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A proof of the Hamiltonian Thom isotopy Lemma

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Abstract. In this note we present a complete proof of the fact that all the submanifolds of a one parameter family of compact symplectic submanifolds inside a compact symplectic manifold are Hamiltonian isotopic.

Keywords: Symplectic submanifolds, Hamiltonian isotopy

MSC 2022 classification: primary 53D05

1 Introduction

In this paper we give a detailed proof of the following statement that is known as a folklore within the symplectic community.

Theorem 1. *Let (M, ω) be a compact symplectic manifold with a smooth family $\{S_t\}_{t\in[0,1]}$ of closed symplectic submanifolds. Then there is a smooth Hamiltonian isotopy $(\rho_t)_{t\in[0,1]}$ such that $\rho_t(S_0) = S_t$, $t \in [0, 1]$.*

This result is part of the lore of the symplectic community, with partial proofs scattered in some places, see for instance [2, 1, 6]. More details on the bibliography will be given in Section 4. The scope of this short note is to present a complete proof, filling eventually all the details.

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Notations

Below are some notations that we are going to use.

We shall consider time dependent objects (vector fields, differential forms, etc.) that depend on a second parameter usually called $s \in [0,1]$. For instance, a time dependent vector field, depending on s is a section $\zeta \in \Gamma([0,1] \times [0,1] \times M, q^*(TM))$, where $q : [0,1] \times [0,1] \times M \to M$ is the projection. We also write this section as $\zeta : [0,1]_t \times [0,1]_s \times M \longrightarrow TM$ and put $\zeta_t^s = \zeta(t,s) \in \Gamma(M, TM)$ for its values. Similar notations will be adopted for other objects as differential forms.

Given a manifold X, a submanifold Y (or a subset), a fiber bundle E over X and a map $f : E \to Z$ (e.g. a form or a differential), the symbol $f \restriction Y$ will be used to denote the restriction of f to $E|_Y$. The symbol $\big|$ is used to denote the classical restriction to subsets.

2 Preliminaries: vector fields, isotopies and Thom Isotopy Lemma

Let M be a smooth manifold; an (ambient) isotopy is a smooth map $\rho : M \times [0,1] \longrightarrow M$ such that $\rho_0 = \mathrm{Id}_M$ and ρ_t is a diffeomorphism for every $t \in [0,1]$. An isotopy is a curve starting from the identity inside $\mathrm{Diff}(M)$. Differentiating it we get a time dependent vector field ζ_t satisfying:

$$\frac{d}{dt}\rho_t = \zeta_t \circ \rho_t, \quad t \in [0,1]. \tag{2.1}$$

Viceversa, given a time dependent vector field ζ_t with $t \in [0,1]$, it is possible, under the usual assumptions on the existence of flows, to find a unique isotopy ρ_t satisfying equation (2.1). This is called the *flow* [1] of ζ_t. If ζ is compactly supported (i.e. ζ_t is compactly supported for every $t \in [0,1]$) then the flow exists.

It is worth to give few details on the construction of ρ_t given ζ_t as it will be used later to construct the Hamiltonian isotopy. We assume ζ_t, $t \in [0,1]$ is compactly supported and put

$$\widehat{M} := M \times [0,1] \subset M \times \mathbb{R}.$$

This is an auxiliary submanifold with boundary; $\widehat{M}$ will be also used next. We have projections $\pi : M \times \mathbb{R} \to \mathbb{R}$ and $\pi_M : M \times \mathbb{R} \to M$ and embeddings $e_t : M \to M \times \mathbb{R}$ defined by $e_t(x) = (x,t)$.

[1] somewhere called the reduced flow of ζ to distinguish with the full flow as a time dependent vector field

Now we extend ζ_t (keeping the same name) to a vector field $\zeta_t \in \Gamma(M \times \mathbb{R}, \pi^*(TM))$ with the property that there is a fixed $\varepsilon > 0$ such that $\zeta_t = 0$ for $t \notin [-\varepsilon, 1+\varepsilon]$. Then we convert it into the standard (autonomous) vector field

$$\widehat{\zeta}(x,t) := \zeta_t(x) + \partial_t \in \Gamma(M \times \mathbb{R}, T(M \times \mathbb{R})).$$

