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On locally homogeneous contact metric manifolds with Reeb flow invariant Jacobi operator

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Abstract. We show that a locally homogeneous, regular contact metric manifold, whose characteristic Jacobi operator is invariant under the Reeb flow, is not compact, provided it admits at least one negative ξ-sectional curvature.

Keywords: locally homogeneous contact metric manifold; regular contact manifold; characteristic Jacobi operator.

MSC 2022 classification: 53C15, 53C25, 53C30.

1 Introduction

Given a contact manifold (M, η), with contact form η, contact distribution $D := ker(\eta) \subset TM$ and Reeb vector field ξ, one can endow M with infinitely many Riemannian metrics g associated to the contact form η in the sense of [1]. The study of the interaction between the contact form and the geometric features of the manifold governed by such a metric is a vast and rich research subject. We recall that an *associated metric* g to η is a Riemannian metric for which there exists a $(1,1)$ tensor field $\varphi : TM \to TM$ such that

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y),$$

for every X, Y vector fields on M. The tensor field φ is uniquely determined by g, and the tensors (φ, ξ, η, g) make up a *contact metric structure* on M.

Recently, in [12] some sufficient conditions are considered which ensure the non-compactness of a contact metric manifold, which involve the symmetric operators:

$$h := \frac{1}{2}\mathcal{L}_\xi \varphi, \quad l := R(-, \xi)\xi. \tag{1.1}$$

In general, the behaviour of these two operators has a strong influence on the geometry of the underlying contact metric manifold. We recall that the vanishing of the operator h characterizes the circumstance that ξ be a Killing vector field; this is true for instance for the widely studied class of Sasakian manifolds, see again [1]. Concerning instead the geometric meaning of the operator l, usually called the *characteristic Jacobi operator* of M, we remark that, if v is a unit tangent vector at a point $p \in M$, orthogonal to ξ_p, then the number $g(lv, v)$ is the sectional curvature of the 2-plane spanned by v and ξ_p; such a curvature is called a *ξ-sectional curvature.*

In particular, in [12] it is proved that a locally homogeneous, regular contact metric manifold with *vanishing* characteristic Jacobi operator must be non-compact. The regularity assumption means that the orbit space M/ξ determined by the flow of the Reeb vector field is smooth and the canonical projection $\pi : M \to M/\xi$ is a submersion (for more details, see for instance [1, Chapter 3]). The class of regular contact manifolds contains the class of *homogeneous* contact manifolds, due to a general result of Boothy-Wang [3].

In this paper, we consider a significant case which is not covered in [12], namely the case where the characteristic Jacobi operator l is invariant under the Reeb flow, i.e.

$$\mathcal{L}_\xi l = 0.$$

This condition was already investigated for instance in [2] and [4].

Our result is the following:

Theorem 1. *Let $(M, \varphi, \xi, \eta, g)$ be a locally homogeneous, regular contact metric manifold. Assume that the characteristic Jacobi operator l is invariant under the Reeb flow, that is $\mathcal{L}_\xi l = 0$.*

If M admits at least one negative ξ-sectional curvature, then it is not compact.

We remark that the assumption concerning the ξ-sectional curvatures is essential here; namely, every compact, homogeneous Sasakian manifold satisfies $\mathcal{L}_\xi l = 0$, the ξ-sectional curvatures being all equal to 1. As a non Sasakian counterexample, one can consider the tangent sphere bundle $T_1\mathbb{S}^m(c)$ of a spherical space form $\mathbb{S}^m(c)$ with sectional curvature $c > 0$, $c \neq 1$, which always admits a homogeneous contact metric structure satisfying the so-called $(\kappa, 0)$ condition, which implies that $l = \kappa(Id - \eta \otimes \xi)$, where κ is a positive constant (a suitable $\mathcal{D}$-homothetic deformation of the standard contact metric structure of $T_1\mathbb{S}^m(c)$). See [1], §7.3 for more information about these examples.

Considering instead the case where $l = \kappa(Id - \eta \otimes \xi)$ with $\kappa < 0$, we have the following immediate corollary:

Corollary 1. *Every locally homogeneous, regular, contact metric manifold with constant negative ξ-sectional curvature is not compact.*

This result follows directly from Theorem 1 because in this case the condition $\mathcal{L}_\xi l = 0$ is satisfied automatically: indeed, by assumption, at each point $p \in M$, the restriction $l_p : D_p \to D_p$ of l_p to the contact subbundle has a unique eigenvalue $\kappa < 0$ with maximal multiplicity, which does not depend on p. Thus $l = \kappa(Id - \eta \otimes \xi)$ and since the Reeb vector field satisfies $\mathcal{L}_\xi \eta = 0$, we also have $\mathcal{L}_\xi l = 0$. We remark that Corollary 1 was already obtained in [12] (see [12, Corollary 4.3]) as a consequence of a general non-compactness result, whose proof makes use of certain deformations of the contact metric structure; the proof given here is more direct. Again a typical example of a homogeneous, non-compact contact metric manifold satsfying the assumptions in this Corollary can be obtained considering a $\mathcal{D}$-homothetic deformation of the standard contact metric structure of the tangent sphere bundle $T_1\mathbb{H}^m(c)$ of a hyperbolic space form with negative sectional curvature $c > -1$.

