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On the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and its dual

Angela A. Albanese
Department of Mathematics and Physics "E. De Giorgi", University of Salento
I-73100 Lecce, Italy
angela.albanese@unisalento.it

Claudio Mele
Department of Mathematics and Physics "E. De Giorgi", University of Salento
I-73100 Lecce, Italy
claudio.mele1@unisalento.it

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Abstract. We prove that the pre-dual $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ of the space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$ is a sequentially retractive (LF)-space.

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Introduction

In the present paper, we deal with the classes $\mathcal{S}_\omega(\mathbb{R}^N)$ of ultradifferentiable rapidly decreasing functions of Beurling type, where ω is a non-quasianalytic weight function in the sense of Braun, Meise and Taylor [9]. Björck [5] introduced in 1966 the global classes $\mathcal{S}_\omega(\mathbb{R}^N)$ of ultradifferentiable functions, using weights ω in the sense of Beurling. In the last years, many authors focused their interest on this setting, studying many properties of $\mathcal{S}_\omega(\mathbb{R}^N)$ (see [6, 7, 8, 11], for instance, and the references therein). In [1, 2], the authors introduced and studied the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ of the slowly increasing functions and the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ of the very slowly increasing functions in the setting of ultradifferentiable function spaces of Beurling type. For instance, in [1, 2], the authors proved that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of the multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of its dual $\mathcal{S}'_\omega(\mathbb{R}^N)$ and that the strong dual $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is the space of the convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of its dual $\mathcal{S}'_\omega(\mathbb{R}^N)$. Schwartz [18] started in 1966 the study of multipliers and convolutors of the space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions, because of the importance of their application to the study of partial differential equations. Many authors also introduced and studied particular aspects of the spaces of multipliers and of convolutors for ultradifferentiable classes of rapidly decreasing functions of Beurling or Roumieu type in the sense of Komatsu [16] (see, f.i., [12, 15]).

In this paper we continue the study of the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ started in [1, 2], showing that it is a sequentially retractive (LF)-space (Section 3) and hence, complete. We point out that from a functional point of view, establishing or not the completeness of an (LF)-space is a very difficult task. It is still an open problem posed by Grothendieck (see, e.g., [4, p.78]) if a regular (LF)-space is complete.

As a first step towards the main result of the paper, we prove in §2.1 a general criterion to establish the sequential retractivity of a quasi-regular (LF)-space (see Proposition 2). In order to apply this criterion to our case, we show in Section 3 that on the dual space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ the strong topology, the projective topology and the topology induced as a subspace of $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ coincide, where $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ denotes the space of all continuous linear operators from $\mathcal{S}_\omega(\mathbb{R}^N)$ into itself, endowed with the topology of the uniform convergence on bounded subsets of $\mathcal{S}_\omega(\mathbb{R}^N)$ (see Proposition 9).

1 Definitions and preliminary results

1.1 (LF)-spaces

Let E be a locally convex Hausdorff space (lcHs, briefly). As usual, we denote by E'_β its topological dual endowed with the strong topology.

An (LF)-space is a lcHs E which is an inductive limit $E = \mathrm{ind}_{\vec{n}}\, E_n$ of an inductive sequence $\{E_n\}_{n\in\mathbb{N}}$ of Fréchet spaces, i.e., $E_n \hookrightarrow E_{n+1}$ continuously for all $n \in \mathbb{N}$, $E = \cup_{n\in\mathbb{N}} E_n$ and E is endowed with the finest locally convex Hausdorff topology such that the inclusions $E \hookrightarrow E_n$ are continuous.

Let us recall some necessary definitions. An (LF)-space $E = \mathrm{ind}_{\vec{n}}\, E_n$ is said to be *quasi-regular* if for every bounded subset B of E, there exist $n \in \mathbb{N}$ and a bounded subset C of E_n such that $B \subseteq \overline{C}$, where the closure is taken in E. We say that $E = \mathrm{ind}_{\vec{n}}\, E_n$ is *regular* if every bounded subset in E is contained and bounded in E_n, for some $n \in \mathbb{N}$. Regular (LF)-spaces are clearly quasi-regular and complete (LF)-spaces are regular.

Moreover, an (LF)-space $E = \mathrm{ind}_{\vec{n}}\, E_n$ is said to be: *compactly retractive* if for every compact subset K of E, there exists $m \in \mathbb{N}$ such that $K \subset E_m$ and it is compact there; *boundedly retractive* if every bounded subset B of E is contained in some step E_n and the topologies of E and E_n coincide on B; *sequentially retractive* if every convergent sequence in E is contained in some step E_n and converges there.

We point out that, due to Grothendieck's factorization theorem, all the conditions introduced above do not depend on the defining inductive spectrum of E. We refer the reader to [19, 20] for more details.

