{1 + |\nabla^{\mathbb{R}^n} \theta_i|^2}} \tag{3.20}$$

and

$$\mathrm{H} \circ \Theta = -\frac{\Delta^{\mathbb{R}^n} \theta_i}{\sqrt{1 + |\nabla^{\mathbb{R}^n} \theta_i|^2}} + \frac{\mathrm{Hess}^{\mathbb{R}^n} \theta_i (\nabla^{\mathbb{R}^n} \theta_i, \nabla^{\mathbb{R}^n} \theta_i)}{\big(\sqrt{1 + |\nabla^{\mathbb{R}^n} \theta_i|^2}\big)^3} \,.$$

Letting and $\rho : \mathbb{R}^n \to [0,1]$ a cut–off function with compact support in Q_{2R} and equal to 1 on $Q_R = Q_i' \cap \Pi_{p_i} M_0$ and setting $A_R = \{(x, \theta_i(x)) \ : \ x \in Q_R\}$, $A_{2R} = \{(x, \theta_i(x)) \ : \ x \in Q_{2R}\}$, we have

$$\|\mathrm{B}\|^p_{L^p(A_R)} = \int_{Q_R} |\mathrm{B} \circ \Theta|^p J\Theta \, dx \leqslant \int_{Q_R} \rho^p |\mathrm{Hess}^{\mathbb{R}^n} \theta_i|^p \, dx = \int_{\mathbb{R}^n} |\rho \mathrm{Hess}^{\mathbb{R}^n} \theta_i|^p \, dx \,, \tag{3.21}$$

as $\mu = J\Theta \, \mathscr{L}^n$ and $J\Theta = \sqrt{1 + |\nabla^{\mathbb{R}^n} \theta_i|^2}$. Then, we estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |\rho \mathrm{Hess}^{\mathbb{R}^n} \theta_i|^p \, dx &\leqslant C \int_{\mathbb{R}^n} |\mathrm{Hess}^{\mathbb{R}^n} (\rho \theta_i)|^p \, dx + C \int_{\mathbb{R}^n} |2 \nabla^{\mathbb{R}^n} \rho \otimes \nabla^{\mathbb{R}^n} \theta_i|^p \, dx \\ &\quad + C \int_{\mathbb{R}^n} |\theta_i \mathrm{Hess}^{\mathbb{R}^n} \rho|^p \, dx \\ &\leqslant C \int_{\mathbb{R}^n} |\mathrm{Hess}^{\mathbb{R}^n} (\rho \theta_i)|^p \, dx + C \,, \end{aligned}$$

where $C = C(M_0, p, \delta)$, as the last two integrals in the first line are clearly bounded by a constant $C = C(M_0, p, \delta)$.

Hence, applying the standard Calderón–Zygmund estimates in $\mathbb{R}^n$ (see [16], for instance) to the last term above, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |\rho \mathrm{Hess}^{\mathbb{R}^n} \theta_i|^p \, dx \\ &\leqslant C \int_{\mathbb{R}^n} |\Delta^{\mathbb{R}^n} (\rho \theta_i)|^p \, dx + C \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\int_{\mathbb{R}^n}|\rho\Delta^{\mathbb{R}^n}\theta_i|^p\,dx + C\int_{\mathbb{R}^n}|2\langle\nabla^{\mathbb{R}^n}\rho\,|\,\nabla^{\mathbb{R}^n}\theta_i\rangle|^p\,dx + C\int_{\mathbb{R}^n}|\theta_i\Delta^{\mathbb{R}^n}\rho|^p\,dx\\
&\leqslant C\int_{\mathbb{R}^n}\Bigl|-\rho(\mathrm{H}\circ\Theta)\sqrt{1+|\nabla^{\mathbb{R}^n}\theta_i|^2}+\frac{\rho\mathrm{Hess}^{\mathbb{R}^n}\theta_i(\nabla^{\mathbb{R}^n}\theta_i,\nabla^{\mathbb{R}^n}\theta_i)}{1+|\nabla^{\mathbb{R}^n}\theta_i|^2}\Bigr|^p\,dx + C\\
&\leqslant C\int_{\mathbb{R}^n}|\rho(\mathrm{H}\circ\Theta)|^p\,dx + C\int_{\mathbb{R}^n}|\rho\mathrm{Hess}^{\mathbb{R}^n}\theta_i(\nabla^{\mathbb{R}^n}\theta_i,\nabla^{\mathbb{R}^n}\theta_i)|^p\,dx + C\\
&\leqslant C\int_{\mathbb{R}^n}|\rho(\mathrm{H}\circ\Theta)|^p\,dx + C\int_{\mathbb{R}^n}|\nabla^{\mathbb{R}^n}\theta_i|^{2p}|\rho\mathrm{Hess}^{\mathbb{R}^n}\theta_i|^p\,dx + C
\end{aligned}
$$

where the constant C depends only on M_0, p and δ (we estimated the last two integrals in the second line with such a constant, as we did above for the Hessian).

If $\delta > 0$ is small enough, then $C|\nabla^{\mathbb{R}^n}\theta_i|^{2p} < 1/2$ and we get

$$
\int_{\mathbb{R}^n}|\rho\mathrm{Hess}^{\mathbb{R}^n}\theta_i|^p\,dx \leqslant 2C\int_{\mathbb{R}^n}|\rho(\mathrm{H}\circ\Theta)|^p\,dx + 2C \leqslant 2C\int_{Q_{2R}}|(\mathrm{H}\circ\Theta)|^p\,dx + 2C
$$

which clearly implies, by formula (3.21),

$$
\begin{aligned}
\|\mathrm{B}\|_{L^p(A_R)} &\leqslant C\int_{Q_{2R}}|(\mathrm{H}\circ\Theta)|^p\,dx + C \leqslant C\int_{Q_{2R}}|(\mathrm{H}\circ\Theta)|^p J\Theta\,dx + C\\
&\leqslant C\bigl(1+\|\mathrm{H}\|^p_{L^p(A_{2R})}\bigr)\,,
\end{aligned}
$$

with $C = C(M_0, p, \delta)$.

Hence, by construction and invariance by isometry,

$$
\|\mathrm{B}\|_{L^p(M\cap Q'_i)} \leqslant C\bigl(1+\|\mathrm{H}\|^p_{L^p(M\cap Q_i)}\bigr) \leqslant C\bigl(1+\|\mathrm{H}\|^p_{L^p(M)}\bigr)\,.
$$

Since the number of hypercubes Q'_i covering M is fixed and $C = C(M_0, p, \delta)$, we obtain the thesis of the theorem. QED

We have an analogous theorem for Schauder estimates, after defining appropriately the Hölder $C^{0,\alpha}$–norm of a tensor T on M, that is,

$$
\|T\|_{C^{0,\alpha}(M)} = \sup_M |T| + [T]_{C^{0,\alpha}(M)}
$$

where we need to give a meaning to the seminorm $[T]_{C^{0,\alpha}(M)}$.

