Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **41** (2021) no. 2, 31–60. doi:10.1285/i15900932v41n2p31

Tensor product semigroups on locally convex spaces

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Received: 30.1.2021; accepted: 19.8.2021.

Abstract. In this paper, we introduce tensor product semigroups of operators on locally convex spaces. The basic properties are presented. We give multiple relations between the tensor product semigroups and its components. The generator of such semigroups is studied.

Keywords: Semigroups of operators, tensor product, locally convex spaces.

MSC 2020 classification: primary 47D03, secondary 47D99

1 Introduction

The aim of this paper is to study a special class of two-parameter semigroups which is the tensor product semigroups in the case of tensor product of two locally convex spaces.

Let E be a locally convex space, a two-parameter family $(T(s,t))_{s,t\geq 0}$ of bounded linear operators in $\mathcal{L}(E)$ is called a two-parameter semigroup of bounded linear operators on E, if it satisfies the following conditions :

(1) $T(0,0) = I$ (I is the identity operator on E)

(2) $T((s_1,t_1)+(s_2,t_2)) = T(s_1,t_1)\,T(s_2,t_2)$ for all $s_1,\ s_2,\ t_1\ ,t_2 \geq 0$

The theory of two-parameter semigroups on Banach spaces is initially studied by E. Hille, R.S. Phillips [6] and N. H. Abdelaziz [1]. In 1992, S. C. Arora and S. Sharma [5] gave a definition of the infinitesimal generator of two-parameter semigroups on Banach spaces. It turns out that this definition does not work but only for one-parameter semigroups. This last point urged S. Al-Sharif and R. Khalil [3] in 2004 to give a definition of the infinitesimal generator of two-parameter semigroup $(T(s,t))_{s,t\geq 0}$ similar to the case of one-parameter semigroup, i.e. in terms of derivative at $(0,0)$.

The tensor product of semigroups $\left(T(t)\,\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ was first introduced by R. Nagel and U, Schlotterbeck [4]. As a one-parameter C_0-semigroup, he showed that its infinitesimal generator is $\overline{A_1 \otimes I + I \otimes A_2}$ where A_1 and A_2 are the infinitesimal generators of the one-parameter C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ respectively.

In 2010, this concept was generalized to the case of two parameters by R. Khalil, R . Al-Mirbati and D. Drissi in their paper [8]. The tensor product semigroups $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$, this time as a semigroup with two parameters, they have defined its infinitesimal generator being the derivative of $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ as a vector valued function of two variable at $(s,t)=(0,0)$. Also they were able to generalize Rainer Nagel's result by showing that for all $a,\ b \geq 0$

$$\overline{aA_1 \otimes I + bI \otimes A_2} = a\left(\overline{A_1 \otimes I}\right) + b\left(\overline{I \otimes A_2}\right)$$

In the following article, we study the tensor product semigroups on locally convex spaces. The section 2 contains a preliminary on the topology of the tensor product of locally convex spaces.

The section 3 deals with the algebraic side of tensor product semigroups, at the end of this section, we will conclude that, in the following, it is sufficient to study the tensor product of semigroups.

The main results of section 4 are Theorems 2 and 3, in which we have proved multiple relation of continuity and equicontinuity between the tensor product semigroups and its components, we will show that $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous (resp. exponentially equicontinuous) if and only if $\left(T(s)\,\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ are locally equicontinuous (resp. exponentially equicontinuous) if and only if $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are locally equicontinuous (resp. exponentially equicontinuous).

In section 5 we will deal with the definition of the infinitesimal generator of tensor product semigroups on locally convex spaces, we will give a new definition of the infinitesimal generator in terms of semi-differentiability which will correct some ambiguity in [8], and will allow to study and develop several properties of a locally equicontinuous (exponentially equicontinuous) tensor product semigroups.