The following theorem due to Wengenroth for (LF)-spaces (see [20, Theorem 6.4]) shows the equivalence of some the properties introduced above.

Theorem 1. *For an (LF)-space $E = \operatorname{ind}_{\vec{n}} E_n$ the following conditions are equivalent:*

(1) E is boundedly retractive;

(2) E is compactly retractive;

(3) E is sequentially retractive.

Furthermore, sequentially retractive (LF)-spaces are complete (see [20, Corollary 2.8]) and hence, regular.

We recall that if $E = \operatorname{ind}_{\vec{n}} E_n$ is an (LF)-space, then the canonical linear map $j\colon E'_\beta \to \operatorname{proj}_{\overleftarrow{n}} (E_n)'_\beta$, $f \mapsto (f_{|E_n})_{n\in\mathbb{N}}$, (where the projective limit is formed with respect to the natural restrictions $(E_{n+1})'_\beta \to (E_n)'_\beta$, $g \mapsto g_{|E_n}$) is a continuous and surjective isomorphism. But in general, j is not an open map. By standard duality arguments, one shows the following fact.

Proposition 1. *Let $E = \operatorname{ind}_{\vec{n}} E_n$ be an (LF)-space. Then E is quasi-regular if, and only, if $E'_\beta = \operatorname{proj}_{\overleftarrow{n}} (E_n)'_\beta$ topologically.*

Proof. See, for example, [14] or [17, §22.6 and 22.7]. QED

Now, we give a simple criterion to establish the sequential retractivity of a quasi-regular (LF)-space.

Proposition 2. *Let $E = \operatorname{ind}_{\vec{n}} E_n$ be a quasi-regular (LF)-space. Suppose that there exists a Fréchet space F such that $E \hookrightarrow F$ continuously and*

(*) for all $n \in \mathbb{N}$ and for every bounded absolutely convex subset B of E_n, the spaces E_{n+1} and F induce the same topology on B.

Then E is a sequentially rectractive (LF)-space.

Proof. Let $\{x_j\}_{j\in\mathbb{N}}$ be a null sequence in E. Then there exists a bounded absolutely convex subset B_0 of E such that $\{x_j\}_{j\in\mathbb{N}} \cup \{0\} \subset B_0$. Since E is a quasi-regular (LF)-space, there exist $n \in \mathbb{N}$ and a bounded absolutely convex subset C of E_n such that $B_0 \subseteq \overline{C}^E$. We observe that by assumption, the sequence $\{x_j\}_{j\in\mathbb{N}}$ also converges to 0 in F.

For a fixed $x \in \overline{C}^E$, there exists a net $\{c_\alpha\}_\alpha \subset C$ such that $c_\alpha \to x$ in E and hence, in F. Accordingly, $\{c_\alpha\}_\alpha$ is a Cauchy net of F. Since E_{n+1} and F induce the same uniformity on C, it follows that $\{c_\alpha\}_\alpha$ is also a Cauchy net of E_{n+1}.

But, E_{n+1} is a Fréchet space. So, we get that there exists $y \in E_{n+1}$ such that $c_\alpha \to y$ in E_{n+1} and hence, in F. Therefore, $x = y$. By the arbitrariness of x, it follows that $B_0 \subset E_{n+1}$ and that B_0 is bounded in E_{n+1}, since $B_0 \subseteq \overline{C}^{E_{n+1}} \subset E_{n+1}$.

Now, E_{n+2} and F induce the same topology on every bounded subset of E_{n+1} and hence, on B_0. Since $\{x_j\}_{j\in\mathbb{N}} \subset B_0$ and $x_j \to 0$ in F, it follows that $x_j \to 0$ in E_{n+2}. This completes the proof. QED

1.2 The spaces $\mathcal{S}_\omega(\mathbb{R}^N)$, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$

We first give the definition of non-quasianalytic weight function in the sense of Braun, Meise and Taylor [9] suitable for the Beurling case, i.e., we also consider the logarithm as a weight function.

Definition 1. A non-quasianalytic weight function is a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(α) there exists $K \geq 1$ such that $\omega(2t) \leq K(1+\omega(t))$ for every $t \geq 0$;

(β) $\int_1^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$;

(γ) there exist $a \in \mathbb{R}$, $b > 0$ such that $\omega(t) \geq a + b\log(1+t)$, for every $t \geq 0$;

(δ) $\varphi_\omega(t) = \omega \circ \exp(t)$ is a convex function.

We recall some known properties of the weight functions that shall be useful in the following (the proofs can be found in the literature):

(1) Condition (α) implies that

$$\omega(t_1 + t_2) \leq K(1 + \omega(t_1) + \omega(t_2)),\ \forall t_1, t_2 \geq 0.$$

Observe that this condition is weaker than subadditivity (i.e., $\omega(t_1 + t_2) \leq \omega(t_1)+\omega(t_2)$). The weight functions satisfying (α) are not necessarily subadditive in general.