It is clear that $\widehat{\zeta}$ has an ordinary flow (it is bounded) $\Phi : (M \times \mathbb{R}) \times \mathbb{R} \longrightarrow M \times \mathbb{R}$ such that

$$\frac{d}{ds}\Phi_s((x,t)) = \widehat{\zeta}(\Phi_s(x,t)), \quad \Phi_0 = \mathrm{Id}_{M \times \mathbb{R}}\,.$$

Since the second component of $\widehat{\zeta}$ is ∂_t, it follows that Φ_s has to be in the form $\Phi_s(x,t) = (\Phi^M(x,t,s), s+t)$ for a smooth family of maps $\Phi^M : (M \times \mathbb{R}) \times \mathbb{R} \to M$. In particular the restriction of the flow to $[0,1]$ preserves $\widehat{M}$. Therefore defining $\rho_t := \Phi^M(\cdot, 0, t)$ for $t \in [0,1]$ we get the flow of ζ_t. In other words the flow is defined by the commutative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\Phi_t} & \widehat{M} \\ {\scriptstyle e_0}\big\uparrow & & \big\downarrow{\scriptstyle \pi_M} \\ M & \xrightarrow[\rho_t]{} & M \end{array}$$

for $t \in [0,1]$.

2.1 Thom isotopy Lemma

This section is a brief reminder on families of submanifolds and the Thom isotopy Lemma. The material is standard and we are following the lecture notes [3].

Definition 1. A family $\{S_t\}_{t \in [0,1]}$ of submanifolds of a manifold M is smooth when we can find a smooth map $F : M \times [0,1] \to L$ with L a smooth manifold and a submanifold $A \hookrightarrow L$ having the following properties.

(1) For every $t \in [0,1]$ the map $f_t := F(\cdot, t) : M \to L$ is transverse to A. This means that

$$df_t(T_x M) + T_{f_t(x)} A = T_{f_t(x)} L, \text{ at every point } x \in M \text{ such that } f_t(x) \in A.$$

(2) $S_t = f_t^{-1}(A)$ for every $t \in [0,1]$.

The transversality assumption in (1) implies that every $f_t^{-1}(A)$ is a smooth submanifold.

Lemma 1 (Thom isotopy Lemma - basic version)**.** *Let M be a smooth manifold equipped with a smooth function $F : M \times [0,1] \to \mathbb{R}$ such that:*

(1) for every $t \in [0,1]$, zero is a regular value of the function $f_t := F(\cdot, t) : M \to \mathbb{R}$.

(2) Every submanifold $S_t = f_t^{-1}(0)$ is compact.

Then there exists an isotopy $\rho_t : M \to M$, $t \in [0,1]$ such that for every $t \in [0,1]$:

$$\rho_t(S_0) = S_t.$$

The same is true for the manifolds with boundary $f_t^{-1}([0,\infty))$.

Proof. We don't give the full proof but a somewhat detailed sketch paying attention to the parts that will be used later. Define

$$\widehat{Z} := F^{-1}(0) \subset \widehat{M},$$

then $S_t = f_t^{-1}(0) \cong \widehat{Z} \cap (M \times \{t\})$. Notice that $\widehat{Z}$ is compact. We prove the thesis by constructing a time dependent vector field ζ_t on M such that the corresponding autonomous vector field $\widehat{\zeta}$ on $\widehat{M}$ is tangent to $\widehat{Z}$. Indeed in this case, the flow Φ_t of $\widehat{\zeta}$ preserves $\widehat{Z}$ while mapping the slice $S_0 \cong \widehat{Z} \cap (M \times \{0\})$ to the t-slice $S_t \cong \widehat{Z} \cap (M \times \{t\})$. Therefore

$$\rho_t = \pi_M \circ \Phi_t \circ e_0,$$

as defined before is the desired isotopy (with the inverse diffeomorphism provided by the opposite vector field $-\widehat{\zeta}$).

We are left with the construction of ζ_t; the condition ensuing that $\widehat{\zeta}$ is tangent to $\widehat{Z}$ is equivalent to the equation:

$$\frac{\partial F}{\partial x}(x,t)\zeta_t(x) + \frac{\partial F}{\partial t}(x,t) = 0, \tag{2.2}$$

at every point $(x,t) \in \widehat{Z}$. Since the partial differential $\dfrac{\partial F}{\partial x}(m,t)$ is surjective at every point $(x_0,t_0) \in \widehat{Z}$, this equation can be locally solved. Every point $z_0 = (x_0,t_0) \in \widehat{Z}$ has a neighborhood U_{z_0} and a vector field $v_{z_0} \in \Gamma(U_{z_0}, \pi_M^*(TM))$ such that (2.2) holds in U_{z_0}.