2 Preliminaries

Let X and Y be locally convex Hausdorff spaces. And let $X \otimes Y$ denote their algebraic tensor product. A semi-norm p on $X \otimes Y$ is called cross-semi-norm, provided there exists continuous seminorms p and q on X and Y such that $p(x \otimes y) = p(x)\, q(y)$ for every $x \otimes y \in X \otimes Y$. We define the two main topologies on the algebraic tensor product $X \otimes Y$ of X and Y.

Let Γ_X (resp Γ_Y) be a family of continuous semi-norms generating the topology on X (resp.on Y).

For semi-norms $p \in \Gamma_X$ and $q \in \Gamma_Y$ on X and Y respectively, let $p \otimes_\pi q$ denote the cross-semi-norm on $X \otimes Y$ defined by

$$p \otimes_\pi q(z) = \inf\left\{ \sum_{i=1}^{n} p(x_i)\, q(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i \ ,\ x_i \in X, y_i \in Y, n \in \mathbb{N} \right\},$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. The projective tensor topology on $X \otimes Y$ is defined by the family of seminorms $\{p \otimes_\pi q \mid p \in \Gamma_X \text{ and } q \in \Gamma_Y\}$, and we have

$$(\forall x \otimes y \in X \otimes Y)\ p \otimes_\pi q(x \otimes y) = p(x)\, q(y).$$

Equipped with this family, the space $X \otimes Y$ is called the projective tensor product and denoted by $X \otimes_\pi Y$. The space $X \hat{\otimes}_\pi Y$ is the completion of $X \otimes_\pi Y$.

Further, for seminorms $p \in \Gamma_X$ and $q \in \Gamma_Y$ on X and Y respectively we can define the following seminorm on $X \otimes Y$

$$p \otimes_\varepsilon q(z) = \sup\left\{ \left| \sum_{i=1}^{n} \langle x_i, \varphi\rangle \langle y_i, \phi\rangle \right| \mid z = \sum_{i=1}^{n} x_i \otimes y_i, \varphi \in U_p^\circ \text{ and } \phi \in U_q^\circ \right\}$$

where U_p° is the polar of the zero-neighborhood $U_p = \{x \in X \mid p(x) \leq 1\}$. The injective tensor topology is generated by the family of seminorms

$$\{p \otimes_\varepsilon q \mid p \in \Gamma_X \text{ and } q \in \Gamma_Y\},$$

and we have

$$(\forall x \otimes y \in X \otimes Y)\ p \otimes_\varepsilon q(x \otimes y) = p(x)\, q(y),$$

so $p \otimes_\varepsilon q$ is also a cross-semi-norm.

The injective tensor product $X \otimes_\varepsilon Y$ is the vector space $X \otimes Y$ endowed with this topology. The completion of $X \otimes_\varepsilon Y$ is denoted by $X \hat{\otimes}_\varepsilon Y$.

Next we will denote the injective ε and the projective topology π by α, also we will denote the corresponding generating family of seminorms

$$\Gamma_{X \otimes_\alpha Y} = \{p \otimes_\alpha q \mid p \in \Gamma_X \text{ and } q \in \Gamma_Y\}.$$

For more details on tensor product spaces and tensor product of operators, we refer to [13], [10].

We also note that seminorms on $X \hat{\otimes}_\alpha Y$ are the continuous extention of seminorms on $X \otimes_\alpha Y$ [7].

The following Proposition is necessary for the following, it also a generalization of the definition of compatible locally convex topology with the tensor product [10], [13].

Proposition 1. *Let X and Y two locally convex Hausdorff spaces. For any equicontinuous subsets $A \subset \mathcal{L}(X)$ and $B \subset \mathcal{L}(Y)$, the set $A \otimes B$ defined by*

$$A \otimes B = \{T \otimes S \mid T \in A \text{ and } S \in B\}$$

is an equicontinuous subset of $\mathcal{L}(X \otimes_\alpha Y)$.