(2) By condition (γ), we have that

$$e^{-\lambda\omega(t)} \in L^p(\mathbb{R}^N),\ \forall \lambda \geq \frac{N+1}{bp}.$$

Given a non-quasianalytic weight function ω, we define the Young conjugate φ^*_ω of φ_ω as the function $\varphi^*_\omega : [0,\infty) \to [0,\infty)$ by

$$\varphi^*_\omega(s) := \sup_{t\geq 0}\{st - \varphi_\omega(t)\}, \quad s \geq 0.$$

There is no loss of generality to assume that ω vanishes on $[0,1]$. Therefore, φ^*_ω is convex and increasing, $\varphi^*_\omega(0) = 0$ and $(\varphi^*_\omega)^* = \varphi_\omega$. Furthermore, $\frac{\varphi^*_\omega(t)}{t}$ is an increasing function in $(0,\infty)$ and in particular, for every $s,t \geq 0$ and $\lambda > 0$ the following chain of inequalities holds true:

$$2\lambda\varphi^*_\omega\left(\frac{s+t}{2\lambda}\right) \leq \lambda\varphi^*_\omega\left(\frac{s}{\lambda}\right) + \lambda\varphi^*_\omega\left(\frac{t}{\lambda}\right) \leq \lambda\varphi^*_\omega\left(\frac{s+t}{\lambda}\right). \tag{1.1}$$

We now introduce the ultradifferentiable function spaces which will be considered in this paper: $\mathcal{S}_\omega(\mathbb{R}^N)$, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$.

Following Björk [5] we define:

Definition 2. Let ω be a non-quasianalytic weight function. We denote by $\mathcal{S}_\omega(\mathbb{R}^N)$ the set of all functions $f \in L^1(\mathbb{R}^N)$ such that $f, \hat{f} \in C^\infty(\mathbb{R}^N)$ and for all $\lambda > 0$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\|\exp(\lambda\omega)\partial^\alpha f\|_\infty < \infty \quad \text{and} \quad \|\exp(\lambda\omega)\partial^\alpha \hat{f}\|_\infty < \infty\,,$$

where $\hat{f}$ denotes the Fourier transform of f. The elements of $\mathcal{S}_\omega(\mathbb{R}^N)$ are called *ω-ultradifferentiable rapidly decreasing functions of Beurling type*. We denote by $\mathcal{S}'_\omega(\mathbb{R}^N)$ the dual of $\mathcal{S}_\omega(\mathbb{R}^N)$ endowed with its strong topology.

The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a Fréchet space with different equivalent systems of seminorms (see [2, Proposition 2.4], [7, Theorem 4.8] and [6, Theorem 2.6]). In the following, we will use the following systems of norms generating the Fréchet topology of $\mathcal{S}_\omega(\mathbb{R}^N)$:

$$q_{\lambda,\mu}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_\infty \exp\left(-\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)\right), \quad \lambda,\mu > 0,\ f \in \mathcal{S}_\omega(\mathbb{R}^N),$$

(equivalently, the sequence of norms $\{q_{m,n}\}_{m,n\in\mathbb{N}}$) or, for any $1 \leq p < \infty$ and $\lambda, \mu > 0$,

$$\sigma_{\lambda,\mu,p}(f) := \left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p^p \exp\left(-p\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)\right)\right)^{\frac{1}{p}}, \quad f \in \mathcal{S}_\omega(\mathbb{R}^N),$$

(equivalently, the sequence of norms $\{\sigma_{m,n,p}\}_{m,n\in\mathbb{N}}$).

We point out that the space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a nuclear Fréchet space, see, f.i., [8, Theorem 3.3] or [11, Theorem 1.1].

We refer the reader to [9] for the definition and the main properties of the ultradifferentiable function spaces $\mathcal{E}_\omega(\Omega)$, $\mathcal{D}_\omega(\Omega)$ and their duals of Beurling type in the sense of Braun, Meise and Taylor. We only recall that for an open subset Ω of $\mathbb{R}^N$, the space $\mathcal{E}_\omega(\Omega)$ is defined as

$$\mathcal{E}_\omega(\Omega) := \{f \in C^\infty(\Omega)\colon p_{K,m}(f) < \infty\ \forall K \Subset \Omega,\ m \in \mathbb{N}\}\,,$$

where

$$p_{K,m}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N} |\partial^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right).$$

$\mathcal{E}_\omega(\Omega)$ is a nuclear Fréchet space with respect to the lc-topology generated by the system of seminorms $\{p_{K,m}\}_{K\Subset\Omega, m\in\mathbb{N}}$ (see [9, Proposition 4.9]). The elements of $\mathcal{E}_\omega(\Omega)$ are called *ω-ultradifferentiable functions of Beurling type* on Ω.

