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Involution semi-braces and the Yang-Baxter equation

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Abstract. The main aim of this paper is to provide set-theoretical solutions of the Yang–Baxter equation that are not necessarily bijective. We use the new structure of *involution semi-brace*, that is a quadruple $(S,+,\cdot,^*)$ with $(S,+)$ a semigroup and $(S,\cdot,^*)$ an involution semigroup satisfying the relation $a(b+c) = ab + a(a^* + c)$, for all $a,b,c \in S$.

Keywords: semi-brace, set-theoretical solution, Yang-Baxter equation, involution

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Introduction

The quantum Yang-Baxter equation first appeared in theoretical physics in a paper by Yang [26] and, independently, in one by Baxter [2].
In [14], Drinfel'd suggested to study set-theoretical solutions of this equation. Specifically, given a non-empty set S, a map $r : S \times S \to S \times S$ such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 := r \times id_S$ and $r_2 := id_S \times r$, is said to be a *set-theoretical solution of the Yang-Baxter equation*, or shortly a *solution* on S.
Determining all the solutions is still an open problem and it has drawn the attention of several mathematicians. We point out the approach based on left braces, algebraic structures introduced by Rump [24] that include the Jacobson radical rings. Rump's paper traced a novel research direction which led to fruitful results, as one can see in the survey by Cedó [9].
Bijective solutions can be produced through a generalization of left braces, the skew left braces, algebraic structures introduced by Guarnieri and Vendramin [16]. A *skew left brace* is a triple $(S,+,\cdot)$ such that $(S,+)$ and $(S,\cdot)$ are groups and

$$a(b+c) = ab - a + ac$$

holds, for all $a, b, c \in S$. If $(S, +)$ is an abelian group, then $(S, +, \cdot)$ is a left brace. Subsequently, these structures have been investigated by many authors and bijective solutions have been produced.
Catino, Colazzo and Stefanelli [4] showed that the algebraic structure of left semi-brace turns out to be a useful tool for producing solutions which are not necessarily bijective. Slight generalizations of semi-braces, under mild assumptions, provide solutions of the Yang-Baxter equation, see e.g., [3], [6], [8], [10], [17], [18].

To determine new solutions, in [7] there were introduced inverse semi-braces that include semi-braces. A *(left) inverse semi-brace* is a triple $(S, +, \cdot)$ such that $(S, +)$ is an arbitrary semigroup, $(S, \cdot)$ is an inverse semigroup, and

$$a(b+c) = ab + a(a^{-1} + c)$$

holds, for all $a, b, c \in S$, where a^{-1} is the inverse of a with respect to the multiplication. We recall that a semigroup $(S, \cdot)$ is inverse if, for each $a \in S$, there exists a unique $a^{-1} \in S$ satisfying $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

If $(S, \cdot)$ is an inverse semigroup, then the inversion $S \longrightarrow S, a \mapsto a^{-1}$ is an involution. Recall that an *involution* on the semigroup $(S, \cdot)$ is an anti-isomorphism of period two, that is, a function $^* : S \longrightarrow S$ satisfying the following conditions

$$(a^*)^* = a \quad \text{and} \quad (ab)^* = b^* a^*,$$

for all $a, b \in S$. Note that in an inverse semigroup there may exist an involution other than the inversion.

The main aim of this paper is to provide set-theoretical solutions of the Yang–Baxter equation, using new structures like inverse semi-braces, namely the involution semi-braces. A triple $(S, +, \cdot, ^*)$ is a *(left) involution semi-brace* if $(S, +)$ is a semigroup (not necessarily commutative), $(S, \cdot, ^*)$ is a regular $*$-semigroup, and

$$a(b+c) = ab + a(a^* + c) \tag{0.1}$$

holds, for all $a, b, c \in S$. Here, according to Nordahl and Scheiblich [20], the *-semigroup $(S, \cdot, ^*)$ is said to be *regular* if $a = aa^*a$, for any $a \in S$.
The solutions will be sought among the maps r associated to involution semi-braces $(S, +, \cdot, ^*)$, that is, $r : S \times S \longrightarrow S \times S$ given by

$$r(a, b) := (\ a(a^* + b),\ (a^* + b)^* b\ ), \tag{0.2}$$

for all $a, b \in S$. Note that, as for inverse semi-braces, if $(S, +, \cdot, ^*)$ is an involution semi-brace, the map r given in (2) is not necessarily a solution. In light of this, we provide sufficient conditions so that the map r is a solution. Our attention is turned to study involution semi-braces for which the map associated is a solution.