Proof. Let $\mathcal{U} = (U_i)_{i \in I}$, and $\mathcal{V} = (V_j)_{j \in J}$ local bases of X and Y.

For the topology π we have that $(\Lambda(U_i \otimes V_j))_{(i,j) \in I \times J}$ defines a local basis of the locally convex space $X \otimes_\pi Y$ [10], $\Lambda(U_i \otimes V_j)$ the balanced and convex hull of the set $U_i \otimes V_j \subseteq X \otimes_\pi Y$.

If $\Lambda(U_1 \otimes V_1)$ is an neighborhood of zero in $X \otimes_\pi Y$ where $U_1 \in \mathcal{U}$ and $V_1 \in \mathcal{V}$, we have that A and B are equicontinuous subsets. Then there exist $U_2 \in \mathcal{U}$ and $V_2 \in \mathcal{V}$ so that for all $T \in A$ and $S \in B$ we have $T(U_2) \subseteq U_1$ and $S(V_2) \subseteq V_1$. Let $z \in \Lambda(U_2 \otimes V_2)$ then $z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ where $x_i \in U_2$, $y_i \in V_2$ and $\sum_{i=1}^{n} |\lambda_i| \leq 1$. therefore

$$T \otimes S(z) = \sum_{i=1}^{n} \lambda_i T x_i \otimes S y_i \in \Lambda(U_1 \otimes V_1).$$

Consequently $T \otimes S(\Lambda(U_2 \otimes V_2)) \subseteq \Lambda(U_1 \otimes V_1)$.

On the other hand, for the topology ε we have that $\left(U_i^\circ \otimes V_j^\circ\right)^\circ_{(i,j) \in I \times J}$ defines a local basis of the locally convex space $X \otimes_\varepsilon Y$ [10]. If $(U_1^\circ \otimes V_1^\circ)^\circ$ is a neighborhood of zero in $X \otimes_\varepsilon Y$ where $U_1 \in \mathcal{U}$ and $V_1 \in \mathcal{V}$. We have that A and

B are equicontinuous subsets, then there exist $U_2 \in \mathcal{U}$ and $V_2 \in \mathcal{V}$ so that for all $T \in A$ and $S \in B$ we have $T(U_2) \subseteq U_1$ and $S(V_2) \subseteq V_1$.

Let $x \in U_2$ for all $T \in A$ we have that $Tx \in U_1$. Hence $|\varphi \circ Tx| \leq 1$ for all $\varphi \in U_1^\circ$, therefore

$$\varphi \circ T \in U_2^\circ \text{ for all } \varphi \in U_1^\circ \text{ and } T \in A.$$

With the same procedure, we get

$$\psi \circ S \in V_2^\circ \text{ for all } \psi \in V_1^\circ \text{ and } S \in B.$$

Let $z = \sum\limits_{i=0}^{n} x_i \otimes y_i \in (U_2^\circ \otimes V_2^\circ)^\circ$, then $|\langle \varphi \circ T \otimes \psi \circ S, z\rangle| \leq 1$ but we have

$$|\langle \varphi \otimes \psi, T \otimes Sz\rangle| = \left|\sum_{i=1}^{n} \varphi \circ T(x_i)\, \psi \circ S(y_i)\right| = |\langle \varphi \circ T \otimes \psi \circ S, z\rangle| \leq 1$$

for all $\varphi \in U_1^\circ$ and $\psi \in V_1^\circ$. Consequently $T \otimes S((U_2^\circ \otimes V_2^\circ)^\circ) \subseteq (U_1^\circ \otimes V_1^\circ)^\circ$. QED

3 Tensor product semigroup

Definition 1. Let X and Y be two locally convex Hausdorff spaces. Let $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ be two families of operators of one parameter in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively. Define a two-parameter semigroup as a vector-valued function of two variables : $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathcal{L}\left(X\hat{\otimes}_\alpha Y\right)$, by $F(s,t) = T(s)\,\hat{\otimes}_\alpha S(t)$ where

$$\forall (x,y) \in X \times Y : T(s)\,\hat{\otimes}_\alpha S(t)\,(x \otimes y) = T(s)\,x \otimes_\alpha S(t)\,y$$

Then $(F(s,t))_{s,t\geq 0}$ is called a tensor product semigroup, (abbreviated T.P.S.) on the locally convex space $X\hat{\otimes}_\alpha Y$.