The spaces $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$, for $1\le p\le\infty$, have been introduced in [1, 2], as follows.

Definition 3. Let ω be a non-quasianalytic weight function.

(a) For $m,n\in\mathbb{N}$ and $1\le p\le\infty$, we denote by $\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ the space of all functions $f\in C^\infty(\mathbb{R}^N)$ such that

$$r_{m,n,\infty}(f) := \sup_{\alpha\in\mathbb{N}_0^N}\sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty, \text{ if } p=\infty,$$

$$r^p_{m,n,p}(f) := \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(-n\omega)\partial^\alpha f\|_p^p \exp\left(-pm\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty, \text{ if } 1\le p<\infty.$$

The space $(\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N), r_{m,n,p})$ is a Banach space for all $m,n\in\mathbb{N}$ and $1\le p\le\infty$.

(b) For $1\le p\le\infty$, the space $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ is defined by

$$\mathcal{O}_{M,\omega,p}(\mathbb{R}^N) := \bigcap_{m=1}^\infty\bigcup_{n=1}^\infty \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$$

and it is endowed with its natural lc-topology, i.e., $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N) = \operatorname{proj}_{\overleftarrow{m}} \mathcal{O}^m_{\omega,p}(\mathbb{R}^N)$ is the projective limit of the (LB)-spaces $\mathcal{O}^m_{\omega,p}(\mathbb{R}^N) := \operatorname{ind}_{\overrightarrow{n}} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$, $m\in\mathbb{N}$. We denote by $\mathcal{O}'_{M,\omega,p}(\mathbb{R}^N)$ the dual of $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$, endowed with its strong topology.

(c) For $1\le p\le\infty$, the space $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ is defined by

$$\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) := \bigcup_{n=1}^\infty\bigcap_{m=1}^\infty \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$$

and it is endowed with its natural lc-topology, i.e., $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) = \operatorname{ind}_{\overrightarrow{n}} \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ is an (LF)-space, where $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N) := \operatorname{proj}_{\overleftarrow{m}} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ is a Fréchet space for any $n\in\mathbb{N}$. We denote by $\mathcal{O}'_{C,\omega,p}(\mathbb{R}^N)$ the dual of $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$, endowed with its strong topology.

In the following, the spaces $\mathcal{O}^m_{n,\omega,\infty}(\mathbb{R}^N)$, $\mathcal{O}^m_{\omega,\infty}(\mathbb{R}^N)$, $\mathcal{O}_{n,\omega,\infty}(\mathbb{R}^N)$, $\mathcal{O}_{M,\omega,\infty}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega,\infty}(\mathbb{R}^N)$ are simply denoted by $\mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$, $\mathcal{O}^m_{\omega}(\mathbb{R}^N)$, $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$, $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ respectively (also $r_{m,n,\infty}$ is simply denoted by $r_{m,n}$), for any $m, n \in \mathbb{N}$. The elements of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, resp.) are called *slowly increasing functions of Beurling type* (*very slowly increasing functions of Beurling type*, resp.).

Recall that the inclusions $\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ are well-defined, continuous with dense range, see [1, Theorems 3.8, 3.9 and 5.2(1)]. Hence, the inclusions $\mathcal{E}'_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{D}'_\omega(\mathbb{R}^N)$ are also well-defined and continuous. Furthemore, the inclusions $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ are continuous.

Concerning the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ introduced above we have the following result (see [2, Proposition 3.8]).

Proposition 3. *Let ω be a non-quasianalytic weight function. Then the following properties are satisfied:*

(1) $\mathcal{O}^m_\omega(\mathbb{R}^N) = \mathcal{O}^m_{\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{n,\omega}(\mathbb{R}^N) = \mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ algebraically and topologically for every $1 \leq p < \infty$ and $m, n \in \mathbb{N}$;

(2) $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ algebraically and topologically for every $1 \leq p < \infty$.

In particular, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$ (see [1, Theorem 4.4]).

Proposition 4. *Let ω be a non-quasianalytic weight function. Then the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$, i.e., $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ if, and only if, $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$ for all $g \in \mathcal{S}_\omega(\mathbb{R}^N)$. Moreover, if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then the linear operator $M_f : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $M_f(g) := fg$, for $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, is continuous.*

Furthemore, by a result of Debrouwere and Neyt [10, Theorem 5.3] we have

Proposition 5. *Let ω be a non-quasianalytic weight function. Then the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is ultrabornological.*

Thanks to Proposition 4, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be identified with the subspace of $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$ whose elements are the multipliers M_f. So, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be endowed with the topology τ_b induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$, where $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ denotes the space of all continuous linear operators from $\mathcal{S}_\omega(\mathbb{R}^N)$ into itself, endowed with the topology of the uniform convergence on bounded subsets of $\mathcal{S}_\omega(\mathbb{R}^N)$. But, by Proposition 5 the induced topology τ_b coincides with the natural topology of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

The space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$ (see [2, Theorem 5.3]).