1 Involution semi-braces: definitions and examples

Recall that, if $(S, \cdot)$ is a semigroup and $^* : S \longrightarrow S$ a unary operation satisfying the following conditions

$$(x^*)^* = x, \quad (xy)^* = y^*x^*,$$

for all $x, y \in S$, then the triple $(S, \cdot, ^*)$ is called $*$-*semigroup*, or *involution semigroup*, and the unary operation $*$ is said to be an *involution.*
Following [20], $(S, \cdot, ^*)$ is said to be *regular* if

$$x = xx^*x,$$

for any $x \in S$. We note that x^* is an inverse of x since $x^*xx^* = (xx^*x)^* = x^*$.
Inverse semigroups $(S, \cdot, ^{-1})$ are clearly included in the class of regular $*$ - semigroups, and moreover, they can be equationally characterized within that class. Indeed, as shown by Schein in [25], the class of all inverse semigroups is defined within the variety of regular $*$–semigroups by the identity $xx^*x^*x = x^*xxx^*$.
Note that, if I is a non-empty set and $S := I \times I$ endowed with the following operation $(x, y) \cdot (u, v) := (x, v)$, for all $x, y, u, v \in X$, i.e., $(S, \cdot)$ is a rectangular band, then $(S, \cdot, ^*)$ is a regular $*$-semigroup, with $(x, y)^* := (y, x)$, for all $x, y \in I$, but $(S, \cdot)$ it is not an inverse semigroup.

The variety of regular $*$-semigroups was considered in many papers and its important subvarieties were characterized, see e.g. [11], [12], [15], [21].
We point out that Reilly, in his short notes [23], fully described regular $*$-semigroups satisfying the identity $xyy^*x^* = xx^*$, introduced by Banaschewski in [1]. We will refer to them as *completely simple regular $*$-semigroups.*

Note that regular $*$ - semigroups are distinguished from * - regular semigroups in the sense of Drazin [13] and Nambooripad and Pastijn [19], as involution semigroups in which for each $a \in S$ there exists $x \in S$ such that

$$a = axa, \quad x = xax, \quad (ax)^* = ax, \quad (xa)^* = xa.$$

Drazin showed that in a *-regular semigroup such an x must be unique, so that it can be denoted by $a^\dagger$. The element $a^\dagger$ is often named the generalized (or Moore-Penrose) inverse of a.

Definition 1. Let S be a set with two operations $+$ and $\cdot$ such that $(S,+)$ is a semigroup (not necessarily commutative) and $(S,\cdot)$ a regular * – semigroup. Then, we say that $(S,+,\cdot,*)$ is a *left involution semi-brace* if

$$a(b+c) = ab + a(a^* + c) \tag{1.3}$$

holds, for all $a,b,c \in S$. We call $(S,+)$ and $(S,\cdot)$ the *additive semigroup* and the *multiplicative semigroup* of $(S,+,\cdot,^*)$, respectively.
A *right involution semi-brace* is defined similarly, by replacing condition (1.3) with $(a+b)c = (a+c^*)c + bc$, for all $a,b,c \in S$.
A two-sided involution semi-brace $(S,+,\cdot)$ is a left involution semi-brace that is also a right involution semi-brace with respect to the same operations $+$ and $\cdot$, and the same involution.

Any arbitrary regular $*$ – semigroup gives easily rise to an involution semi-brace, as we will show in the next example.