Lemma 1. *Let $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ be two families of operators of one parameter in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively. Then*

(1) $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is T.P.S. on $X \otimes_\alpha Y$ if and only if $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is T.P.S. on $X\hat{\otimes}_\alpha Y$.

(2) We have that $(T(s) \otimes I)_{s\geq 0}$ is a one-parameter semigroup on $X \otimes_\alpha Y$ if and only if $\left(T(s)\,\hat{\otimes}_\alpha I\right)_{s\geq 0}$ is a one-parameter semigroup on $X\hat{\otimes}_\alpha Y$.

Lemma 2. *Let X be a locally convex Hausdorff space and $(T(s))_{s\geq 0}$ be a one-parameter family of operators in $\mathcal{L}(X)$. Then the following are equivalent :*

(1) $(T(s))_{s\geq 0}$ *is a one-parameter semigroup on* X.

(2) $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ *is a one-parameter semigroup on* $X\hat{\otimes}_\alpha Y$.

(3) $\left(I\hat{\otimes}_\alpha T(s)\right)_{s\geq 0}$ *is a one-parameter semigroup on* $Y\hat{\otimes}_\alpha X$.

Proof. (1) $\Rightarrow$ (2) trivial.
(2) $\Rightarrow$ (3) Let

$$\begin{aligned} u : X \otimes Y &\to Y \otimes X \\ \sum_{i=1}^{n} x_i \otimes y_i &\mapsto \sum_{i=1}^{n} y_i \otimes x_i, \end{aligned}$$

so u is an isomorphism. Hence $\left(u\left(T(s)\hat{\otimes}_\alpha I\right)\right)_{s\geq 0}$ is a one-parameter semigroup on $Y\hat{\otimes}_\alpha X$. Let $z = \sum_{i=0}^{n} y_i \otimes x_i \in Y \otimes_\alpha X$, then we have

$$\begin{aligned} I \otimes T(0)\, z &= \sum_{i=1}^{n} y_i \otimes T(0)\, x_i = u\left(\sum_{i=1}^{n} T(0)\, x_i \otimes y_i\right) \\ &= u\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = z. \end{aligned}$$

On the other hand, for $s_1,\ t_1,\ s_2,\ t_2 \geq 0$ and $z = \sum_{i=0}^{n} y_i \otimes x_i \in Y \otimes_\alpha X$, we have

$$\begin{aligned} [I \otimes T(s_1 + s_2)]\, z &= \sum_{i=1}^{n} y_i \otimes T(s_1 + s_2)\, x_i = u\left[\sum_{i=1}^{n} T(s_1 + s_2)\, x_i \otimes y_i\right] \\ &= u\left[\sum_{i=1}^{n} (T(s_1) \otimes I)(T(s_2) \otimes I)\, x_i \otimes y_i\right] \\ &= u\left[\sum_{i=1}^{n} T(s_1)\, T(s_2)\, x_i \otimes y_i\right] \\ &= \sum_{i=1}^{n} y_i \otimes T(s_1)\, T(s_2)\, x_i \\ &= [I \otimes T(s_1)]\, [I \otimes T(s_2)]\, z. \end{aligned}$$

(3) $\Rightarrow$ (1) Let $x \in X$ and $y \in Y$ such that $y \neq 0$, so we have :

$$\begin{aligned}(T(0)x - x) \otimes y &= T(0)x \otimes y - x \otimes y \\ &= u[y \otimes T(0)x] - x \otimes y \\ &= 0,\end{aligned}$$

which implies that

$$\varphi(T(0)x - x)y = 0$$

for all $\varphi \in X'$, therefore $T(0)x = x$.

