Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **42** (2022) no. 2, 1–6. doi:10.1285/i15900932v42n2p1

Inequalities related to the S-Divergence

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Received: 18.3.2022; accepted: 28.10.2022.

Abstract. The S-Divergence is a distance like function on the convex cone of positive definite matrices, which is motivated from convex optimization. In this paper, we will prove some inequalities for Kubo-Ando means with respect to the square root of the S-Divergence.

Keywords: S-Divergence; Kubo-Ando means; positive definite matrices

MSC 2022 classification: primary 15A45, secondary 47B65

1 Introduction

Let $\mathbb{H}_n$ denote the set of all $n \times n$ Hermitian matrices. The set of all positive definite (henceforth *positive*) matrices in $\mathbb{H}_n$ is denoted by $\mathbb{P}_n$. The *Frobenius norm* of a matrix A is $\|A\|_F = \sqrt{\operatorname{tr}(A^*A)}$, while $\|A\|$ denoted the operator norm.

The set $\mathbb{P}_n$ is a well-studied differentiable Riemannian manifold, with the Riemannian metric given by the differential form $ds = \|A^{-1/2} dA A^{-1/2}\|_F$. The metric induces the *Riemannian distance* (for more information, one can see, e.g., [2, Chapter 6]):

$$\delta_R(A, B) := \| \log(B^{-1/2} A B^{-1/2}) \|_F, \quad \forall\, A, B > 0. \tag{1.1}$$

Motivated from convex optimization, one can define the *S-Divergence*:

$$\delta_S^2(A, B) = \log\det(\frac{A + B}{2}) - \frac{1}{2} \log\det(AB), \ \forall\, A, B > 0. \tag{1.2}$$

[i]This work is partially supported by NSF of China (12171251)

Sra exhibited several properties related to the Riemannian distance δ_R (see [20]). Note that the S-divergence δ_S^2 is non-negative definite and symmetric, but not a *metric*. Indeed, Sra prove that δ_S is a metric on $\mathbb{P}_n$ (see [20, Theorem 3.1]).

Note that the equality $\log\det A = \operatorname{Tr}\log A$ holds for all $A \in \mathbb{P}_n$, by the argument of [16, p.28], we have that

$$\begin{aligned}\delta_S^2(A,B) &= \log\det\left(\frac{A^{-1/2}BA^{-1/2}+I}{2}\right) - \frac{1}{2}\log\det(A^{-1/2}BA^{-1/2}) \\ &= \operatorname{Tr}[\log\left(\frac{A^{-1/2}BA^{-1/2}+I}{2}\right) - \log(A^{-1/2}BA^{-1/2})^{1/2}]. \qquad (1.3)\end{aligned}$$

It follows that for any $\lambda > 0$, we have that $\delta_S(\lambda A, \lambda B) = \delta_S(A,B)$.

Many authors consider the inequalities related to the various means (see [4, 9, 11, 12, 13]). In this paper, we will work on this problem and prove some inequalities related to the geometric mean, spectral geometric mean and Wasserstein mean under the S-divergence.

2 Inequalities related to various means

In this section, we will prove some inequalities related to some Kubo-Ando means. For positive matrices A and B, recall that the *geometric mean* $A\sharp B$ is defined by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The geometric mean has a lot of attractive properties (see, e.g., [1, 14]). In the following theorem, we list the properties of the S-divergence used in the paper (see See [20, Theorem 4.1, Theorem 4.5 and Corollary 4.10]).

Theorem 1. *δ_S has the following properties:*

(i) $A\sharp B$ is the equidistant from A and B, that is,

$$\delta_S(A, A\sharp B) = \delta_S(B, A\sharp B).$$

(ii) If A, B are positive definite and $t \in [0,1]$, we have that

$$\delta_S^2(A^t, B^t) \le t\delta_S^2(A,B).$$

(iii) If X, Y are positive definite and A is positive semidefinite, $\beta = \lambda_{min}(A)$, then

$$\delta_S^2(A+X, A+Y) \le \delta_S^2(\beta I + X, \beta I + Y) \le \delta_S^2(X,Y).$$

Suppose that $t \in [0,1]$, then one can define the *Wasserstein mean* of $A, B \in \mathbb{P}_n$ by

$$\begin{aligned} A \diamond_t B &= (1-t)^2 A + t^2 B + t(1-t)[A^{1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2} \\ &\quad + A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{1/2}] \\ &= (1-t)^2 A + t^2 B + t(1-t)[(AB)^{1/2} + (BA)^{1/2}] \\ &= A^{-1/2}[(1-t)A + t(A^{1/2}BA^{1/2})^{1/2}]^2 A^{-1/2}. \end{aligned}$$

Bhatia, Jain and Lim [3, p.180] proved that $A \diamond_t B$ is the natural parametrisation of the geodesic joining A and B associated Riemannian distance

$$\langle Y, Z \rangle_A = \sum_{i,j} \alpha_i \frac{\mathrm{Re}\overline{y_{ji}} z_{ji}}{(\alpha_i + \alpha_j)^2},$$

where $A = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a positive definite matrix.