Proposition 6. *Let ω be a non-quasianalytic weight function satisfying the condition* $\log(1+t) = o(\omega(t))$ *as $t \to \infty$. Then $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$, i.e., $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ if, and only if, $T \star f \in \mathcal{S}_\omega(\mathbb{R}^N)$ for all $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Moreover, the linear operator $C_T : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $C_T(f) := T \star f$, for $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, is continuous.*

We point out that by [2, Theorem 5.3,(1) $\Rightarrow$ (2)], for any non-quasianalytic weight function ω, if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, then the convolutor $C_T \in \mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$. Hence, $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ can be always identified with a subspace of $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$, whose elements are the convolutors C_T.

Finally, we recall that if ω is a non-quasianalytic weight function satisfying the condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$, the Fourier transform is a topological isomorphism from $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ onto $\mathcal{O}_{M,\omega}(R^N)$ (see [2, Theorem 6.1]). In particular, for all $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ the convolution $T \star f$ satisfies the following property:

$$\mathcal{F}(T \star f) = \hat{f}\mathcal{F}(T). \tag{1.2}$$

We conclude this section by showing that the Fourier transform is also a topological isomorphism when $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is endowed with the topology τ_b induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$

Proposition 7. *Let ω be a non-quasianalytic weight function satisfying the condition* $\log(1+t) = o(\omega(t))$ *as $t \to \infty$. Then the Fourier transfom $\mathcal{F}$ is a topological isomorphism from $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ onto $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$.*

Proof. For a fixed net $\{T_i\}_{i\in I} \subset \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, we observe that in view of (1.2) the following equality holds true:

$$\mathcal{F}(T_i \star g) = \hat{g}\mathcal{F}(T_i), \quad g \in \mathcal{S}_\omega(\mathbb{R}^N),\ i \in I.$$

Moreover, $T_i \star g \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $i \in I$. Since $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}_\omega(\mathbb{R}^N)$ onto itself, if we set $f_i := \mathcal{F}(T_i) \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ for $i \in I$, then $T_i \to T$ in $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ for some $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ if, and only if, $T_i \star g \to T \star g$ in $\mathcal{S}_\omega(\mathbb{R}^N)$ uniformly on bounded subsets of $\mathcal{S}_\omega(\mathbb{R}^N)$ if, and only if, $\mathcal{F}(T_i \star g) \to \mathcal{F}(T \star g)$ in $\mathcal{S}_\omega(\mathbb{R}^N)$ uniformly on bounded subsets of $\mathcal{S}_\omega(\mathbb{R}^N)$ if, and only if, $\hat{g}f_i \to \hat{g}f$ in $\mathcal{S}_\omega(\mathbb{R}^N)$ uniformly on bounded subsets of $\mathcal{S}_\omega(\mathbb{R}^N)$, where $f := \mathcal{F}(T)$, if, and only if, $f_i \to f$ in $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, being the topology of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ equal to the one induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ as observed above. QED

2 $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially retractive (LF)-space

The aim of this section is to establish that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially retractive Montel (LF)-space. To do this, we collect some auxiliary results.

Lemma 1. *Let $m, n \in \mathbb{N}$ and $1 < p < \infty$. Then the following properties are satisfied:*

(1) The Banach space $\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ is reflexive;

(2) $T \in (\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N))'$ if, and only if, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$ and there exists $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \subset L^{p'}(\mathbb{R}^N, \exp(n\omega(x))dx)$ such that

$$T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha \quad \text{and} \quad \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_{p'}^{p'} \exp\left(p' m \varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty, \tag{2.3}$$

being p' the conjugate exponent of p.

Moreover, the space $(\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N))'$ endowed with the norm

$$r'_{m,n,p}(T) := \inf\left\{\left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_{p'}^{p'} \exp\left(p' m \varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p'}}\right\},$$

where the infimum is taken over all $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \subset L^{p'}(\mathbb{R}^N, \exp(n\omega(x))dx)$ satisfying (2.3), is the strong dual of the Banach space $\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$.