On the other hand, we have for $s_1,\ t_1,\ s_2,\ t_2 \geq 0$ and $x \otimes y \in X \otimes_\alpha Y$ such that $y \neq 0$:

$$\begin{aligned}(T(s_1+s_2)x - T(s_1)T(s_2)x) \otimes y &= T(s_1+s_2)x \otimes y - T(s_1)T(s_2)x \otimes y \\ &= u[y \otimes T(s_1+s_2)x] - T(s_1)T(s_2)x \otimes y \\ &= u[y \otimes T(s_1)T(s_2)x] - T(s_1)T(s_2)x \otimes y \\ &= 0,\end{aligned}$$

we obtain that

$$\varphi(T(s_1+s_2)x - T(s_1)T(s_2)x)y = 0$$

for all $\varphi \in X'$, therefore $T(s_1+s_2)x = T(s_1)T(s_2)x$. QED

Example 1. As an example, if $(T(s))_{s\geq 0}$, $(S(t))_{t\geq 0}$ are one-parameter semigroups of operators in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively.

Then the family $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. on $X\hat{\otimes}_\alpha Y$ called tensor product of semigroups.

Proof. We have by a previous Lemma that $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ are one-parameter semigroups on $X\hat{\otimes}_\alpha Y$ and $Y\hat{\otimes}_\alpha Y$ respectively. and we have for all $s, t \geq 0$

$$T(s)\hat{\otimes}_\alpha S(t) = \left(T(s)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t)\right) = \left(I\hat{\otimes}_\alpha S(t)\right)\left(T(s)\hat{\otimes}_\alpha I\right)$$

so for all $s_1,\ s_2,\ t_1,\ t_2 \geq 0$

$$\begin{aligned}T(s_1+s_2)\hat{\otimes}_\alpha S(t_1+t_2) &= \left(T(s_1+s_2)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t_1+t_2)\right) \\ &= \left(T(s_1)\hat{\otimes}_\alpha I\right)\left(T(s_2)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t_1)\right)\left(I\hat{\otimes}_\alpha S(t_2)\right) \\ &= \left(T(s_1)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t_1)\right)\left(T(s_2)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t_2)\right) \\ &= \left(T(s_1)\hat{\otimes}_\alpha S(t_1)\right)\left(T(s_2)\hat{\otimes}_\alpha S(t_2)\right).\end{aligned}$$

QED

Lemma 3. *Let X and Y be two locally convex Hausdorff spaces, let $a,\ c \in X$ and $b,\ d \in Y$ be a nonzero vectors.*

If $a \otimes b = c \otimes d$, then there exists $\beta \in \mathbb{R}^*$ such that:

$$a = \beta c \text{ and } b = \frac{1}{\beta} d.$$

Proof. We have $a \otimes b = c \otimes d$, then

$$\varphi(a)\, b = \varphi(c)\, d$$

for all $\varphi \in X'$, in particular for a $\psi \in X'$ such that $\psi(c) \neq 0$, that is : $\frac{\psi(a)}{\psi(c)} b = d$. It is clear that $\psi(a) \neq 0$, so we choose $\beta = \frac{\psi(a)}{\psi(c)}$. On the other hand, we have :

$$a \otimes b = c \otimes \beta b$$

Thus $(a - \beta c) \otimes b = 0$, then

$$\varphi(a - \beta c)\, b = 0$$

for all $\varphi \in X'$, consequently $a = \beta c$. QED

Theorem 1. *Let X and Y be two locally convex Hausdorff spaces and $(T(s))_{s\geq 0}$, $(S(t))_{t\geq 0}$ be two one-parameter families of operators in $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively. Then the family $\left(T(s)\, \hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. on $X \hat{\otimes}_\alpha Y$ if and only if there is a unique $\beta \in \mathbb{R}^*$, and a unique one-parameter semigroups $\left(\tilde{T}(s)\right)_{s\geq 0}$, $\left(\tilde{S}(t)\right)_{t\geq 0}$ on X, Y respectively, such that :*

$$\tilde{T}(s) = \beta T(s) \text{ and } \tilde{S}(t) = \frac{1}{\beta} S(t) \text{ for all } s,t \geq 0.$$