If T is an m–form (hence, a covariant m–tensor), one possibility is to "extend the action" of the tensor T from the bundle $\oplus^m TM$ of covariant m-–tensors on M to the one of the whole "ambient" $\mathbb{R}^{n+1}$ by means of the orthogonal projection on the tangent bundle TM (as we identify T_xM with a vector subspace of $T_x\mathbb{R}^{n+1} \approx \mathbb{R}^{n+1}$, for every $x \in M$). To give an example, if $T = \mathrm{B}$, letting

$\pi_x : \mathbb{R}^{n+1} \to T_xM$ be the orthogonal projection on the tangent space of M, for every $x \in M$, we can define the "extension" of B (without relabeling it) by considering at every $x \in M$ the bilinear form $\mathrm{B}_x : \oplus^2 T_x\mathbb{R}^{n+1} \approx \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ as $\mathrm{B}_x(v, w) = \mathrm{B}_x(\pi_x(v), \pi_x(w))$. Extending analogously a general m–form T from operating on $\oplus^m TM$ to $\oplus^m T\mathbb{R}^{n+1}$, its norm as a multilinear functional is unchanged at every point $x \in M$ and we can then consider its components $T_{j_1 \dots j_m}$ in the canonical basis of $\mathbb{R}^{n+1}$ to define

$$\begin{aligned} [T]_{C^{0,\alpha}(M)} &= \sum_{j_1,\dots,j_m=1}^{n+1} [T_{j_1\dots j_m}]_{C^{0,\alpha}(M)} \\ &= \sum_{j_1,\dots,j_m=1}^{n+1} \sup_{\substack{x,y\in M \\ x\neq y}} \frac{|T_{j_1\dots j_m}(x) - T_{j_1\dots j_m}(y)|}{|x-y|^\alpha} \,. \end{aligned}$$

Finally, if the tensor is of general type (it has also contravariant components), we "transform" it in a covariant one by means of the musical isomorphisms (see [15], for instance) and then proceed as above. Anyway, in the following all the tensors will be covariant.

Remark 3. This "global", partially coordinate–free definition (only the canonical coordinates of $\mathbb{R}^{n+1}$ are involved, not any coordinate chart for M) is useful in general, but in our special case of families of hypersurfaces which are representable as graphs on a fixed one, we can also consider an *equivalent* Hölder seminorm by means of the local description of M with the hypercubes Q_i, which is more convenient for our computations. For any m–form T on M, we set (in the notation of the proof of Theorem 2)

$$\begin{aligned} [T]_{C^{0,\alpha}(V)} &= \sum_{j_1,\dots,j_m=1}^{n} [T_{j_1\dots j_m} \circ \Theta]_{C^{0,\alpha}(\Theta^{-1}(V))} \\ &= \sum_{j_1,\dots,j_m=1}^{n} \sup_{\substack{x,y\in \Theta^{-1}(V) \\ x\neq y}} \frac{|T_{j_1\dots j_m}(\Theta(x)) - T_{j_1\dots j_m}(\Theta(y))|}{|x-y|^\alpha} \,, \end{aligned}$$

for every open set $V \subseteq M \cap Q_i$, where $T_{j_1 \dots j_m}$ are the components of T in the parametrization $x \mapsto \Theta(x) = x + \theta_i(x)e_{n+1}$. Then, we define

$$[T]_{C^{0,\alpha}(M)} = \sum_{i=1}^{k} [T]_{C^{0,\alpha}(A_R)},$$

by means of the finite family of sets A_R (whose number is fixed) covering $M \in \mathfrak{C}^1_\delta(M_0)$.

Theorem 3. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $\alpha \in (0,1]$. Then, if $\delta > 0$ is small enough, there exists a constant $C(M_0,\alpha,\delta)$ such that the following geometric Schauder estimate holds,*

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C(M_0,\alpha,\delta)\big(1+\|\mathrm{H}\|_{C^{0,\alpha}(M)}\big)$$

for every $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$.

Proof. In the same setting and notation of the proof of Theorem 2, for every hypercube Q_i, the function θ_i belongs to $C^{1,\alpha}(Q_{2R})$, with $\|\theta_i\|_{C^{1,\alpha}(Q_{2R})} \leqslant 2\delta$. Then, keeping into account Remark 3, we deal with $\|\mathrm{B}\|_{C^{0,\alpha}(A_R)}$, which satisfies

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} = \|\mathrm{B}\circ\Theta\|_{C^{0,\alpha}(Q_R)} = \left\|\frac{\mathrm{Hess}^{\mathbb{R}^n}\theta_i}{\sqrt{1+|\nabla^{\mathbb{R}^n}\theta_i|^2}}\right\|_{C^{0,\alpha}(Q_R)} \leqslant C\,\|\theta_i\|_{C^{2,\alpha}(Q_R)}\,, \tag{3.22}$$

by equality (3.20) and since $Q_R = \Theta^{-1}(A_R)$, by construction.

Hence, by the standard Schauder estimates in $Q_{2R} = \Theta^{-1}(A_{2R})$ (see [16], for instance), we get

$$\begin{aligned}
&\|\theta_i\|_{C^{2,\alpha}(Q_R)}\\
&\leqslant C\,\|\Delta^{\mathbb{R}^n}\theta_i\|_{C^{0,\alpha}(Q_{2R})} + C\|\theta_i\|_{C^{1,\alpha}(Q_{2R})}\\
&\leqslant C\left\|-(\mathrm{H}\circ\Theta)\sqrt{1+|\nabla^{\mathbb{R}^n}\theta_i|^2} + \frac{\mathrm{Hess}^{\mathbb{R}^n}\theta_i(\nabla^{\mathbb{R}^n}\theta_i,\nabla^{\mathbb{R}^n}\theta_i)}{1+|\nabla^{\mathbb{R}^n}\theta_i|^2}\right\|_{C^{0,\alpha}(Q_{2R})} + C\\
&\leqslant C\,\|\mathrm{H}\circ\Theta\|_{C^{0,\alpha}(Q_{2R})} + C\,\|\nabla^{\mathbb{R}^n}\theta_i\|^2_{C^{0,\alpha}(Q_{2R})}\|\mathrm{Hess}^{\mathbb{R}^n}\theta_i\|_{C^{0,\alpha}(Q_{2R})} + C\\
&\leqslant C\,\|\mathrm{H}\circ\Theta\|_{C^{0,\alpha}(Q_{2R})} + C\delta^2\|\theta_i\|_{C^{2,\alpha}(Q_{2R})} + C\,,
\end{aligned}$$

where the constant C depends only on M_0, α and δ, as $\|\theta_i\|_{C^{1,\alpha}(Q_{2R})} \leqslant 2\delta$. This estimate clearly implies, by formula (3.22) and equality (3.20),

$$\|\mathrm{B}\|_{C^{0,\alpha}(A_R)} \leqslant C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + C\delta^2\|\mathrm{B}\|_{C^{0,\alpha}(M)} + C$$

and since the family of sets A_R covering $M \in \mathfrak{C}^1_\delta(M_0)$ is finite and its number is fixed, we conclude

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + C\delta^2\|\mathrm{B}\|_{C^{0,\alpha}(M)} + C\,,$$

with a constant C depending only on M_0, α and δ (and we can clearly choose C to be monotonically increasing with δ).