Finally the compactness of $\widehat{Z}$ implies that we find a finite cover $\{U_{z_1}, ..., U_{z_n}\}$ of $\widehat{Z}$ with corresponding vector fields $v_{z_1}, ..., v_{z_n}$. If we define $A := \widehat{M} \setminus \widehat{Z}$, then

$\{A, U_{z_1}, ..., U_{z_n}\}$ is a cover of the whole $\widehat{M}$. Let $\{\rho_A, \rho_{z_1}, ..., \rho_{z_n}\}$ be a subordinated partition of unity; one easily checks that the vector field

$$\zeta := \sum_{j=1}^{n} \rho_{z_j} v_{z_j}$$

is a well defined time dependent vector field on the whole of M (i.e. is a vector field on $\widehat{M}$ without the ∂_t - component) and has the desired property that the associated vector field $\widehat{\zeta}$ stays tangent to $\widehat{Z}$. QED

Remark 1. We have presented the result in the case of a family of hypersurfaces i.e. the map F is real valued. The conclusion remains valid, with the same proof for a smooth family in the sense of Definition 1.

3 The Moser trick

We follow the book [2, 5] for basic facts in symplectic geometry. A symplectic manifold is a couple (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is a closed, non-degenerate 2-form. Non-degenerateness means that there is an induced isomorphism of vector bundles $TM \longrightarrow T^*M$ by $v \mapsto \omega(v, \cdot)$. In particular M is even dimensional: $\dim M = 2k$ and oriented (by ω) because $\omega^k/(k!)$ is a volume form.

Example 1. A standard example is the cotangent bundle T^*M of every manifold. In this case, the symplectic structure is exact: $\omega = -d\alpha$ where $\alpha \in \Omega(T^*M)$ is a specific 1-form, the canonical *Liouville form*

$$\alpha(v) = \varphi(d_\varphi \pi(v)), \qquad v \in T_\varphi(T^*M), \quad \varphi \in T^*M,$$

with $\pi : T^*M \to M$ the projection. In local coordinates $(x_1, ..., x_n, \xi_1, ..., \xi_n) : T^*U \to \mathbb{R}^{2n}$ induced by coordinates $(x_1, ..., x_n) : U \to \mathbb{R}^n$ on the base, we have: $\alpha = \sum_i \xi_i dx_i$ so that

$$\omega = dx_i \wedge d\xi_i. \tag{3.3}$$

The previous example is in some *local sense* universal. Indeed by the Darboux Theorem [2, Theorem 8.1], every symplectic manifold is locally equivalent to the cotangent bundle of an open set in the euclidean space. The equivalence here is given by symplectomorphisms. A symplectomorphism $\varphi : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ between two symplectic manifolds is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*(\omega_2) = \omega_1$. A standard way to produce symplectomorphisms is via symplectic or Hamiltonian isotopies.

Now we recall the Moser trick. This admits several versions and can be regarded more as a method than a single result. Here we follow closely [5] presenting the general procedure in the form of a lemma. In Section 4 we will prove the parameter dependent relative form.

Lemma 2 (Moser trick)**.** *Let M be a manifold with a family $\omega_t \in \Omega^2(M)$, of symplectic forms, defined for $t \in [0,1]$ such that*

$$\frac{d}{dt}\omega_t = d\sigma_t,$$

for every $t \in [0,1]$, where σ_t is a smooth family of one forms. Then there is a unique time dependent vector field ζ_t, $t \in [0,1]$ satisfying The Moser equation

$$\sigma_t + \iota(\zeta)\omega_t = 0. \tag{3.4}$$

The flow ρ_t of ζ_t has the property that at every time t for which it exists:

$$\rho_t^*\omega_t = \omega_0.$$

In particular if M is compact $\rho_1 : (M,\omega_0) \to (M,\omega_1)$ is a symplectomorphism.