Proof. If $\beta = 1$, then $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ define one-parameter semigroups, therefore $\left(T(s)\, \hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. on $X \hat{\otimes}_\alpha Y$.

If $\beta \neq 1$, then $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are not semigroups of operators, since $T(0) = \frac{1}{\beta}\tilde{T}(0) = \frac{1}{\beta} I \neq I$ and $S(0) = \beta \tilde{S}(0) = \beta I \neq I$, even though $\left(T(s)\, \hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. $X \hat{\otimes}_\alpha Y$,

because :

$$T(0) \otimes S(0) = \frac{1}{\beta} I \otimes \beta I = I \otimes I,$$

and for all $s_1,\ s_2,\ t_1,\ t_2 \geq 0$

$$\begin{aligned} T(s_1+s_2)\otimes S(t_1+t_2) &= \frac{1}{\beta}\tilde{T}(s_1+s_2)\otimes \beta\tilde{S}(t_1+t_2) \\ &= \tilde{T}(s_1)\tilde{T}(s_2)\otimes \tilde{S}(t_1)\tilde{S}(t_2) \\ &= \left[\tilde{T}(s_1)\otimes \tilde{S}(t_1)\right]\left[\tilde{T}(s_2)\otimes \tilde{S}(t_2)\right] \\ &= \left[T(s_1)\otimes S(t_1)\right]\left[T(s_2)\otimes S(t_2)\right]. \end{aligned}$$

Consequently $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. on $X\hat{\otimes}_\alpha Y$.

To show the inverse, *we* suppose that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a T.P.S. on $X\hat{\otimes}_\alpha Y$. Then $T(0)\otimes_\alpha S(0) = I\otimes_\alpha I$,

and according to the previous Lemma, there exist $\gamma\in\mathbb{R}^*$ such that $T(0) = \gamma I$ and $S(0) = \frac{1}{\gamma}I$.

Define the families $\left(\tilde{T}(s)\right)_{s\geq 0}$ and $\left(\tilde{S}(t)\right)_{t\geq 0}$ from $\mathbb{R}^+$ to $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ respectively by $\tilde{T}(s) = \frac{1}{\gamma}T(s)$, and $\tilde{S}(t) = \gamma S(t)$, $s,\ t\geq 0$.

We show that $\left(\tilde{T}(s)\right)_{s\geq 0}$ and $\left(\tilde{S}(t)\right)_{t\geq 0}$ define two one-parameter semigroups.

We have : $\tilde{T}(0) = \frac{1}{\gamma}T(0) = \frac{1}{\gamma}\gamma I = I$ et $\tilde{S}(0) = \gamma S(0) = \gamma\frac{1}{\gamma}I = I$. On the other hand let $s_1,\ s_2\in\mathbb{R}^+$ and $x\in X$. Then for any $y\in Y\backslash\{0\}$ there exist a continuous seminorm q on Y such that $q(y)\neq 0$. Hence, for any seminorm p on X:

$$\begin{aligned} & p\left(\tilde{T}(s_1+s_2)x - \tilde{T}(s_1)\tilde{T}(s_2)x\right) \\ &= \frac{1}{q(y)}p\otimes_\alpha q\left[\left(\tilde{T}(s_1+s_2)x - \tilde{T}(s_1)\tilde{T}(s_2)x\right)\otimes y\right] \\ &= \frac{1}{q(y)}p\otimes_\alpha q\left[\left(\tilde{T}(s_1+s_2)\otimes I - \tilde{T}(s_1)\tilde{T}(s_2)\otimes I\right)(x\otimes y)\right] \\ &= \frac{1}{q(y)}p\otimes_\alpha q\left[\left(\frac{1}{\gamma}T(s_1+s_2)\otimes\gamma S(0) - \tilde{T}(s_1)\tilde{T}(s_2)\otimes I\right)(x\otimes y)\right] \\ &= \frac{1}{q(y)}p\otimes_\alpha q\left[\left(T(s_1+s_2)\otimes S(0) - \tilde{T}(s_1)\tilde{T}(s_2)\otimes I\right)(x\otimes y)\right] \\ &= \frac{1}{q(y)}p\otimes_\alpha q\left[\left(\tilde{T}(s_1)\tilde{T}(s_2)\otimes I - \tilde{T}(s_1)\tilde{T}(s_2)\otimes I\right)(x\otimes y)\right] \\ &= 0, \end{aligned}$$

therefore

$$\tilde{T}(s_1+s_2) = \tilde{T}(s_1)\tilde{T}(s_2) \text{ for all } s_1,\ s_2\geq 0$$

Similarly $\left(\tilde{S}(t)\right)_{t\geq 0}$ satisfies the semigroup property.

Hence $\left(\tilde{T}(s)\right)_{s\geq 0}$ and $\left(\tilde{S}(t)\right)_{t\geq 0}$ are two one-parameter semigroups on X and Y respectively. $\boxed{QED}$

Remark 1. Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a T.P.S. on $X\hat{\otimes}_\alpha Y$, then there exists $\beta \in \mathbb{R}^*$ and two one-parameter semigroups $\left(\tilde{T}(s)\right)_{s\geq 0}$, $\left(\tilde{S}(t)\right)_{t\geq 0}$ on X and Y respectively such that :

$$\tilde{T}(s) = \beta T(s) \text{ and } \tilde{S}(t) = \frac{1}{\beta} S(t) \text{ for all } s,\, t \geq 0,$$

therefore

$$T(s)\hat{\otimes}_\alpha S(t) = \tilde{T}(s)\hat{\otimes}_\alpha \tilde{S}(t),$$

for all $s,\, t \in \mathbb{R}^+$, then it is enough to study $\left(\tilde{T}(s)\hat{\otimes}_\alpha \tilde{S}(t)\right)_{s,t\geq 0}$.

In the following, we will note the tensor product of semigroup $\left(\tilde{T}(s)\hat{\otimes}_\alpha \tilde{S}(t)\right)_{s,t\geq 0}$ by $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$.

4 Continuity and equicontinuity of the T.P.S. and its components

Definition 2. Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a T.P.S. on the locally convex space $X\hat{\otimes}_\alpha Y$ then :

(1) We say that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is strongly continuous if for all $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$ we have

$$\lim_{(s,t)\to(s_0,t_0)} p\hat{\otimes}_\alpha q\left(T(s)\hat{\otimes}_\alpha S(t)z - T(s_0)\hat{\otimes}_\alpha S(t_0)z\right) = 0$$

for all $z \in X\hat{\otimes}_\alpha Y$ and $s_0,\, t_0 \geq 0$, with $(s,t) \to (0^+, 0^+)$ if $s_0 = t_0 = 0$.

(2) We say that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is a C_0-T.P.S. on $X\hat{\otimes}_\alpha Y$ if for all $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$ we have

$$\lim_{(s,t)\to(0^+,0^+)} p\hat{\otimes}_\alpha q\left(T(s)\hat{\otimes}_\alpha S(t)z - I\hat{\otimes}_\alpha Iz\right) = 0$$

for all $z \in X\hat{\otimes}_\alpha Y$.