Then, if $\delta > 0$ is small enough, we have $C\delta^2\,\|\mathrm{B}\|^2_{C^{0,\alpha}(M)} < \|\mathrm{B}\|^2_{C^{0,\alpha}(M)}/2$, hence we get

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant 2C\,\|\mathrm{H}\|_{C^{0,\alpha}(M)} + 2C\,,$$

that is,

$$\|\mathrm{B}\|_{C^{0,\alpha}(M)} \leqslant C\left(1 + \|\mathrm{H}\|_{C^{0,\alpha}(M)}\right),$$

where the constant C depends only on M_0, α and δ, which is the thesis of the theorem. QED

We now deal with families of n–dimensional graph hypersurfaces in $M \in \mathfrak{C}^1_\delta(M_0)$ over M_0 with a uniform bound $\|\mathrm{B}\|_{L^\infty(M)}$ on the second fundamental form.

Arguing again in the same setting and notation of the proof of Theorem 2, for $p \in (1, +\infty)$ and any C^2–function $u : M \to \mathbb{R}$ (or $u \in W^{2,p}(M)$), we have

$$\|\nabla^2 u\|_{L^p(M)} \leqslant C \sum_{i=1}^{k} \|\nabla^2(u\rho_i)\|_{L^p(M\cap Q_i)} \tag{3.23}$$

(here ∇ is the Levi–Civita connection of M) and, for every C^2 function $v : M \to \mathbb{R}$, with compact support in $M \cap Q_i$, there holds

$$\begin{aligned}\int_{M\cap Q_i} |\nabla^2 v(y)|^p\, d\mu(y) &= \int_{\mathbb{R}^n} \bigl|(\nabla^2 v)(x + \theta_i(x)\nu_i)\bigr|^p J\Theta(x)\, dx \\ &\leqslant C(\delta) \int_{\mathbb{R}^n} \bigl|(\nabla^2 v)(x + \theta_i(x)\nu_i)\bigr|^p\, dx\,, \end{aligned}\tag{3.24}$$

as $J\Theta = \sqrt{1 + |\nabla^{\mathbb{R}^n}\theta_i|^2} \leqslant 1 + 2\delta$.

In the coordinates given by the parametrization Θ, the coefficients of the metric g of M (induced by $\mathbb{R}^{n+1}$) in $M \cap Q_i$ are

$$g_{\ell m}(\Theta(x)) = \delta_{\ell m} + \frac{\partial \theta_i}{\partial x_\ell}(x)\frac{\partial \theta_i}{\partial x_m}(x)\,,$$

hence, they and the ones of the inverse matrix are bounded by a constant depending only on M_0 and δ. By formula (1.1), the Christoffel symbols of the Levi–Civita connection ∇ satisfy

$$|\Gamma^s_{\ell m}(\Theta(x))| \leqslant C \sum_{p,q,r=1}^{n} \Bigl|\frac{\partial (g_{pq}\circ\Theta)}{\partial x_r}(x)\Bigr| = C \sum_{p,q,r=1}^{n} \Bigl|\frac{\partial^2 \theta_i}{\partial x_r \partial x_p}(x)\frac{\partial \theta_i}{\partial x_q}(x)\Bigr|\,. \tag{3.25}$$

Then, recalling the first formula (1.5),

$$\begin{aligned}\Bigl|\frac{\partial^2 \theta_i}{\partial x_\ell \partial x_m}(x)\Bigr| &= \Bigl|\frac{\partial^2 \Theta}{\partial x_\ell \partial x_m}(x)\Bigr| \\ &= \Bigl|\Gamma^s_{\ell m}(\Theta(x))\frac{\partial \Theta}{\partial x_s}(x) - \mathrm{B}_{\ell m}(\Theta(x))\nu(\Theta(x))\Bigr| \\ &\leqslant C|\Gamma^s_{\ell m}(\Theta(x))|\,\Bigl|\frac{\partial \Theta}{\partial x_s}(x)\Bigr| + |\mathrm{B}_{\ell m}(\Theta(x))| \\ &\leqslant C|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)|\,|\nabla^{\mathbb{R}^n}\theta_i(x)|\,(1 + |\nabla^{\mathbb{R}^n}\theta_i(x)|) + |\mathrm{B}(\Theta(x))|\,,\end{aligned}$$

where in the last passage we estimated the Christoffel symbols by means of inequality (3.25). As $|\nabla^{\mathbb{R}^n}\theta_i| \leqslant 2\delta$, we conclude

$$\begin{aligned}|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)| &\leqslant C|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)|\,|\nabla^{\mathbb{R}^n}\theta_i(x)| + C|\mathrm{B}(\Theta(x))|\\ &\leqslant C|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)|\delta + C|\mathrm{B}(\Theta(x))|\end{aligned}$$

with a constant C depending only on δ, which implies, if δ is smaller than $1/2C$, the estimate

$$|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)| \leqslant 2C(M_0,\delta)|\mathrm{B}(\Theta(x))|\,,$$

for every $x \in Q_i \cap \Pi_{p_i} M \subseteq \mathbb{R}^n$.

By the first formula (3.25), it follows

$$|\Gamma^s_{\ell m}(\Theta(x))| \leqslant C|\mathrm{Hess}^{\mathbb{R}^n}\theta_i(x)|\,|\nabla^{\mathbb{R}^n}\theta_i| \leqslant C\delta|\mathrm{B}(\Theta(x))|$$

with $C = C(\delta)$, then computing schematically, we have

$$(\nabla^2 v)(\Theta(x)) = \mathrm{Hess}^{\mathbb{R}^n}(v\circ\Theta)(x) - \Gamma(\Theta(x)) \star \nabla^{\mathbb{R}^n}(v\circ\Theta)(x)\,, \tag{3.26}$$

hence,

$$|(\nabla^2 v)(\Theta(x))| \leqslant C|\mathrm{Hess}^{\mathbb{R}^n}(v\circ\Theta)(x)| + C\delta|\mathrm{B}(\Theta(x))|\,|\nabla^{\mathbb{R}^n}(v\circ\Theta)(x)|\,.$$