Proof. The equation (3.4) is solved by the nondegenerateness of any ω_t. Let ρ_t be the flow of the unique solution ζ_t; we have

$$\begin{aligned}\frac{d}{dt}\rho_t^*\omega_t &= \rho_t^*\left(\frac{d}{dt}\omega_t + \iota(\zeta_t)d\omega_t + d\iota(\zeta_t)\omega_t\right)\\ &= \rho_t^*(d\sigma_t + d\iota(\zeta_t)\omega_t))\\ &= d\rho_t^*(\sigma_t + \iota(\zeta_t)\omega_t) = 0.\end{aligned}$$

QED

Remark 2. We have presented the proof in a somewhat reversed order. Often one first looks for an isotopy such that $\rho_t^*\omega_t = \omega_0$, then by differentiation, Moser equation appears. Moreover note that the main assumption on the differential forms can be written as $\frac{d}{dt}[\omega_t] = 0$ (de Rham cohomology class). In this case a smooth family of potentials σ_t can be found. A quick proof, as indicated in [5, Theorem 3.2.4], uses basic Hodge theory.

The next results are the parameter versions of two consequences of the Moser method. Again we follow closely [5] checking that all their proofs remain valid with an extra parameter.

We begin with the trivial observation that the Moser trick can be performed with a parameter. Let M be a (non necessarily compact) manifold and ω_t^s for $(t,s) \in [0,1] \times [0,1]$ a family of symplectic forms with exact t-derivative:

$$\frac{d}{dt}\omega_t^s = d\sigma_t^s, \quad t,s \in [0,1], \tag{3.5}$$

for a smooth family $(\sigma_t^s)_{s,t}$ of time dependent 1-forms. Non degenerateness implies that there is a unique s-family of time dependent vector fields solving The Moser equation:

$$\iota(\zeta_t^s)\omega_t^s + \sigma_t^s = 0.$$

If M is compact we can find a smooth, two parameters family of diffeomorphisms ψ_t^s such that

$$\psi_0^s = \mathrm{Id}, \quad (\psi_t^s)^*\omega_0^s = \omega_t^s, \quad t,s \in [0,1].$$

In general, the result holds for all the values of s such that the time dependent vector field $\zeta(t,s)$ admits a (local) flow.

The next result is the parametric version of [5, Lemma 3.2.1].

Proposition 1 (Moser isotopy with parameter, [5])**.** *Let M be a $2n$-dimensional manifold with a compact submanifold X. Assume that $(\omega_0^s)_{s\in[0,1]}, (\omega_1^s)_{s\in[0,1]}$ are two s-families of closed 2-forms such that for every $s \in [0,1]$:*

(1) $\omega_0^s \restriction X = \omega_1^s \restriction X$ as differential forms (i.e. on the whole $T_X M$)

(2) $\omega_0^s \restriction X$ and $\omega_1^s \restriction X$ are non degenerate on $T_X M$.

Then there exists a neighborhood U_0 of X, a family of open neighborhoods $\{U_s\}_{s\in[0,1]}$ of X and a family of diffeomorphisms $\psi^s : U_0 \longrightarrow U_s = \psi^s(U_0)$ such that

$$\psi^s|_X = \mathrm{Id}_X, \quad \omega_1^s = (\psi^s)^*\omega_0^s, \quad s \in [0,1].$$

Moreover choices can be made in a way that $d\psi^s \restriction X = \mathrm{Id}_{T_X M}$ for every $s \in [0,1]$.

Proof. The result follows from Moser method once we find a neighborhood U_0 of X and a smooth family $(\sigma^s)_{s\in[0,1]}$ of 1-forms on U_0 such that

$$\omega_1^s - \omega_0^s = d\sigma^s \text{ on } U_0 \quad \text{and} \quad \sigma^s \restriction X = 0 \text{ on } T_X M,$$

for every $s \in [0,1]$.

This granted, the family

$$\omega_t^s := \omega_0^s + t d\sigma^s$$

is symplectic on X for every s, t. It remains symplectic, for every s, t on an open set $U_0 \supset X$ where we can find a vector field ζ_t^s such that $\iota(\zeta_t^s)\omega_t^s + \sigma^s = 0$ for $t, s \in [0, 1]$. Moreover the crucial property:

$$V_t^s|_X = 0$$

for every t, s holds. Shrinking U_0 if required, we can apply The Moser method (smooth in s) around X.