Proof. It is straightforward. QED

Lemma 2. *Let ω be a non-quasianalytic weight function and $1 \le p < \infty$. Then the inductive spectrum $\{\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)\}_{n\in\mathbb{N}}$ is reduced, i.e., for all $n \in \mathbb{N}$ the space $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ is dense in $\mathcal{O}_{n+1,\omega,p}(\mathbb{R}^N)$.*

Proof. The result follows from [2, Proposition 4.2(3) and Remark 4.3]. QED

In view of Lemmas 1 and 2, for any $1 < p < \infty$ the family $\{\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)\}_{n\in\mathbb{N}}$ is a reduced inductive spectrum of reflexive Fréchet spaces. Accordingly, the strong dual $\mathcal{O}'_{n+1,\omega,p}(\mathbb{R}^N)$ of $\mathcal{O}_{n+1,\omega,p}(\mathbb{R}^N)$ is continuously included in the strong dual $\mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)$ of $\mathcal{O}_{n,\omega,p}(\mathbb{R}^N)$ for all $n \in \mathbb{N}$, and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N) = \mathcal{O}'_{C,\omega,p}(\mathbb{R}^N) = \cap_{n\in\mathbb{N}} \mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)$. So, we can consider on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ the projective topology τ_{pr} defined by the projective spectrum $\{\mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)\}_{n\in\mathbb{N}}$, i.e., $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr}) = \operatorname{proj}_{\overleftarrow{n}} \mathcal{O}'_{n,\omega,p}(\mathbb{R}^N)$. The topology τ_{pr} does not depend on p, because of the fact that $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ for all $1 < p < \infty$. As it is well known, the inclusion $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta) \hookrightarrow (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr})$ is continuous, where τ_β denotes the strong topology of $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. In the following, we show that also $\tau_{pr} = \tau_\beta$ on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. To see this, we first establish the following fact, which is valid for every non-quasianalytic weight function ω.

Proposition 8. *Let ω be a non-quasianalytic weight function. Then the inclusion $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr}) \hookrightarrow (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ is continuous, where τ_b denotes the locally convex topology induced on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$.*

Proof. For all $n \in \mathbb{N}$, set $U_n := \{f \in \mathcal{S}_\omega(\mathbb{R}^N) \colon q_{n,n}(f) \leq 1\}$. Then $\{U_n\}_{n\in\mathbb{N}}$ is a basis of closed absolutely convex 0-neighborhoods of $\mathcal{S}_\omega(\mathbb{R}^N)$. Taking into account that $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr}) = \mathrm{proj}_{\overleftarrow{n}}\, \mathcal{O}'_{n,\omega,2}(\mathbb{R}^N)$, it suffices to show that for all $n \in \mathbb{N}$ and every bounded, closed absolutely convex subset B of $\mathcal{S}_\omega(\mathbb{R}^N)$, the set

$$M(B, U_n) := \{T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \colon T \star f \in U_n \ \forall f \in B\}$$

contains $\overset{\circ}{C}$ (the polar taken in $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$), for some bounded subset C of $\mathcal{O}_{n',\omega,2}(\mathbb{R}^N)$ and some $n' \geq n$. To prove this, for a fixed $n \in \mathbb{N}$, we observe that if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, then $T \in \mathcal{O}'_{n',\omega,2}(\mathbb{R}^N)$ with $n' = [Kn] + 1 \geq n$ (where K is the constant appearing in Definition 1(α)). Hence, $T \in (\mathcal{O}^m_{n',\omega,2}(\mathbb{R}^N))'$, for some $m \in \mathbb{N}$ which can be always supposed greater or equal than n'. So, by Lemma 1, there exists $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \subset L^2(\mathbb{R}^N, \exp(n'\omega(x))\, dx)$ such that $T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha$ and $\sum_{\alpha\in\mathbb{N}_0^N} \| \exp(n'\omega) f_\alpha\|_2^2 \exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty$. Now, arguing in a similar way as in the proof of [2, Theorem 5.3], we obtain for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, $\beta \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$, that

$$|\partial^\beta(T \star f)(x)| \leq \sum_{\alpha\in\mathbb{N}_0^N} \| \exp(n'\omega) f_\alpha\|_2 \| \exp(-n'\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_2,$$

where

$$\begin{aligned} \| \exp(-n'\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_2^2 &\leq c^2 \exp\left(-\frac{2n'\omega(x)}{K}\right) \| \exp(n'\omega)\partial^{\alpha+\beta} f\|_2^2 \\ &\leq c^2 \exp\left(-2n\omega(x)\right) \| \exp(n'\omega)\partial^{\alpha+\beta} f\|_2^2, \end{aligned}$$

with $c := \exp(n')$, being $n' \geq Kn$. Therefore, we get for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, $\beta \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned} \exp\left(n\omega(x)\right) |\partial^\beta(T \star f)(x)| &\leq c \sum_{\alpha\in\mathbb{N}_0^N} \| \exp(n'\omega) f_\alpha\|_2 \| \exp(n'\omega)\partial^{\alpha+\beta} f\|_2 \\ &\leq c \left(\sum_{\alpha\in\mathbb{N}_0^N} \| \exp(n'\omega) f_\alpha\|_2^2 \exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}} \times \\ &\times \left(\sum_{\alpha\in\mathbb{N}_0^N} \| \exp(n'\omega)\partial^{\alpha+\beta} f\|_2^2 \exp\left(-2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Due to (1.1), it follows that