Applying the Calderón–Zygmund inequality in $\mathbb{R}^n$, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\big|(\nabla^2 v)(x+\theta_i(x)\nu_i)\big|^p\,dx &\leqslant C\int_{\mathbb{R}^n}|\mathrm{Hess}^{\mathbb{R}^n}[v(x+\theta_i(x)\nu_i)]|^p\,dx\\ &\quad + C\delta\int_{\mathbb{R}^n}|\mathrm{B}(\Theta(x))|^p\,|\nabla^{\mathbb{R}^n}[v(x+\theta_i(x)\nu_i])|^p\,dx\\ &\leqslant C\int_{\mathbb{R}^n}|\Delta^{\mathbb{R}^n}[v(x+\theta_i(x)\nu_i)]|^p\,dx\\ &\quad + C(\delta)\int_{\mathbb{R}^n}|\mathrm{B}(\Theta(x))|^p\,|\nabla v(\Theta(x))|^p\,dx\,.\\ &\leqslant C\int_{\mathbb{R}^n}|\Delta^{\mathbb{R}^n}[v(x+\theta_i(x)\nu_i)]|^p\,dx\\ &\quad + C(\delta)\int_{M\cap Q_i}|\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\,,\end{aligned} \tag{3.27}$$

arguing as in estimate (2.12) to get the last inequality.

Contracting equation (3.26) with the inverse of the metric and estimating, we have

$$|\Delta^{\mathbb{R}^n}(v\circ\Theta)(x)| \leqslant C|(\Delta v)(\Theta(x))| + C\delta|(\mathrm{B}\circ\Theta)(x)|\,|\nabla^{\mathbb{R}^n}(v\circ\Theta)(x)|$$

thus, by inequalities (3.24) and (3.27), we obtain

$$\begin{aligned}\int_{M\cap Q_i}|\nabla^2 v(y)|^p\,d\mu(y) &\leqslant C\int_{\mathbb{R}^n}|(\Delta v)(x+\theta_i(x)\nu_i)|^p\,dx\\ &\qquad + C\int_{M\cap Q_i}|\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\\ &\leqslant C\int_{M\cap Q_i}|\Delta v(y)|^p\,d\mu(y)\\ &\qquad + C\int_{M\cap Q_i}|\mathrm{B}(y)|^p|\nabla v(y)|^p\,d\mu(y)\,,\end{aligned}$$

with $C = C(M_0, p, \delta)$, arguing again as above.

Getting back to inequality (3.23), we conclude

$$\begin{aligned}\|\nabla^2 u\|^p_{L^p(M)} &\leqslant C\sum_{i=1}^k\|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\\ &\leqslant C\sum_{i=1}^k\int_{M\cap Q_i}|\Delta(u\rho_i)|^p\,d\mu + C\int_{M\cap Q_i}|\mathrm{B}|^p|\nabla(u\rho_i)|^p\,d\mu\\ &\leqslant C\sum_{i=1}^k\int_{M\cap Q_i}|\Delta u|^p\,d\mu + C\int_{M\cap Q_i}\big(|u|^p+|\nabla u|^p\big)\,d\mu\\ &\leqslant C\int_M|\Delta u|^p\,d\mu + C\int_M\big(|u|^p+|\nabla u|^p\big)\,d\mu\,, \end{aligned}\tag{3.28}$$

with $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^\infty(M)})$. Interpolating the integral of $|\nabla u|^p$ between $\|\nabla^2 u\|_{L^p(M)}$ and $\|u\|_{L^p(M)}$ by means of the uniform Gagliardo–Nirenberg inequalities of the previous section, we obtain the following theorem.

Theorem 4. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$ and $p \in (1, +\infty)$. Then, if $\delta > 0$ is small enough, there exists a constant C which depends only on M_0, p, δ and $\|\mathrm{B}\|_{L^\infty(M)}$ such that the following Calderón–Zygmund inequality holds,*

$$\|\nabla^2 u\|_{L^p(M)} \leqslant C\|\Delta u\|_{L^p(M)} + C\|u\|_{L^p(M)} \tag{3.29}$$

hence,

$$\|u\|_{W^{2,p}(M)} \leqslant C\|\Delta u\|_{L^p(M)} + C\|u\|_{L^p(M)}\,, \tag{3.30}$$

for every hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$ and $u \in W^{2,p}(M)$.

Remark 4. Notice that if $p < n$, we can modify the chain of inequalities (3.28) as follows:

$$\begin{aligned}
\|\nabla^2 u\|^p_{L^p(M)} &\leqslant C\sum_{i=1}^k \|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu + C\int_{M\cap Q_i} |\mathrm{B}|^p|\nabla(u\rho_i)|^p\,d\mu\\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu\\
&\quad + C\Big(\int_{M\cap Q_i} |\mathrm{B}|^n\,d\mu\Big)^{p/n}\Big(\int_{M\cap Q_i} |\nabla(u\rho_i)|^{np/(n-p)}\,d\mu\Big)^{(n-p)/n}\\
&\leqslant C\sum_{i=1}^k \int_{M\cap Q_i} |\Delta(u\rho_i)|^p\,d\mu + C\|\mathrm{B}\|^p_{L^n(M\cap Q_i)}\|\nabla^2(u\rho_i)\|^p_{L^p(M\cap Q_i)}\,.
\end{aligned}$$

Hence, arguing as before, it is easy to conclude that inequalities (3.29) and (3.30) hold with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^n(M)})$, if $\delta > 0$ is small enough. Moreover, since we have seen in Theorem 2 that a control on $\|\mathrm{H}\|_{L^n(M)}$ implies a control on $\|\mathrm{B}\|_{L^n(M)}$, we have uniform Calderón–Zygmund inequalities for families of n–dimensional graph hypersurfaces over M_0, with mean curvature uniformly bounded in $L^n(M)$.

With a similar argument, computing as in Theorem 3, we have analogous Schauder estimates for $C^{2,\alpha}$ functions $u : M \to \mathbb{R}$, with $M \in \mathfrak{C}^{1,\alpha}_\delta(M_0)$ and $\delta > 0$ small enough,

$$\|u\|_{C^{2,\alpha}(M)} \leqslant C\|\Delta u\|_{C^{0,\alpha}(M)} + C\|u\|_{C^{0,\alpha}(M)}\,, \tag{3.31}$$

where the constant C depends only on M_0, $\alpha \in (0,1]$, δ and $\|\mathrm{B}\|_{C^{0,\alpha}(M)}$ (or $\|\mathrm{H}\|_{C^{0,\alpha}(M)}$, by Theorem 3).

Remark 5. Localizing and computing in coordinates (see Remark 3), it is easy to generalize estimates (3.29), (3.30) and (3.31) also to tensors, under the same hypotheses. The same holds also for all the estimates of the previous section (see [20] for an example of how this can be done).

3.1 Geometric higher order Calderón–Zygmund estimates

We let M_0 as above and $p > 1$, we want now to deal with $\|\nabla^k \mathrm{B}\|_{L^p(M)}$, assuming that we have a uniform bound on $\|\mathrm{H}\|_{L^q(M)}$ with $q > n$, where M

is any n–dimensional graph hypersurface over M_0 in $\mathfrak{C}^1_\delta(M_0)$, if $\delta > 0$ is small enough.