2 Preliminaries

A contact form η on an odd-dimensional manifold M is a globally defined 1-form such that $d\eta$ restricts to a non-degenerate skew-symmetric bilinear form on $D_x := Ker(\eta_x) \subset T_xM$ for each point $x \in M$. Given a contact manifold (M, η), we have that $TM = D \oplus \mathbb{R}\xi$, where $D = Ker(\eta)$ is the contact subbundle and $\mathbb{R}\xi$ is the 1-dimensional distribution spanned by the Reeb vector field of η, which is the unique vector field ξ on M such that:

$$d\eta(\xi, -) = 0, \quad \eta(\xi) = 1.$$

If g is an associated metric to η, it is known (see e.g. [1]) that:

$$\nabla_\xi \xi = 0, \tag{2.2}$$

$$\nabla_\xi \varphi = 0, \tag{2.3}$$

$$h\varphi + \varphi h = 0, \tag{2.4}$$

$$\nabla \xi = -\varphi - \varphi h, \tag{2.5}$$

$$\nabla_\xi h = \varphi(I - h^2 - l), \tag{2.6}$$

where h and l are the symmetric operators defined by (1.1), and $I = Id_{TM}$.

A contact metric manifold is said to be *locally homogeneous* provided given any two points p and q there exists a local automorphism $f : U \to V$, such that $f(p) = q$, where U and V are open neighbourhoods of p and q; by definition, f is a diffeomorphism which preserves the contact form η and the metric g, restricted to U and V respectively. We remark that every such local automorphism f must also preserve the tensor field φ, the Reeb vector field ξ and both the operators h and l.

3 Proof of the result

Consider a locally homogeneous, regular contact metric manifold M satisfying the assumptions in the statement of Theorem 1. The hypothesis that M is locally homogeneous guarantees that the symmetric operator l has constant eigenvalues, with constant multiplicities. Indeed, if $p, q \in M$ and $f : U \to V$ is a local automorphism such that $f(p) = q$, then $(df)_p \circ l_p = l_q \circ (df)_p$ holds true. Moreover, since we are assuming that M admits at least one negative ξ-sectional curvature at some point $p \in M$, at least one of the eigenvalues λ is negative and there exists a unit tangent vector $v \in D_p$ such that $l(v) = \lambda v$. Consider the vector subbundle E of TM defined by:

$$E := Ker(l - \lambda I) \subset D.$$

According to $\mathcal{L}_\xi l = 0$, we have that

$$[\xi, \Gamma E] \subset \Gamma E, \tag{3.7}$$

where ΓE denotes the module of smooth sections of E.

By regularity, the space $B := M/\xi$ of maximal integral curves of ξ is a smooth manifold and the natural projection $\pi : M \to B$ is a submersion, whose fibers are tangent to ξ. We claim that the vector v can be extended to a vector field $Y \in \mathfrak{X}(M)$, which is a section of the distribution E, and such that:

$$[Y, \xi] = 0.$$

Indeed, according to (3.7) the distribution E is projectable; let E' its projection onto B. Consider the vector $u = \pi_*(v)$. Then u can be extended to a smooth section Z of E'; this vector field Z admits a unique lift Y to a section of D. Indeed, for each point $x \in M$ set:

$$Y_x := (d\pi)_x^{-1}(Z_{\pi(x)}) \in D_x,$$

which is well-defined since $(d\pi)_x : D_x \to T_{\pi(x)}B$ is a linear isomorphism.

By construction, we have that $Y_p = v$ and Y is a section of E. Moreover, Y is invariant under the flow $\{\psi_t\}$ of ξ, because for each t, $(\psi_t)_*Y$ is again a section of the contact subbundle D and $\pi \circ \psi_t = \pi$. Hence $[\xi, Y] = 0$.

Now, we prove by contradiction that M is not compact. Assuming the contrary, let $\gamma : \mathbb{R} \to M$ be the maximal integral curve of ξ passing through p. We shall denote by X' the covariant derivative of a smooth vector field X along γ; moreover, for every vector field $Z \in \mathfrak{X}(M)$, we shall use the same symbol Z to denote its restriction to γ, so that $Z' = \nabla_\xi Z$ holds true along γ.

Consider the unique parallel vector field X along γ, such that $X(0) = v$. Observe that $g(X, \xi) = 0$ along the curve, because, according to (2.2), ξ is parallel along γ and $g(v, \xi_p) = 0$.

We now define a smooth function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) := g(Y_{\gamma(t)}, X(t)),$$

or, more succinctly, $f = g(Y, X)$. Hence, by definition, $f(0) = 1$. Since M is assumed to be compact, the norm $||Y||$ must be bounded on M, and thus f must also be bounded; indeed:

$$|f(t)| \leq ||Y_{\gamma(t)}|| \cdot ||X(t)|| = ||Y_{\gamma(t)}||.$$

Since $[Y, \xi] = 0$, along γ we have, according to (2.5):

$$Y' = \nabla_\xi Y = -\varphi Y - \varphi h Y = -\varphi(I + h)Y. \tag{3.8}$$

Moreover, taking (2.6) into account we get:

$$\begin{aligned} (hY)' &= \nabla_\xi hY = (\nabla_\xi h)Y + hY' = \\ &= \varphi(I - h^2 - l)Y - h\varphi(I + h)Y, \end{aligned}$$

which can be rewritten, using (2.4), as follows:

$$(hY)' = \varphi(I + h - l)Y. \tag{3.9}$$

Now, being X parallel along γ, computing f' we obtain, using (3.8):

$$f' = g(Y', X) = g((I + h)Y, \varphi X).$$

Moreover, according to (2.3), we have that φX is also parallel along γ; hence we can compute f'' in a similar fashion:

$$f'' = g(Y' + (hY)', \varphi X) = -g(\varphi l Y, \varphi X) = -g(lY, X),$$

where (3.8) and (3.9) have been used. But since Y a section of E we have $lY = \lambda Y$, so that in conclusion the second derivative of f satisfies:

$$f'' = -\lambda g(Y, X),$$

namely

$$f'' = -\lambda f.$$

Since $\lambda < 0$, we have reached a contradiction because f is bounded and $f \neq 0$.

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