Theorem 2. *For any $A, B \in \mathbb{P}_n$ and any $t \in (0,1)$, we have that*

$$\delta_S^2(A, A \diamond_t B) \geq 2\delta_S^2(I, (1-t)I + tA^{-1} \sharp B).$$

Proof. Let $C = A^{1/2}BA^{1/2}$. By Theorem 1, we can derive that

$$\begin{aligned} & \delta_S^2(A, A \diamond_t B) \\ = \ & \delta_S^2(A^2, [(1-t)A + t(A^{1/2}BA^{1/2})^{1/2}]^2) \\ \geq \ & 2\delta_S^2(A, (1-t)A + t(A^{1/2}BA^{1/2})^{1/2}) \\ = \ & 2\delta_S^2(I, (1-t)I + tA^{-1} \sharp B). \end{aligned}$$

QED

Remark 1. For A and B, when put $C = A^{1/2}BA^{1/2}$, we just can prove that

$$\begin{aligned} & \delta_S^2(B, A \diamond_t B) \\ = \ & \delta_S^2(C, ((1-t)A + tC^{1/2})^2) \\ = \ & 2\delta_S^2(C^{1/2}, (1-t)A + tC^{1/2}). \end{aligned}$$

Moreover, one can define the *spectral geometric mean* between positive matrices A and B:

$$A \natural B = (A^{-1} \sharp B)^{1/2} A (A^{-1} \sharp B)^{1/2}$$

(we refer [14] for more details). It is easy to see that $\delta_S^2(A^{-1} \sharp B, A \natural B) = \delta_S^2(I, A)$.

Proposition 1. *For any positive matrices A and B, we have that*

$$\delta_S^2(I, A\natural B) \leq \frac{1}{2}\delta_S^2(B, A^{-1}).$$

Proof. By the definition, one can derive that

$$\begin{aligned}
\delta_S^2(I, A\natural B) &= \delta_S^2((A^{-1}\sharp B)^{-1}, A) = \delta_S^2(A^{-1}\sharp B, A^{-1}) \\
&= \delta_S^2((A^{1/2}BA^{1/2})^{1/2}, I) \\
&\leq \frac{1}{2}\delta_S^2(A^{1/2}BA^{1/2}, I) \\
&= \frac{1}{2}\delta_S^2(B, A^{-1}).
\end{aligned}$$

QED

More generally, one can define weighted spectral geometric mean for $0 \leq t \leq 1$. See, e.g., [15]. Let A, B be positive matrices, the *weighted spectral geometric mean* is defined by

$$A\natural_t B = (A^{-1}\sharp B)^t A (A^{-1}\sharp B)^t.$$

By the definition, it is easy to prove the following properties:

Lemma 1. *For any $s, t \in [0, 1]$ and any positive matrices A, B, we have that*

$$\delta_S^2(A\natural_s B, A\natural_t B) = \delta_S^2(A, A\natural_{t-s} B).$$

When $1/2 < t < 1$, we have

$$\begin{aligned}
&\delta_S^2(A^{-1}\sharp B, A\natural_t B) \\
=\ &\delta_S^2(I, (A^{-1}\sharp B)^{t-1/2} A (A^{-1}\sharp B)^{t-1/2}) \\
=\ &\delta_S^2(I, A\natural_{t-1/2} B).
\end{aligned}$$

On the other hand, to give a universal estimate, we can prove the following inequality.

Theorem 3. *If $t \neq 1/2$, for any positive matrices A, B, we have*

$$\delta_S^2(A^{-1}\sharp B, A\natural_t B) \leq \frac{|1-2t|}{2}\delta_S^2(B, A^{(3-2t)/(1-2t)}).$$

Proof. When $0 < t < 1/2$, it follows from the properties of S-divergence δ_S that

$$\begin{aligned} & \delta_S^2(A^{-1}\sharp B, A\natural_t B) \\ = & \delta_S^2((A^{-1}\sharp B)^{1-2t}, A) \\ \leq & (1-2t)\delta_S^2(A^{-1}\sharp B, A^{1/(1-2t)}) \\ = & (1-2t)\delta_S^2((A^{1/2}BA^{1/2})^{1/2}, A^{(2-2t)/(1-2t)}) \\ \leq & \frac{1-2t}{2}\delta_S^2(A^{1/2}BA^{1/2}, A^{(4-4t)/(1-2t)}) \\ = & \frac{1-2t}{2}\delta_S^2(B, A^{(3-2t)/(1-2t)}). \end{aligned}$$

When $1/2 < t < 1$, by a similar argument, we have that

$$\delta_S^2(A^{-1}\sharp B, A\natural_t B) \leq \frac{2t-1}{2}\delta_S^2(B, A^{(3-2t)/(1-2t)}).$$

QED

Remark 2. We also can derive that

$$\delta_S^2(A^{-1}\sharp B, A\natural_t B) = \delta_S^2((A^{-1}\sharp B)^{1-2t}, A).$$

Remark 3. Note that $A\natural_t B$ is the solution of the equation $(A^{-1}\sharp B)^t = A^{-1}\sharp X$, then we have that

$$\begin{aligned} & \delta_S^2(A, A\natural_t B) \\ = & \delta_S^2(A^{1/2}AA^{1/2}, A^{1/2}(A\natural_t B)A^{1/2}) \\ \geq & 2\delta_S^2(A, (A^{1/2}(A\natural_t B)A^{1/2})^{1/2}) \\ = & 2\delta_S^2(I, A^{-1}\sharp(A\natural_t B)) \\ = & 2\delta_S^2(I, (A^{-1}\sharp B)^t). \end{aligned}$$

Acknowledgement

We would like to express my appreciation to the referee for his/her kind efforts and valuable comments.

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