Theorem 5. *Let $M_0 \subseteq \mathbb{R}^{n+1}$ be a smooth, compact hypersurface, embedded in $\mathbb{R}^{n+1}$. Then, for any $q > n$, if $\delta > 0$ is small enough, there exists a constant C which depends only on M_0, p, q, δ and $\|\mathrm{H}\|_{L^q(M)}$, such that the following geometric higher order Calderón–Zygmund inequality holds, for $p \in (1, n)$,*

$$\|\nabla^k \mathrm{B}\|_{L^p(M)} \leqslant C\big(1 + \|\nabla^k \mathrm{H}\|_{L^p(M)}\big)$$

hence,

$$\|\mathrm{B}\|_{W^{k,p}(M)} \leqslant C\big(1 + \|\mathrm{H}\|_{W^{k,p}(M)}\big)\,,$$

for any hypersurface $M \in \mathfrak{C}^1_\delta(M_0)$ and $k \in \mathbb{N}$.

Moreover, the same inequalities hold for any $p \in (1, +\infty)$ with a constant C depending only on M_0, p, δ and $\|\mathrm{B}\|_{L^\infty(M)}$.

Proof. We first deal with the case $p \in (1, n)$. Fixed $k \in \mathbb{N}$, by means of inequality (3.29), which holds with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^n(M)})$, by Remark 4 and taking into account Remark 5, we have

$$\begin{aligned}
\|\nabla^k \mathrm{B}\|_{L^p(M)} &= \|\nabla_{i_1}\nabla_{i_2}(\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B})\|_{L^p(M)}\\
&\leqslant C\|\Delta(\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B})\|_{L^p(M)} + C\|\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)}\\
&= C\|g^{\ell m}\nabla_\ell\nabla_m\nabla_{i_3}\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_m\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\|\mathrm{Riem} \star \nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\nabla\mathrm{Riem} \star \nabla^{k-3}\mathrm{B}\|_{L^p(M)}\\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\nabla_m\cdots\nabla_{i_k}\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\|\mathrm{Riem} \star \nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\nabla\mathrm{Riem} \star \nabla^{k-3}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\|\nabla^2\mathrm{Riem} \star \nabla^{k-4}\mathrm{B}\|_{L^p(M)}\\
&\qquad\qquad\qquad\cdots\\
&\leqslant C\|g^{\ell m}\nabla_\ell\nabla_{i_3}\nabla_{i_4}\cdots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} \star \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\leqslant C\|g^{\ell m}\nabla_{i_3}\nabla_\ell\nabla_{i_4}\cdots\nabla_{i_k}\nabla_m\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem} \star \nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\qquad\qquad\qquad\cdots
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|g^{\ell m}\nabla_{i_3}\nabla_{i_4}\cdots\nabla_{i_k}\nabla_\ell\nabla_m \mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&= C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s\mathrm{Riem}\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}
\end{aligned}
$$

where the symbol $T\star S$ (following Hamilton [17]) denotes a tensor formed by a sum of terms each one given by some contraction of the pair T, S with the inverse of the metric g^{ij}. A very useful property of such $\star$ product is that $|T\star S|\leqslant C|T||S|$ where the constant C depends only on the "algebraic structure" of $T\star S$, moreover, it clearly holds $\nabla T\star S = \nabla T\star S + T\star\nabla S$.

By formula (1.7) for the Riemann tensor, we can then write $\mathrm{Riem} = \mathrm{B}\star\mathrm{B}$, hence

$$
\begin{aligned}
\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leqslant C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{s=0}^{k-2}\|\nabla^s(\mathrm{B}\star\mathrm{B})\star\nabla^{k-2-s}\mathrm{B}\|_{L^p(M)}\\
&\leqslant C\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{\substack{s,r,t\in\mathbb{N}\\ s+r+t=k-2}}\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)}\,. \qquad (3.32)
\end{aligned}
$$

Now, by Simons' identity (1.6), we have

$$
\nabla^{k-2}\Delta\mathrm{B} = \nabla^k\mathrm{H} + \nabla^{k-2}(\mathrm{H}\mathrm{B}^2) - \nabla^{k-2}(|\mathrm{B}|^2\mathrm{B})\,,
$$

hence,

$$
\|\nabla^{k-2}\Delta\mathrm{B}\|_{L^p(M)} \leqslant \|\nabla^k\mathrm{H}\|_{L^p(M)} + C\sum_{\substack{s,r,t\in\mathbb{N}\\ s+r+t=k-2}}\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)}\,.
$$

Using this estimate in inequality (3.32), we conclude

$$
\begin{aligned}
\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leqslant C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}\\
&\quad + C\sum_{\substack{s,r,t\in\mathbb{N}\\ s+r+t=k-2}}\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)}\,.
\end{aligned}
$$

We now estimate any of the terms in the last sum as follows: we have

$$
\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)} \leqslant\ C\|\nabla^s\mathrm{B}\|_{L^{\alpha p}(M)}\|\nabla^r\mathrm{B}\|_{L^{\beta p}(M)}\|\nabla^t\mathrm{B}\|_{L^{\gamma p}(M)}\,, \qquad (3.33)
$$

with

$$\alpha = \frac{k+1}{s+1}\,, \qquad \beta = \frac{k+1}{r+1}\,, \qquad \gamma = \frac{k+1}{t+1}\,,$$

hence, $1/\alpha+1/\beta+1/\gamma = 1$, as $s+r+t = k-2$. Moreover, using the interpolation estimates (2.17) (extended to tensors – see Remark 5), there hold

$$\begin{aligned}
\|\nabla^s \mathrm{B}\|_{L^{p\alpha}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\alpha}\|\mathrm{B}\|^{1-\overline{\theta}_\alpha}_{L^n(M)}\\
\|\nabla^r \mathrm{B}\|_{L^{p\beta}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\beta}\|\mathrm{B}\|^{1-\overline{\theta}_\beta}_{L^n(M)}\\
\|\nabla^t \mathrm{B}\|_{L^{p\gamma}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\overline{\theta}_\gamma}\|\mathrm{B}\|^{1-\overline{\theta}_\gamma}_{L^n(M)}
\end{aligned}$$

with $\overline{\theta}_\alpha = \frac{s+1}{k+1}$, $\overline{\theta}_\beta = \frac{r+1}{k+1}$ and $\overline{\theta}_\gamma = \frac{t+1}{k+1}$, determined by

$$\begin{aligned}
\frac{1}{p\alpha} &= \frac{s}{n} + \overline{\theta}_\alpha\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\overline{\theta}_\alpha}{n}\\
\frac{1}{p\beta} &= \frac{r}{n} + \overline{\theta}_\beta\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\overline{\theta}_\beta}{n}\\
\frac{1}{p\gamma} &= \frac{t}{n} + \overline{\theta}_\gamma\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\overline{\theta}_\gamma}{n}.
\end{aligned}$$