Finally the construction of σ^s is performed exactly in the same way as in loc. cit. using the homotopy operator on a tubular neighborhood. Since the difference $\omega_1^s - \omega_0^s$ vanishes identically on X, we can find σ^s such that all its coefficients with their first partial derivatives vanish on X on every coordinate chart (see [4, Appendix 1.]). Such a property implies $d\psi^s \restriction X = \mathrm{Id}_{T_X M}$. QED

Now comes the Weinstein symplectic neighborhood Theorem, the parametric version of [5, Theorem 3.4.10].

Proposition 2 (Weinstein symplectic neighborhood Theorem with parameter). *For $j = 0, 1$ let (M_i, ω_i) symplectic manifolds with compact symplectic submanifolds S_i. Let $\Phi^s : \nu_{S_0} \longrightarrow \nu_{S_1}$, $s \in [0, 1]$ be a family of isomorphisms of the corresponding symplectic normal bundles covering a family of symplectomorphisms $\phi^s : (S_0, \omega_0) \longrightarrow (S_1, \omega_1)$. There is a neighborhood U_0 of S_0 and a family of symplectomorphisms*

$$\psi^s : U_0 \longrightarrow \psi^s(U_0)$$

such that: $\psi^s(U_0)$ is an open neighborhood of S_1, ψ^s extends ϕ^s and $d\psi^s|_{\nu_{S_0}} = \Phi^s$.

Proof. Choose Riemannian metrics and open neighborhoods V_j of S_j such that the exponential maps induce diffeomorphisms: $\exp_j : \nu_{S_j} \longrightarrow V_j$. Let Ψ^s be the family of diffeos defined by the diagram:

$$\begin{array}{ccc} \nu_{S_0} & \xrightarrow{\exp_0} & V_0 \\ {\scriptstyle \Phi^s}\downarrow & & \downarrow{\scriptstyle \Psi^s} \\ \nu_{S_1} & \xrightarrow[\exp_1]{} & V_1 \end{array}$$

Then the Ψ^s constitute a family of diffeomorphisms that extend ϕ^s and induce Φ^s (via the normal derivative). In this way, we may apply the Moser Isotopy Theorem, Proposition 1 to V_0 equipped with the forms ω_0 and $(\Psi^s)^*\omega_1$. Such forms are equal on the whole $T_{S_0}M_0$. Of course to preserve the normal data the diffeomorphism provided by the Moser isotopy should be chosen with differential equal to the identity on the submanifold. QED

4 Hamiltonian Thom isotopy

Let (M, ω) be a symplectic manifold. For every function $f \in C^\infty(M)$, the Hamiltonian vector field associated to f is defined uniquely by the equation:

$$\iota(V_f)\omega = df.$$

The flow of an Hamiltonian vector field preserves the symplectic structure. A vector field is said Hamiltonian if arises from a function in this way.

Let now $\rho : M \times [0,1] \to M$ be an isotopy; if every ρ_t is a symplectomorphism then ρ is called a *symplectic isotopy.* A special class of symplectic isotopies are the *Hamiltonian isotopies*, the ones with ζ_t Hamiltonian for every t. In this case a smooth time dependent Hamiltonian $H : M \times [0,1] \longrightarrow \mathbb{R}$ such that $\iota(\zeta_t)\omega = dH(\cdot, t)$ can be found. Of course if $H^1(M, \mathbb{R}) = 0$, every symplectic isotopy is Hamiltonian.
Here we prove Theorem 1. We state it again for ease of reading.

Theorem 2. *Let (M, ω) be a compact symplectic manifold with a smooth family $\{S_t\}_{t\in[0,1]}$ of closed symplectic submanifolds. Then there is a smooth Hamiltonian isotopy $(\rho_t)_{t\in[0,1]}$ such that $\rho_t(S_0) = S_t$, $t \in [0,1]$.*

As said in the introduction this is part of the lore of the symplectic community. Let's list shortly the main references.

- C. Da Silva [2] states the result without the Hamiltonian property of the isotopy i.e. ρ_t is a symplectic isotopy.
- Auroux [1, Proposition 4] shows that there is a *continuous* family of symplectomorphisms $(\rho_t)_t$ such that $\rho_0 = \mathrm{Id}$ and $\rho_t(S_0) = S_t$, $t \in [0,1]$. By checking that the Moser method and its consequences, as the Weinstein symplectic neighborhood, can be performed with respect to a parameter, the same proof shows that the family of symplectomorphisms is smooth. (Actually it seems that this proof requires such smoothness.)
- Siebert and Tian [6, Proposition 0.3] prove Theorem 1 in dimension 4. Their proof uses complex coordinates, a fact that can be replaced by the existence, a priori, of a symplectic isotopy ρ_t mapping S_0 to S_t.