Example 1. If $(S,\cdot)$ is a regular *-semigroup and $(S,+)$ is a right zero semigroup or a left zero semigroup, then S is a two-sided involution semi-brace, which we call *trivial involution semi-braces*. Clearly, if $|S| > 1$, then such trivial involution semi-braces are not isomorphic.

Example 2. Let $(S,\cdot,*)$ be a regular $*$-semigroup, e an idempotent element of S and set $a+b = be$, for all $a,b \in S$. Then, it easy to check that S is a left involution semi-brace. Note that, if e is central, then $(S,+,\cdot,^*)$ is also a right involution semi-brace.

On the other hand, the numerous subvarieties allow one to obtain new examples.

Example 3. Let $(S,\cdot,^*)$ be a semigroup belonging to the variety of semilattice * - semigroups, that is the variety defined by identity $xx^*y = xx^*$ [11]. If we define $a+b = aa^*$, for all $a,b \in S$, then $(S,+,\cdot,^*)$ is an involution semi-brace.

Other examples of involution semi-braces can be obtained by using the well-known general construction of the *involutorial Plonka sum of algebra*, introduced in [22]. Here, we give the basic construction restricted to the case of involution semi-braces.

Theorem 1. *Let Y be a semilattice *–semigroup, $\{S_\alpha \mid \alpha \in Y\}$ a family of disjoint involution semi-braces and a bijection $*$ on $\bigcup\{S_\alpha \mid \alpha \in Y\}$. For each pair α,β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ be a homomorphism of involution semi-braces such that*

(1) $\phi_{\alpha,\alpha}$ is the identical automorphism of S_α, for every $\alpha \in Y$,

(2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, *for all* $\alpha, \beta, \gamma \in Y$ *such that* $\alpha \geq \beta \geq \gamma$,

(3) $^* : S_\alpha \longrightarrow S_{\alpha^*}$ *is an involution semi-braces anti-isomorphism, for any* $\alpha \in Y$,

(4) $\phi_{\alpha^*,\beta^*}(a) = (\phi_{\alpha,\beta}(a^*))^*$ *for all* $\alpha, \beta, \in Y$, $\alpha \geq \beta$ *and* $a \in S_{\alpha^*}$.

Then, $S = \bigcup\{S_\alpha \mid \alpha \in Y\}$ *endowed by the addition and the multiplication defined by*

$$a + b = \phi_{\alpha,\alpha\beta}(a) + \phi_{\beta,\alpha\beta}(a),$$

$$ab = \phi_{\alpha,\alpha\beta}(a)\phi_{\beta,\alpha\beta}(a),$$

for every $a \in S_\alpha$ *and* $b \in S_\beta$*, is an involution semi-brace.*
Such an involution semi-brace is said to be a strong semilattice S of involution semi-braces S_α *and is denoted by* $S = [Y;\, S_\alpha, \phi_{\alpha,\beta}]$.

2 Solutions associated to involution semi-braces

In this section, we deal with solutions associated to (left) involution semi-braces and we provide sufficient conditions to obtain them.

Let us note that if a is an element of S, the map $\lambda_a : S \longrightarrow S, \quad x \mapsto a(a^*+x)$ is an endomorphism of the semigroup $(S, +)$ and $\lambda_{ab}(x) = ab(ab)^* + \lambda_a\lambda_b(x)$, for all $a, b, x, y \in S$. Indeed,

$$\lambda_a(x + y) = a(a^* + x + y) = a(a^* + x) + a(a^* + y) = \lambda_a(x) + \lambda_a(y)$$

and

$$\begin{aligned} \lambda_{ab}(x) &= ab((ab)^* + x) \\ &= a(b(ab)^* + b(b^* + x) \\ &= ab(ab)^* + a(a^* + \lambda_b(x)) \\ &= ab(ab)^* + \lambda_a\lambda_b(x). \end{aligned}$$

If b is an element of S, we denote by ρ_b the map from S into itself defined by $\rho_b(x) = (x * +b)^*b$, for each $x \in S$.