Noticing that $\overline{\theta}_\alpha \in (s/k, 1)$, $\overline{\theta}_\beta \in (r/k, 1)$ and $\overline{\theta}_\gamma \in (t/k, 1)$, if we choose θ_α, θ_β and θ_γ such that

$$\frac{s}{k} < \theta_\alpha < \overline{\theta}_\alpha = \frac{s+1}{k+1}, \quad \frac{r}{k} < \theta_\beta < \overline{\theta}_\beta = \frac{r+1}{k+1} \quad \text{and} \quad \frac{t}{k} < \theta_\gamma < \overline{\theta}_\gamma = \frac{t+1}{k+1}\,,$$

respectively close to $\overline{\theta}_\alpha$, $\overline{\theta}_\beta$ and $\overline{\theta}_\gamma$, the uniquely determined values q_α, q_β and q_γ satisfying

$$\begin{aligned}
\frac{1}{p\alpha} &= \frac{s}{n} + \theta_\alpha\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\alpha}{q_\alpha}\\
\frac{1}{p\beta} &= \frac{r}{n} + \theta_\beta\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\beta}{q_\beta}\\
\frac{1}{p\gamma} &= \frac{t}{n} + \theta_\gamma\Big(\frac{1}{p} - \frac{k}{n}\Big) + \frac{1-\theta_\gamma}{q_\gamma}
\end{aligned}$$

must be close to n, thus properly choosing θ_α, θ_β and θ_γ, as above, we have that q_α, q_β and q_γ are smaller than $q > n$. Hence, by the interpolation estimates again, we have

$$\begin{aligned}
\|\nabla^s \mathrm{B}\|_{L^{p\alpha}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\alpha}\|\mathrm{B}\|^{1-\theta_\alpha}_{L^{q_\gamma}(M)}\\
\|\nabla^r \mathrm{B}\|_{L^{p\beta}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\beta}\|\mathrm{B}\|^{1-\theta_\beta}_{L^{q_\beta}(M)}\\
\|\nabla^t \mathrm{B}\|_{L^{p\gamma}(M)} &\leqslant C\big(\|\nabla^k \mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\gamma}\|\mathrm{B}\|^{1-\theta_\gamma}_{L^{q_\gamma}(M)}\,.
\end{aligned}$$

Then, since $\|\mathrm{B}\|_{L^{q_\alpha}(M)}$, $\|\mathrm{B}\|_{L^{q_\beta}(M)}$ and $\|\mathrm{B}\|_{L^{q_\gamma}(M)}$ are bounded by $C\|\mathrm{B}\|_{L^q(M)}$, being the three exponents smaller that q (the volumes are equibounded for all $M \in \mathfrak{C}^1_\delta(M_0)$), we get

$$\begin{aligned}
\|\nabla^s\mathrm{B}\|_{L^{p_\alpha}(M)} &\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\alpha}\|\mathrm{B}\|^{1-\theta_\alpha}_{L^q(M)}\\
\|\nabla^r\mathrm{B}\|_{L^{p_\beta}(M)} &\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\beta}\|\mathrm{B}\|^{1-\theta_\beta}_{L^q(M)}\\
\|\nabla^t\mathrm{B}\|_{L^{p_\gamma}(M)} &\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\theta_\gamma}\|\mathrm{B}\|^{1-\theta_\gamma}_{L^q(M)}\,,
\end{aligned}$$

Letting

$$\Theta = (\theta_\alpha + \theta_\beta + \theta_\gamma) < \frac{s+1}{k+1} + \frac{r+1}{k+1} + \frac{t+1}{k+1} = 1\,,$$

as $s+r+t = k-2$, putting these estimates in inequality (3.33) and recalling Theorem 2, we conclude

$$\begin{aligned}
\|\nabla^s\mathrm{B}\star\nabla^r\mathrm{B}\star\nabla^t\mathrm{B}\|_{L^p(M)} &\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta}\,\|\mathrm{B}\|^{3-\Theta}_{L^q(M)}\\
&\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta}\big(1+\|\mathrm{H}\|_{L^q(M)}\big)^{3-\Theta}\\
&\leqslant C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta}\,, \qquad (3.34)
\end{aligned}$$

with $C = C(M_0, p, \delta, \|\mathrm{H}\|_{L^n(M)}, \|\mathrm{H}\|_{L^q(M)}) = C(M_0, p, \delta, \|\mathrm{H}\|_{L^q(M)})$, as $q > n$.

Hence, by means of Young inequality, as $\Theta < 1$, we estimate

$$\begin{aligned}
\|\nabla^k\mathrm{B}\|_{L^p(M)} &\leqslant C\|\nabla^k\mathrm{H}\|_{L^p(M)}\\
&\qquad + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\big(\|\nabla^k\mathrm{B}\|_{L^p(M)} + \|\mathrm{B}\|_{L^p(M)}\big)^{\Theta}\\
&\leqslant C\|\nabla^k\mathrm{H}\|_{L^p(M)}\\
&\qquad + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\varepsilon\|\nabla^k\mathrm{B}\|_{L^p(M)} + C\|\mathrm{B}\|_{L^p(M)} + C\,,
\end{aligned}$$

then choosing $\varepsilon > 0$ such that $C\varepsilon < 1/2$, after "absorbing" in the left hand side the term $C\varepsilon\|\nabla^k\mathrm{B}\|_{L^p(M)}$ and estimating $\|\mathrm{B}\|_{L^p(M)}$ with $C(1 + \|\mathrm{H}\|_{L^p(M)})$, we obtain

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leqslant C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)} + C\|\mathrm{H}\|_{L^p(M)} + C\,.$$

The term $\|\nabla^{k-2}\mathrm{B}\|_{L^p(M)}$ can be treated analogously, by interpolation between $\|\nabla^k\mathrm{B}\|_{L^p(M)}$ and $\|\mathrm{B}\|_{L^p(M)}$ (it is actually easier to deal with it) and $\|\mathrm{H}\|_{L^p(M)} \leqslant C(M_0, p, q, \delta)\|\mathrm{H}\|_{L^q(M)}$, hence we finally have the desired estimate

$$\|\nabla^k\mathrm{B}\|_{L^p(M)} \leqslant C\|\nabla^k\mathrm{H}\|_{L^p(M)} + C\,,$$

with $C = C(M_0, p, q, \delta, \|\mathrm{H}\|_{L^q(M)})$, for any $M \in \mathfrak{C}^1_\delta(M_0)$ with $\delta > 0$ small enough.