We will prove Theorem 1 with the following steps:

(1) we first show that the proof of Auroux includes smooth dependence in time;

(2) adapt Siebert–Tian proof given a smooth symplectic isotopy ρ_t for granted.

We are ready to add smoothness to the proof of Auroux [1, Proposition 4].

Proposition 3. *Let* (M, ω) *a compact symplectic manifold with a smooth family* $\{S_t\}_{t\in[0,1]}$ *of closed symplectic submanifolds. Then there is a smooth symplectic isotopy* $(\rho_t)_{t\in[0,1]}$ *such that* $\rho_t(S_0) = S_t$, $t \in [0, 1]$.

Proof. Denote with $\iota_t : S_t \hookrightarrow M$ the inclusions. By the Thom isotopy Lemma there is a smooth isotopy $\varphi : M \times [0, 1] \longrightarrow M$ with $\varphi_t(S_0) = S_t$ that combined with Moser stability (with smooth parameter) produces a smooth family of symplectomorphisms

$$\psi^t : (S_0, \iota_0^*\omega) \longrightarrow (S_t, \iota_t^*\omega).$$

Such a family is covered by a smooth family $N^\omega S_0 \longrightarrow N^\omega S_t$ of isomorphisms between the symplectic normal bundles of the submanifolds S_t. Indeed one first considers the principal bundle $P \longrightarrow S_0 \times [0, 1]$ of the normal symplectic frames of the submanifolds; the structure group being the symplectic group. Parallel transport with respect to a connection on P gives a family of isomorphisms of principal bundles $P\big|_{S_0} \longrightarrow P\big|_{S_t}$. This induces isomorphisms on all the associated bundles such as the symplectic normal bundles. By proposition 2 we end up with a smooth family $\psi^t : U_0 \longrightarrow U_t$ of symplectomorphisms between tubular neighborhoods $U_t \supset S_t$. Let now $\mu_t : M \longrightarrow M$ be any smooth family of diffeomorphisms of the ambient extending ψ^t and, following closely [1], put:

$$\omega_t := \mu_t^*\omega, \quad \Omega_t = -\frac{d}{dt}\omega_t.$$

Assume we can find a vector field ζ_t such that:

- the forms $\alpha_t := \iota(\zeta_t)\omega_t$ satisfy $d\alpha_t = \Omega_t$,
- $\zeta_t\big|_{S_0}$ is tangent to S_0 for every t.

Then the proof is completed for if σ_t denotes the flow of ζ_t, let $\rho_t := \mu_t \circ \sigma_t$; we have $\mathcal{L}_{\zeta_t}\omega_t = \Omega_t$ by the Cartan formula so that:

$$\frac{d}{dt}\rho_t^*\omega = \frac{d}{dt}(\sigma_t^*\omega_t) = (\sigma_t)^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{\zeta_t}\omega_t\right) = 0.$$

This means that ρ_t is a family of symplectomorphisms mapping S_0 to S_t.

Let us show how to find ζ_t or equivalently α_t. By construction ω_t is constant on U_0 so that the condition: $\zeta_t\big|_{S_0} \in \Gamma(TS_0)$ means exactly that $N^\omega S_0 \subset \operatorname{Ker}\alpha_t$. Now all the ω_t are cohomologous, which implies $[\Omega_t] = 0$ in $H^2(M, \mathbb{R})$. A smooth family of potentials $\beta_t \in \Omega^1(M)$ on the entire M, such that $d\beta_t = \Omega_t$ can be found. The smoothness following exactly by the argument in the proof of [5, Theorem 3.17]. On U_0 we have $d\beta_t = \Omega_t = 0$ i.e. the forms β_t define classes in $H^1(U_0, \mathbb{R})$. Now, let $\pi : U_0 \longrightarrow S_0$ the projection of a tubular neighborhood satisfying $T_x\pi^{-1}(x) = N^{\omega}_x S_0$ at every point $x \in S_0$ and define:

$$\gamma_t := (\iota_0 \circ \pi)^* \beta_t \in \Omega^1(U_0)$$

with $\iota_0 : S_0 \hookrightarrow M$ the inclusion. Since $(\iota_0 \circ \pi)^*$ is the identity in cohomology we have $[\gamma_t] = [\beta_t\big|_{U_0}]$ in $H^1(U_0, \mathbb{R})$. It follows that there is a smooth family of functions $f_t \in C^\infty(U_0, \mathbb{R})$ with $\gamma_t = \beta_t + df_t$ in U_0. Let g_t be any family of functions on the whole M extending f_t and put $\alpha_t := \beta_t + dg_t$. We have found our α_t. Indeed $d\alpha_t = d\beta_t = \Omega_t$ and from $\alpha_t \restriction U_0 = \gamma_t$ it follows that $N^{\omega}_x S_0 \subset \ker(\alpha_t \restriction x)$ at every point x in S_0. QED

We finally prove Theorem 1 following the idea of the proof of [6, Proposition 0.3].

Proof of Theorem 1. Following the proof of lemma 1, it is enough to construct a Hamiltonian function $H : M \times I \to \mathbb{R}$ such that, with $h_t := H(\cdot, t) : M \to \mathbb{R}$, the vector field

$$\widehat{V}_{h_t} = V_{H_t} + \partial_t$$

is tangent to $\widehat{S} = \{(x,t) \in M \times [0,1] : x \in S_t\}$. To this end, for every $z = (p_0, t_0) \in M \times I$ we will find a neighborhood $U_z = U_{p_0} \times U_{t_0}$ and a local Hamiltonian function $H_z : U_z \to \mathbb{R}$ such that, again denoting by $h_{z,t} := H_z(\cdot, t) : U_{p_0} \to \mathbb{R}$, two conditions are satisfied:

(1) $V_{h_{z,t}} + \partial_t$ is tangent to $U_z \cap \widehat{S}$,

(2) H_z vanishes on $U_z \cap \widehat{S}$.

The global Hamiltonian H will then be defined using a partition of unity $\{\theta_z\}_{z \in M \times I}$ subordinated to the cover $\{U_z\}_{z \in M \times I}$, i.e. we will set $H := \sum_z \mathrm{d}\theta_z H_z$. Its Hamiltonian vector field (at every fixed t) for such H will satisfy:

$$\iota(V_{h_t})\omega = \sum_z \mathrm{d}\theta_z h_{z,t} + \theta_z \mathrm{d}h_{z,t}. \tag{4.6}$$

Condition (2) ensures that along $\widehat{S}$ the first summand in (4.6) vanishes; together with condition (1), this ensures that V_{h_t} is tangent to $\widehat{S}$.

We proceed now with the construction of such functions H_z using a symplectic isotopy $\rho_t : M \longrightarrow M$ with $t \in [0,1]$ such that $\rho_t(S_t) = S_0$. This is (or better its inverse) provided by proposition 3.

For every point $z = (p_0, t_0)$ in the open set $(M \times [0,1]) \setminus \widehat{S}$ we pick a small neighborhood $U_z = U_{p_0} \times U_{t_0}$ disjoint from $\widehat{S}$ and we set $H_z \equiv 0$.

For every point $z = (p_0, 0) \in \widehat{S}$ we proceed as follows. Since S_0 is symplectic, we can find a neighborhood $U_{p_0} \subset M$ and coordinates $\Theta : U_{p_0} \longrightarrow \mathbb{R}^n_x \times \mathbb{R}^n_y$ such that $U_{p_0} \cap S_0$ is described by $\{x_j = y_j = 0,\ j = m+1, ..., n\}$ and

$$\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \quad \text{with} \quad e_0^*(\omega) = \sum_{i=1}^{m} \mathrm{d}x_i \wedge \mathrm{d}y_i. \tag{4.7}$$

Indeed this is (again) consequence of Weinstein symplectic neighborhood Theorem, or better it's non compact version [4, Theorem 15.2]. One first locally trivializes the symplectic normal bundle, then applies the symplectic neighborhood Theorem to split the symplectic structure as a product, then applies the standard Darboux Theorem on the two factors. In alternative, the Carathéodory–Jacobi–Lie Theorem [4, Theorem 17.2] can be used.