Definition 2. Let $(S, +, \cdot, ^*)$ be an involution semi-brace. We call the map $r : S \times S \longrightarrow S \times S$ given by

$$r(a, b) := (\ a(a^* + b),\ (a^* + b)^*b\ ) \tag{2.4}$$

for all $a, b \in S$, the *map associated* to $(S, +, \cdot, ^*)$.
Following the notation introduced above, we can write $r(a,b) = (\lambda_a(b),\ \rho_b(a))$, for all $a, b \in S$.

Example 4. If $(S, +, \cdot, ^*)$ is the trivial involution semi-brace in Example 1 with $(S, +)$ a right zero semigroup, the map r associated to $(S, +, \cdot, ^*)$ given by $r(a,b) = (ab, b^*b)$, for all $a, b \in S$, is an idempotent solution.
Similarly, if $(S, +, \cdot, ^*)$ is the trivial left involution semi-brace with $(S, +)$ left zero semigroup, the map r associated to $(S, +. \cdot, ^*)$ given by $r(a,b) = (aa^*, ab)$, for all $a, b \in S$, is an idempotent solution too.

Example 5. Let $(S, +, \cdot, ^*)$ be the involution semi-brace of Example 2 with $a + b := be$, for all $a, b \in S$, where e is an arbitrary idempotent of $(S, \cdot)$.
Then, the map r associated to $(S, +, \cdot, ^*)$ is given by $r(a,b) = (abe, e^*b^*b)$, for all $a, b \in S$.
If $(S, \cdot, ^*)$ is a completely simple regular *-semigroup, that is, a regular *–semigroup with identity $xyy^*x^* = xx^*$, then r is a solution. Indeed,

$$\begin{aligned}
r_1r_2r_1(a,b,c) &= (abee^*b^*bce,\ e^*(e^*b^*bce)^*e^*b^*bce,\ e^*c^*c) = (abce,\ e^*e, e^*c^*c),\\
r_2r_1r_2(a,b,c) &= (abce,\ e^*(bce)^*bce,\ e^*(e^*c^*c)^*e^*c^*c) = (abce, e^*e, e^*c^*cc^*c)\\
&= (abce,\ e^*e, e^*e^*c),
\end{aligned}$$

for all $a, b, c \in S$.

Example 6. Let $(S, +, \cdot, ^*)$ be the involution semi-brace of Example 3 with $a + b := aa^*$ and $aa^*b = aa^*$, for all $a, b \in S$. The map r associated to $(S, +, \cdot, ^*)$ given by $r(a,b) = (a,\ aa^*)$, for all $a, b \in S$ is a solution.

Now, our aim is to show that if $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$ is a strong semilattice of involution semi-braces such that every S_α has a solution r_α, for every $\alpha \in Y$, the map associated to S is a solution. This result is a consequence of a more general construction technique on solutions, obtained in [6, Theorem 4.1].

Lemma 1. *Let Y be a semilattice, let $\{(X_\alpha, r_\alpha) \mid \alpha \in Y\}$ be a family of dijoint solutions indexed by Y such that for each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there is a map $\phi_{\alpha,\beta} : X_\alpha \longrightarrow X_\beta$. Let X be the union*

$$X = \bigcup \{X_\alpha \mid \alpha \in Y\}$$

and $r : X \times X \longrightarrow X \times X$ the map defined by

$$r(x,y) = r_{\alpha\beta}(\phi_{\alpha,\alpha\beta}(x),\ \phi_{\beta,\alpha\beta}(y)),$$

for all $x \in X_\alpha$ and $y \in X_\beta$. Then (X, r) is a solution if the following conditions are satisfied:

(1) $\phi_{\alpha,\alpha}$ is the identity map of X_α, for every $\alpha \in Y$,

(2) $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$,

(3) $(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})r_\alpha = r_\beta(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})$, for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$.

We call the pair (X, r) a strong semilattice of solutions (X_α, r_α) indexed by Y.