If $p \in (1, +\infty)$, we argue as before, but using directly inequality (3.29), which holds with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^\infty(M)})$ and getting inequality (3.34) with a constant $C = C(M_0, p, \delta, \|\mathrm{B}\|_{L^\infty(M)})$, by simply choosing a suitably large $q > n$ and estimating $\|\mathrm{B}\|_{L^q(M)}$ with $C\|\mathrm{B}\|_{L^\infty(M)}$. The rest of the proof goes in the same way, estimating all the terms $\|\mathrm{B}\|_{L^q(M)}$ and $\|\mathrm{H}\|_{L^q(M)}$ with $C\|\mathrm{B}\|_{L^\infty(M)}$. QED

4 Other inequalities

Let M_0 be a smooth and compact hypersurface embedded in $\mathbb{R}^{n+1}$, bounding a domain E_0 and $\varepsilon > 0$ the width of a tubular neighborhood N_ε of M_0. For any $\delta \in (0, \varepsilon)$, we consider the family $\mathcal{C}^1_\delta(E_0)$, defined as

$$\left\{ E = \Psi(E_0) \; : \; \begin{array}{l} \Psi : \overline{E}_0 \to \overline{E} \text{ is a diffeomorphism with } \|\Psi - \mathrm{Id}\|_{C^1(E_0)} < \delta \\ \Psi(x) = x + \psi(x)\nu_0(x) \text{ for every } x \in M_0 \text{ and } \|\psi\|_{C^1(M_0)} < \delta \end{array} \right\}$$

where ν_0 is the unit normal vector field pointing outward of M_0.

Then, the Jacobian of the map $\Psi : \overline{E}_0 \to \overline{E}$ (and also the tangential one of its restriction to M_0) is bounded from above and from below by some constants which depend only on δ and the second fundamental form of M_0 (see Section 2 for details).

It clearly follows that if $E \in \mathcal{C}^1_\delta(E_0)$, then $M = \partial E = \Psi(M_0) \in \mathfrak{C}^1_\delta(M_0)$. Moreover, if $M \in \mathfrak{C}^1_{\delta'}(M_0)$, then there exists a smooth function $\psi : M_0 \to \mathbb{R}$ with $\|\psi\|_{C^1(M_0)} < \delta'$, such that $M = \{x + \psi(x)\nu(x) \; : \; x \in M_0\}$, then we can construct a smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$ as follows (E is the domain bounded by M):

$$\Psi(x) = \begin{cases} x & \text{if } x \in E_0 \setminus N_\varepsilon \\ x + \zeta(d_0(x)/\varepsilon)\psi(\pi_0(x))\nabla^{\mathbb{R}^{n+1}} d_0(x) & \text{if } x \in \overline{E}_0 \cap N_\varepsilon \end{cases}$$

where d_0 is the signed distance function from M_0 (which is negative in E_0) and $t \mapsto \zeta(t)$ is a smooth monotone nondecreasing function, defined on $\mathbb{R}$, such that it is equal to 1 if $t \geqslant 0$ and to 0 if $t \leqslant -1/2$, with $|\zeta'(t)| \leqslant 3$, for every $t \in \mathbb{R}$. So, it follows

$$\begin{aligned} \|\Psi - \mathrm{Id}\|_{C^1(E_0)} &= \|\zeta(d_0(\cdot)/\varepsilon)\psi(\pi_0(\cdot))\nabla^{\mathbb{R}^{n+1}} d_0(\cdot)\|_{C^1(\overline{E}_0 \cap N_\varepsilon)} \\ &\leqslant C(M_0, \varepsilon)\|\psi\|_{C^1(M_0)} \,. \end{aligned}$$

Hence, fixed any $\delta \in (0, \varepsilon)$, depending the constant C only on M_0 and ε, possibly choosing δ' small enough, the set E belongs to $\mathcal{C}^1_\delta(E_0)$.

We now discuss some uniform inequalities involving also the domains which are bounded by the hypersurfaces.

4.1 Trace inequalities

Letting E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and any $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$), it is well known that the *trace* of any function $u \in H^1(E)$ (a real function on $M = \partial E$, which we still simply denote by u, that coincides with the restriction of u to M, if $u \in C^0(\overline{E})$) is well defined and that the following *trace inequality* holds (see [24, Chapter 4, Proposition 4.5]),

$$\|u\|^2_{H^{1/2}(M)} \leqslant C_E \int_E u^2 + |\nabla u|^2 \, dx\,, \tag{4.35}$$

which implies

$$\|u - \widetilde{u}\|^2_{H^{1/2}(M)} \leqslant C_E \int_E |\nabla u|^2 \, dx\,,$$

where $\widetilde{u} = \fint_E u \, dx$ (see also [14, 19]). We want to show that these inequalities hold with uniform constants $C(M_0, \delta)$, for every $E \in \mathcal{C}^1_\delta(E_0)$.

Expressing $\|u\|^2_{H^{1/2}(M)}$ by means of the Gagliardo $W^{1/2,2}$–seminorm of a function $u \in L^2(M)$ and setting $\Phi = \Psi|_{M_0} : M_0 \to M$, we have

$$\begin{aligned}
\|u\|^2_{H^{1/2}(M)} &= \|u\|_{L^2(M)} + [u]^2_{W^{1/2,2}(M)}\\
&= \|u\|_{L^2(M)} + \int_M \int_M \frac{|u(y) - u(y^*)|^2}{|y - y^*|^{n+1}} \, d\mu(y) \, d\mu(y^*)\\
&\leqslant C\|u \circ \Phi\|_{L^2(M_0)}\\
&\quad + \int_{M_0} \int_{M_0} \frac{|u(\Phi(x)) - u(\Phi(x^*))|^2}{|\Phi(x) - \Phi(x^*)|^{n+1}} \, J\Phi(x) J\Phi(x^*) \, d\mu_0(x) d\mu_0(x^*)\\
&\leqslant C\|u \circ \Psi\|_{L^2(M_0)}\\
&\quad + C \int_{M_0} \int_{M_0} \frac{|u(\Psi(x)) - u(\Psi(x^*))|^2}{|x - x^*|^{n+1}} \, d\mu_0(x) d\mu_0(x^*)\\
&\leqslant C_{E_0} \int_{E_0} |u(\Psi(x))|^2 + |\nabla^0(u \circ \Psi(x))|^2 \, dx\\
&\leqslant C \int_E u^2 + |\nabla u|^2 \, dx = C\|u\|^2_{H^1(E)}\,,
\end{aligned} \tag{4.36}$$

where the constant C depends only on E_0 (we applied inequality (4.35) for E_0 in passing from the fourth to the fifth line) and δ (in bounding $|d\Psi|$, $|d\Phi|$, $J\Psi$ and $J\Phi$ above and below away from zero).

Remark 6. With a similar argument, we can show the following generalization of this inequality, with a uniform constant

$$\|u\|_{H^{s-1/2}(M)} \leqslant C(E_0, s, \delta)\|u\|_{H^s(E)}$$

(see again [24, Chapter 4, Proposition 4.5]), for $s \in (1/2, 3/2)$.