The above coordinate system induces coordinates in $\widehat{M}$ of the form:

$$\widehat{\Theta} : U_z := \{(p,t) : \rho_t(p) \in U_{p_0}\} \longrightarrow \mathbb{R}^n_{\widehat{x}} \times \mathbb{R}^n_{\widehat{y}} \times [0,1]_\tau$$

with:

$$\widehat{x}_i(p,t) := x_i(\rho_t(p)), \quad \widehat{y}_i(p,t) := y_i(\rho_t(p)), \quad \tau(p,t) = t.$$

In such coordinates we have: $\widehat{S} \cap U_z = \{\widehat{x}_i = 0 = \widehat{y}_i,\ i = m+1, ..., 0\}$ so that ∂_τ is tangent to $\widehat{S}$. Let ζ_t be the vector field associated to ρ_t, i.e: $\dfrac{d}{dt}\rho_t = \zeta_t \circ \rho_t$ for $t \in [0,1]$, with coordinate representation $\sum_{j=1}^n \xi_t^j \partial_{x_j} + \eta_t^j \partial_{y_j}$ in U_{p_0} given by

$$\xi_t^j(p) = \frac{d}{ds}\Big|_{s=t} x_j\big(\rho_s(\rho_t^{-1}(p))\big), \quad \eta_t^j(p) = \frac{d}{ds}\Big|_{s=t} y_j\big(\rho_s(\rho_t^{-1}(p))\big).$$

Using the inclusions $e_t : M \hookrightarrow M \times \{t\} \subset \widehat{M}$ at time t, locally defined vector fields on M can be considered on $\widehat{M}$ omitting (when the context is clear) further notations. In this sense:

$$d\left(\rho_t^{-1}\right) \partial_{x_i}\big|_p = \partial_{\widehat{x}_i}\big|_{(p,t)} \tag{4.8}$$

and similar identities for $\widehat{y}_j$. Since the vector field ∂_t on $\widehat{M}$ is given by $\partial_t\big|_{(p,t)} = \dfrac{d}{ds}\Big|_{s=t}\gamma_p(s)$ with $\gamma_p(s) = (p,s)$, we have, over U_z the formula:

$$\partial_t\big|_{(p,t)} = \partial_\tau + \sum_{j=1}^{n} \xi_t^j(\rho_t(p))\partial_{\widehat{x}_j}\big|_{(p,t)} + \eta_t^j(\rho_t(p))\partial_{\widehat{y}_j}\big|_{(p,t)}. \tag{4.9}$$

We define, in U_z the local Hamiltonian by:

$$H_z(p,t) = \sum_{j=m+1}^{n} \eta_t^j(\rho_t(p))\, \widehat{x}_j(p,t) - \xi_t^j(\rho_t(p))\, \widehat{y}_j(p,t). \tag{4.10}$$

Clearly H_z vanishes on $U_z \cap \widehat{S}$, i.e. it satisfies condition (2) above. As for condition (1), since every ρ_t is a symplectomorphism the Hamiltonian vector fields are ρ_t-related:

$$V_{h_t}\big|_p = d_q\left(\rho_t^{-1}\right)\left(V_{k_t}\big|_q\right), \quad q = \rho_t(p),$$

where k_t is the local Hamiltonian function $k_t : U_{p_0} \longrightarrow \mathbb{R}$ defined by

$$k_t = \sum_{j=m+1}^{n} \eta_j^t\, x_j - \xi_j^t\, y_j.$$

For the Hamiltonian vector field of k_t we compute

$$V_{k_t}\big|_{S_0} = -\sum_{j=m+1}^{n} \left(\xi_j^t\, \partial_{x_j} + \eta_j^t\, \partial_{y_j}\right)\big|_{S_0}.$$

It follows, using formulas (4.8) and (4.9) that at every point $(p,t) \in \widehat{S} \cap U_z$ we have:

$$(V_{h_t} + \partial_\tau)\big|_{(p,t)} = \partial_\tau\big|_{(p,t)} + \sum_{j=1}^{m} \xi_t^j(\rho_t(p))\partial_{\widehat{x}_j}\big|_{(p,t)} + \eta_t^j(\rho_t(p))\partial_{\widehat{y}_j}\big|_{(p,t)},$$

which is manifestly tangent to $\widehat{S}$. The proof is completed because the open sets U_z in the form discussed above constitute an open covering of $\widehat{S}$. QED

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