$$q_{m,n}(T\star f)=\sup_{\beta\in\mathbb{N}_0^N}\sup_{x\in\mathbb{R}^N}\exp\left(n\omega(x)\right)|\partial^\beta(T\star f)(x)|\exp\left(-m\varphi^*_\omega\left(\frac{|\beta|}{m}\right)\right)$$

$$\leq c\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)f_\alpha\|_2^2\exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}}\times$$

$$\times\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)\partial^{\alpha+\beta}f\|_2^2\exp\left(-2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)-2m\varphi^*_\omega\left(\frac{|\beta|}{m}\right)\right)\right)^{\frac{1}{2}}$$

$$\leq c\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)f_\alpha\|_2^2\exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}}\times$$

$$\times\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)\partial^{\alpha+\beta}f\|_2^2\exp\left(-4m\varphi^*_\omega\left(\frac{|\alpha+\beta|}{2m}\right)\right)\right)^{\frac{1}{2}}$$

$$=c\sigma_{2m,n',2}(f)\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)f_\alpha\|_2^2\exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}}.$$

Since $q_{n,n}\leq q_{m,n}$ as $m\geq n$, we get via the above inequalities that

$$q_{n,n}(T\star f)\leq c\sigma_{2m,n',2}(f)\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n'\omega)f_\alpha\|_2^2\exp\left(2m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{2}}$$

for every $f\in\mathcal{S}_\omega(\mathbb{R}^N)$. Therefore, by passing to the infimum as indicated in Lemma 1, we obtain for every $f\in\mathcal{S}_\omega(\mathbb{R}^N)$ that

$$q_{n,n}(T\star f)\leq c\sigma_{2m,n',2}(f)r'_{m,n',2}(T).$$

Fixed a bounded closed absolutely convex subset B of $\mathcal{S}_\omega(\mathbb{R}^N)$, we get that

$$\sup_{f\in B}q_{n,n}(T\star f)\leq c\left(\sup_{f\in B}\sigma_{2m,n',2}(f)\right)r'_{m,n',2}(T). \tag{2.4}$$

Now, for each $m\geq n'$, set $\lambda_m:=c\sup_{f\in B}\sigma_{2m,n',2}(f)<\infty$ and $V_{m,n'}:=\{f\in\mathcal{O}_{n',\omega,2}(\mathbb{R}^N)\colon r_{m,n',2}(f)\leq 1\}$. Then $C:=\cap_{m\geq n'}\lambda_mV_{m,n'}$ is a bounded closed absolutely convex subset of $\mathcal{O}_{n',\omega,2}(\mathbb{R}^N)$ and hence, of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. We claim that $\overset{\circ}{C} \subseteq M(B, U_n)$. To show this, we first observe that $T \in \overset{\circ}{V}_{m,n'}$ (the polar taken in $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$) if, and only if, $r'_{m,n',2}(T) \leq 1$ (so, $T \in (\mathcal{O}^m_{n',\omega,2}(\mathbb{R}^N))'$). Now, if $T \in \Gamma(\cup_{m\geq n'}\lambda_m^{-1} \overset{\circ}{V}_{m,n'})$, where Γ denotes the absolutely convex hull, then $T = \sum_{m\geq n'} \alpha_m \lambda_m^{-1} T_m$, with $T_m \in \overset{\circ}{V}_{m,n'}$ for all $m \geq n'$ and $\sum_{m\geq n'} |\alpha_m| \leq 1$. Thus, we can apply (2.4) to obtain that

$$\begin{aligned}\sup_{f\in B} q_{n,n}(T \star f) &\leq \sum_{m\geq n'} |\alpha_m|\lambda_m^{-1} \sup_{f\in B} q_{n,n}(T_m \star f) \\ &\leq \sum_{m\geq n'} |\alpha_m|\lambda_m^{-1}\lambda_m r'_{m,n',2}(T_m) \leq \sum_{m\geq n'} |\alpha_m| \leq 1.\end{aligned}$$

This means that $T \in M(B, U_n)$. Since $M(B, U_n)$ is $\sigma := \sigma(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \mathcal{O}_{C,\omega}(\mathbb{R}^N))$-closed, it follows that $\overset{\circ}{C} \subseteq M(B, U_n)$, being $\overset{\circ}{C} = \overline{\Gamma(\cup_{m\geq n'}\lambda_m \overset{\circ}{V}_{m,n'})}^{\sigma}$. QED

As an immediate consequence of Proposition 8, combined with Remark 7, we obtain the following result.