4.2 Inequalities for harmonic extensions

We let E_0, M_0, $\varepsilon > 0$ and $\delta > 0$ as above and $E \in \mathcal{C}^1_\delta(E_0)$ (with associated smooth diffeomorphism $\Psi : \overline{E}_0 \to \overline{E}$), with $M = \partial E \in \mathfrak{C}^1_\delta(M_0)$.

We denote by $u : E \to \mathbb{R}$ the harmonic extension of a function $f : M \to \mathbb{R}$ in $H^{1/2}(M)$ to E. We aim to show that the following inequality (see [24, Chapter 5, Proposition 1.7])

$$\|u\|_{H^1(E)} \leqslant C_E \|f\|_{H^{1/2}(M)}\,, \tag{4.37}$$

which implies

$$\int_E |\nabla u|^2\,dx \leqslant C_E \|f\|^2_{H^{1/2}(M)}\,,$$

for every $E \in \mathcal{C}^1_\delta(E_0)$, with uniform constants $C = (E_0, \delta)$.

Arguing as above, in formula (4.36), we end up with the following inequalities:

$$\begin{aligned}
\|u\|_{H^1(E)} &\leqslant C(E_0,\delta)\|u\circ\Psi\|_{H^1(E_0)}\\
\|u\circ\Psi\|_{H^1(E_0)} &\leqslant C_{E_0}\|f\circ\Psi\|_{H^{1/2}(M_0)} = C_{E_0}\|f\circ\Phi\|_{H^{1/2}(M_0)}\\
\|f\circ\Phi\|_{H^{1/2}(M_0)} &\leqslant C(M_0,\delta)\|f\|_{H^{1/2}(M)}
\end{aligned}$$

where the second estimate is given by inequality (4.37) for E_0. Putting them together, we have the conclusion.

Remark 7. As above, we also have the following generalization, for $s \in [1/2, 3/2)$,

$$\|u\|_{H^{s+1/2}(E)} \leqslant C(E_0, s, \delta)\|f\|_{H^s(M)}$$

(see again [24, Chapter 5, Proposition 1.7]).

5 Some remarks

We collect here some remarks about the conclusions of the previous sections.

- All the constants depend on the geometric properties of M_0, in particular on the maximal width of a tubular neighborhood, its volume and its second fundamental form. Hence, uniformly controlling such quantities gives uniform estimates for larger families of hypersurfaces, see [7, 8, 9, 11, 18] for a deeper and detailed discussion).

- Notice that for Sobolev, Poincaré, interpolation, trace and "harmonic extension" inequalities, we do not ask $\delta > 0$ to be small, but just $\delta < \varepsilon$, while for the Calderón–Zygmund–type inequalities, that we worked out in Section 3, a smallness condition on δ is necessary for the conclusions.

- All the inequalities holds uniformly also for families of immersed–only hypersurfaces (non necessarily embedded), if they can be expressed as graphs on a fixed compact, smooth hypersurface, possibly immersed–only too.

- It is easy to see that everything we did still works also if the ambient is a *flat*, complete Riemannian manifold, in particular in any flat torus $\mathbb{T}^n$. With some effort, the results can be generalized to graph hypersurfaces in any complete Riemannian manifold, then the constants also depends on the geometry (in particular, on the curvature) of such an ambient space.

Acknowledgements. We would like to thank to the anonymous referee for the detailed reading and careful check of the article. His corrections and suggestions were greatly useful in improving the quality of our paper.

References

[1] M. Abate, F. Tovena: Geometria differenziale, Springer, 2011.

[2] E. Acerbi, N. Fusco, V. Julin, M. Morini: *Non linear stability results for the modified Mullins–Sekerka flow and the surface diffusion flow*, J. Diff. Geom., **113**, 1–53, 2019.

[3] E. Acerbi, N. Fusco, M. Morini: *Minimality via second variation for a nonlocal isoperimetric problem*, Comm. Math. Phys., **322**, 515–557, 2013.

[4] R. A. Adams, J. F. Fournier: Sobolev spaces (second edition), Pure Appl. Math., vol. 140, Elsevier/Academic Press, 2013.

[5] T. Aubin: Some nonlinear problems in Riemannian geometry, Springer, 1998.

[6] R. Benedetti: Lectures on differential topology, Graduate Studies in Mathematics, vol. 218, American Mathematical Society, 2021.

[7] P. Breuning: *Compactness of immersions with local Lipschitz representation*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, **29**, no. 4, 545–572, 2012.

[8] P. Breuning: *C^1–regularity for local graph representations of immersions*, Trans. Amer. Math. Soc., **365**, no. 12, 6185–6198, 2012.

[9] P. Breuning: *Immersions with bounded second fundamental form*, J. Geom. Anal., **25**, no. 2, 1344–1386, 2015.

[10] S. Della Corte, A. Diana, C. Mantegazza: *Global existence and stability for the modified Mullins–Sekerka flow and surface diffusion flow*, Math. Engineering, **4**, Paper n.054, 104 pp, 2022.

[11] S. Delladio: *On hypersurfaces in* $\mathbb{R}^{n+1}$ *with integral bounds on curvature*, J. Geom. Anal, **11**, no. 1, 17–42, 2001.

[12] F. Demengel, G. Demengel, R. Erné: Functional spaces for the theory of elliptic partial differential equations, Universitext, Springer, 2012.

[13] A. Diana, N. Fusco, C. Mantegazza: *Stability for the surface diffusion flow*, ArXiv Preprint Server – http://arxiv.org, 2023.

[14] L. C. Evans: Partial differential equations, second ed., Graduate Studies in Mathematics, American Mathematical Society, 2010.

[15] S. Gallot, D. Hulin, J. Lafontaine: Riemannian geometry, Springer, 1990.

[16] D. Gilbarg, N. S. Trudinger: Elliptic partial differential equations of second order, Springer, 1977.

[17] R. S. Hamilton: *Three–manifolds with positive Ricci curvature*, J. Diff. Geom., **17**, no. 2, 255–306, 1982.

[18] J. Langer: *A compactness theorem for surfaces with* L_p*–bounded second fundamental form*, Math. Ann., **270** , 223–234, 1985.

[19] G. Leoni: A first course in Sobolev spaces: Second edition, 2017.

[20] C. Mantegazza: *Smooth geometric evolutions of hypersurfaces*, Geom. Funct. Anal., **12**, no. 1, 138–182, 2002.

[21] E. Di Nezza, G. Palatucci, E. Valdinoci: Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math., **136**, no. 5, 521–573, 2012.

[22] T. Runst, W. Sickel: Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations, De Gruyter Series in Nonlinear Analysis and Applications, Walter de Gruyter and Co., 1996.

[23] J. Simons: *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2), **88**, 62–105, 1968.

[24] M. E. Taylor: Partial differential equations I : Basic theory, Applied Mathematical Sciences, vol. 115, Springer, 2011.