Now, as a consequence of this lemma, we obtain the following result.

Theorem 2. *Let $S = [Y; S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of involution semi-braces. If S_α has r_α as a solution, for every $\alpha \in Y$, then the map r given by*

$$r(a, b) = r_{\alpha\beta}(\phi_{\alpha,\alpha\beta}(a), \phi_{\beta,\alpha\beta}(b)),$$

for all $a \in S_\alpha$ and $b \in S_\beta$, is a solution on S.

Proof. For any $\alpha \in Y$, let $r_\alpha : S_\alpha \times S_\alpha \longrightarrow S_\alpha \times S_\alpha$ be the solution associated to the involution semi-brace S_α, i.e., the map defined by $r_\alpha(a, b) = (a(a^* + b), (a^* + b)^* b)$, for all $a, b \in S_\alpha$. Since S is a strong semilattice of involution semi-braces, $\phi_{\alpha,\alpha}$ is the identical automorphism of S_α and $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$. Hence, the conditions (1) and (2) in Lemma 1 are satisfied. Moreover, let $\alpha, \beta \in Y$ such that $\alpha \geq \beta$. Since $\phi_{\alpha,\beta}$ is a homomorphism of involution semi-braces, for all $a, b \in S_\alpha$ it follows that

$$\begin{aligned}(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})r_\alpha(a, b) &= (\phi_{\alpha,\beta}(a(a^* + b), \phi_{\alpha,\beta}(a^* + b)^* b))\\ &= (\phi_{\alpha,\beta}(a)((\phi_{\alpha,\beta}(a))^* + \phi_{\alpha,\beta}(b)),\ ((\phi_{\alpha,\beta}(a))^* + \phi_{\alpha,\beta}(b))^* \phi_{\alpha,\beta}(b))\\ &= r_\beta(\phi_{\alpha,\beta}(a),\ \phi_{\alpha,\beta}(b))\\ &= r_\beta(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta})(a, b).\end{aligned}$$

Hence, the condition (3) in Lemma 1 holds. Therefore, we can consider the strong semilattice Y of solutions r_α, i.e., the map r defined by

$$r(a, b) = r_{\alpha\beta}(\phi_{\alpha,\alpha\beta}(a),\ \phi_{\beta,\alpha\beta}(b))$$

for all $a \in S_\alpha$ and $b \in S_\beta$. Finally, we note that

$$\begin{aligned}r(a, b) &= (\phi_{\alpha,\alpha\beta}(a)((\phi_{\alpha,\alpha\beta}(a))^* + \phi_{\alpha,\alpha\beta}(b)),\ ((\phi_{\alpha,\alpha\beta}(a))^* + \phi_{\alpha,\alpha\beta}(b))^* \phi_{\alpha,\alpha\beta}(b))\\ &= (a(a^* + b),\ (a^* + b)^* b),\end{aligned}$$

for all $a \in S_\alpha$ and $y \in S_\beta$. $\boxed{QED}$

Note that, if $(S,+,\cdot)$ is a left semi-brace with $(S,+)$ a left cancellative semigroup, then the map r is a solution (see [4, Theorem 9]). Let us recall that not every left semi-brace gives rise to solutions. In this context, in [5, Theorem 3] a characterization has been proved. For left inverse semi-braces, only sufficient conditions are provided to obtain solutions (see [7, Theorem 7]).

In the following, we provide sufficient conditions to obtain solutions through left involution semi-braces.

Theorem 3. *Let r be the map associated to a left involution semi-brace $(S,+,\cdot,*)$. If the following are satisfied*

(1) $(a+b)\,(a+b)^*\,(a+bc) = a+bc$,

(2) $\lambda_a(b)^* + \lambda_{\rho_b(a)}\,(c) = \lambda_a(b)^* + \lambda_{(a^*+b)^*}\lambda_b\,(c)$,

(3) $\rho_b\,(a)^* + c = (b^*+c)\left(\rho_{\lambda_b(c)}\,(a)^* + \rho_c\,(b)\right)$,

for all $a,b,c\in S$, then the map r is a solution.