Proposition 9. *Let ω be a non-quasianalytic weight function satisfying the condition $\log(1 + t) = o(\omega(t))$ as $t \to \infty$. Then $\tau_\beta = \tau_{pr} = \tau_b$ on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. In particular, $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta) = (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr})$.*

Proof. The facts that $\mathcal{F}$ is a topological isomorphism from the space $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ onto the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is ultrabornological (see [10]) and $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta) \hookrightarrow (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_{pr}) \hookrightarrow (\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_b)$ continuously and that the space $(\mathcal{O}'_{C,\omega}(\mathbb{R}^N), \tau_\beta)$ has a strict web (see [13, Proposition IV.3.3]), permit to apply the De Wilde's open mapping theorem to conclude that $\tau_\beta = \tau_{pr} = \tau_b$. QED

To show the sequential retractivity of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, it remains to observe the following fact.

Lemma 3. *Let ω be a non-quasianalytic weight function and let $n \in \mathbb{N}$. Then the spaces $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on the bounded subsets of $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$.*

Proof. Let B be a bounded subset of $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$. We can assume that B is absolutely convex. Hence, we only have to show that the topology of $\mathcal{E}_\omega(\mathbb{R}^N)$ induces a finer filter of 0-neighborhoods in B than the topology of $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$. So, let $m \in \mathbb{N}$ and $\varepsilon > 0$. Then the set $U := \{f \in \mathcal{O}_{n+1,\omega}(\mathbb{R}^N) \colon r_{m,n+1}(f) < \varepsilon\}$ is a 0-neighborhood of $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$. Since B is a bounded subset of $\mathcal{O}_{n,\omega}(\mathbb{R}^N)$, there exists $C_m > 0$ such that $r_{m,n}(f) \leq C_m$ for all $f \in B$. Let $M > 0$ such that $\exp(-\omega(x)) < \frac{\varepsilon}{C_m}$ for every $|x| \geq M$ and $V := \{f \in \mathcal{E}_\omega(\mathbb{R}^N) \colon p_{K,m}(f) < \varepsilon\}$, where $K := \{x \in \mathbb{R}^N \colon |x| \leq M\}$. We have that $V \cap B \subseteq U \cap B$. Indeed, if $f \in V \cap B$, then

$$|\partial^\alpha f(x)| \leq C_m \exp\left(n\omega(x) + m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \varepsilon \exp\left((n+1)\omega(x) + m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)$$

for every $\alpha \in \mathbb{N}_0^N$ and $|x| \geq M$. Moreover,

$$|\partial^\alpha f(x)| < \varepsilon \exp\left(m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) \leq \varepsilon \exp\left((n+1)\omega(x) + m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)$$

for every $\alpha \in \mathbb{N}_0^N$ and $|x| \leq M$. It follows that $r_{m,n+1}(f) < \varepsilon$ and hence, $f \in U \cap B$. QED

We now are ready to show the main result of the paper, i.e., that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially retractive (LF)-space and hence, complete. The result should be compared with [12, Corollary 3.9(ii)] valid in the setting of ultradifferentiable classes defined according to Komatsu, by pointing out that such a result and the method of proof do not apply to our case.

Theorem 2. *Let ω be a non-quasianalytic weight function satisfying $\log(1+t) = o(\omega(t))$ as $t \to \infty$. Then $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially rectractive Montel (LF)-space. In particular, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a complete (LF)-space.*

Proof. By Proposition 9 combined with Proposition 1, we get that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a quasi-regular (LF)-space. On the other hand, $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is continuously included in the Fréchet space $\mathcal{E}_\omega(\mathbb{R}^N)$ and satisfies condition $(*)$ of Proposition 2 with respect to $\mathcal{E}_\omega(\mathbb{R}^N)$, as proved in Lemma 3. Therefore, we can apply Proposition 2 to conclude that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a sequentially retractive (LF)-space and hence, complete (see [20, Corollary 2.8]).

It remains to show that $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a Montel (LF)-space, i.e., its bounded sets are relatively compact. This follows by observing that, if B is a bounded subset of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, then there exists $n \in \mathbb{N}$ such that $B \subset \mathcal{O}_{n,\omega}(\mathbb{R}^N)$ and bounded there. Applying again Lemma 3, we get that $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$ and $\mathcal{E}_\omega(\mathbb{R}^N)$ induce the same topology on B and hence, B is relatively compact in $\mathcal{O}_{n+1,\omega}(\mathbb{R}^N)$, being $\mathcal{E}_\omega(\mathbb{R}^N)$ a Montel Fréchet space. Therefore, B is also relatively compact in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. QED

Finally, we recall that for any $m \in \mathbb{N}$ the (LB)-space $\mathcal{O}^m_\omega(\mathbb{R}^N)$ is also a sequentially retractive Montel space and hence, complete, as shown in [3, Theorems 1 and 2].

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