Proof. It is a routine computation to verify that the map r associated to S given by $r(a,b) = (\lambda_a(b), \rho_b(a))$ is a solution if and only if they hold

$$\begin{aligned}
&\lambda_a\lambda_b(c) = \lambda_{\lambda_a(b)}\lambda_{\rho_b(a)}\,(c)\\
&\lambda_{\rho_{\lambda_b(c)}(a)}\rho_c\,(b) = \rho_{\lambda_{\rho_b(a)}(c)}\lambda_a\,(b)\\
&\rho_c\rho_b(a) = \rho_{\rho_c(b)}\rho_{\lambda_b(c)}\,(a)\,,
\end{aligned}$$

for all $a,b,c\in S$. Thus, if $a,b,c\in S$, we have that

$$\begin{aligned}
\lambda_{\lambda_a(b)}\lambda_{\rho_b(a)}\,(c) &= \lambda_a\,(b)\left(\lambda_a\,(b)^* + \lambda_{\rho_b(a)}\,(c)\right)\\
&= \lambda_a\,(b)\left(\lambda_a\,(b)^* + \lambda_{(a^*+b)^*}\lambda_b\,(c)\right) && \text{by 2.}\\
&= \lambda_a\,(b)\left((a^*+b)^*\,a^* + (a^*+b)^*\,(a^*+b+\lambda_b\,(c))\right)\\
&= \lambda_a\,(b)\,(a^*+b)^*\,(a^*+\lambda_b\,(c)) && \text{by (1.3)}\\
&= a\,(a^*+b)\,(a^*+b)^*\,(a^*+b\,(b^*+c))\\
&= a\,(a^*+b\,(b^*+c)) && \text{by 1.}\\
&= a\,(a^*+\lambda_b\,(c)) = \lambda_a\lambda_b\,(c)\,.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
\lambda_{\rho_{\lambda_b(c)}(a)}\rho_c(b) &= \rho_{\lambda_b(c)}(a)\left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)\\
&= \left(a^* + \lambda_b(c)\right)^* \lambda_b(c)\left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)\\
&= \left(a^* + \lambda_b(c)\right)^* b\left(b^* + c\right)\left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)\\
&= \left(a^* + \lambda_b(c)\right)^* b\left(\rho_b(a)^* + c\right) && \text{by 3.}\\
&= \left(a^* + \lambda_b(c)\right)^* \left(a^* + b\right)\left(a^* + b\right)^* b\left(\rho_b(a)^* + c\right) && \text{by 1.}\\
&= \left(\left(a^* + b\right)\left(a^* + \lambda_b(c)\right)\right)^* \rho_b(a)\left(\rho_b(a)^* + c\right)\\
&= \left(\left(a^* + b\right)a^* + \lambda_{(a^*+b)^*}\lambda_b(c)\right)^* \lambda_{\rho_b(a)}(c) && \text{by (1.3)}\\
&= \left(\lambda_a(b)^* + \lambda_{(a^*+b)^*}\lambda_b(c)\right)^* \lambda_{\rho_b(a)}(c)\\
&= \left(\lambda_a(b)^* + \lambda_{\rho_b(a)}(c)\right)^* \lambda_{\rho_b(a)}(c) && \text{by 2.}\\
&= \rho_{\lambda_{\rho_b(a)}(c)}\lambda_a(b).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\rho_c\rho_b(a) &= \left(\rho_b(a)^* + c\right)^* c\\
&= \left(\left(b^* + c\right)\left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)\right)^* c && \text{by 3.}\\
&= \left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)^* \left(b^* + c\right)^* c\\
&= \left(\rho_{\lambda_b(c)}(a)^* + \rho_c(b)\right)^* \rho_c(b)\\
&= \rho_{\rho_c(b)}\rho_{\lambda_b(c)}(a).
\end{aligned}$$

Therefore, the map r is a solution on the involution semi-brace S. $\boxed{QED}$

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