(3) We say that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is equicontinuous if for all $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$, there exist $p_1\hat{\otimes}_\alpha q_1 \in \Gamma_{X\hat{\otimes}_\alpha Y}$ and $M > 0$ such that

$$p\hat{\otimes}_\alpha q\left(T(s)\hat{\otimes}_\alpha S(t)\, z\right) \leq M p_1\hat{\otimes}_\alpha q_1(z)$$

for all $z \in X\hat{\otimes}_\alpha Y$.

(4) We say that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous if for $s_0, t_0 \geq 0$ fixed, the familly

$$\left\{T(s)\hat{\otimes}_\alpha S(t) \mid 0 \leq s \leq s_0\ ,\ 0 \leq t \leq t_0\right\}$$

is equicontinuous.

(5) We say that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is exponentially equicontinuous if there exists $a \geq 0$ such that $\left(e^{-a(s+t)}T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is equicontinuous.

Lemma 4. *Let $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ be a T.P.S. on $X\otimes_\alpha Y$, then the following are equivalent :*

(1) $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous.

(2) $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$ are strongly continuous.

(3) $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are strongly continuous.

Proof. (1) $\Rightarrow$ (2) Suppose that $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous, in particular for $(T(s)\otimes_\alpha S(0))_{s\geq 0}$ and $(T(0)\otimes_\alpha S(t))_{t\geq 0}$.

(2) $\Rightarrow$ (3) Let $y \in Y\setminus\{0\}$ then there exists a continuous seminorm $q \in \Gamma_Y$ such that $q(y) \neq 0$. For all continuous seminorm $p \in \Gamma_X$ and $x \in X$ we get

$$\lim_{s\to s_0} p(T(s)x - x) = \lim_{s\to s_0} \frac{1}{q(y)} p\otimes_\alpha q\left[(T(s)\otimes_\alpha I)(x\otimes y) - (x\otimes y)\right] = 0$$

for all $s_0 \geq 0$. Therefore $(T(s))_{s\geq 0}$ is strongly continuous, and the same for $(S(t))_{t\geq 0}$.

(3) $\Rightarrow$ (1) Let $p\otimes_\alpha q \in \Gamma_{X\otimes_\alpha Y}$ and $z = \sum\limits_{i=0}^{n} x_i\otimes y_i \in X\otimes_\alpha Y$, we have for s,

$t,\ s_0,\ t_0 \geq 0$:

$$\begin{aligned}
&p \otimes_\alpha q\left[T(s) \otimes_\alpha S(t)\, z - T(s_0) \otimes_\alpha S(t_0)\, z\right] \\
&\leq \sum_{i=0}^{n} p \otimes_\alpha q\left[T(s)\, x_i \otimes_\alpha S(t)\, y_i - T(s_0)\, x_i \otimes_\alpha S(t_0)\, y_i\right] \\
&= \sum_{i=0}^{n} p \otimes_\alpha q\left[\begin{array}{c} T(s)\, x_i \otimes_\alpha S(t)\, y_i - T(s)\, x_i \otimes_\alpha S(t_0)\, y_i + T(s)\, x_i \otimes_\alpha S(t_0)\, y_i \\ -T(s_0)\, x_i \otimes_\alpha S(t_0)\, y_i \end{array}\right] \\
&= \sum_{i=0}^{n} p \otimes_\alpha q\left[T(s)\, x_i \otimes_\alpha (S(t)\, y_i - S(t_0)\, y_i) + (T(s)\, x_i - T(s_0)\, x_i) \otimes_\alpha S(t_0)\, y_i\right] \\
&\leq \sum_{i=0}^{n} p(T(s)\, x_i)\, q(S(t)\, y_i - S(t_0)\, y_i) + p(T(s)\, x_i - T(s_0)\, x_i)\, q(S(t_0)\, y_i)
\end{aligned}$$

for all $s_0,\ t_0 \geq 0$. Hence, when $(s,t) \to (s_0,t_0)$, we get that $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous. QED

Proposition 2. *Let $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ be a T.P.S. on $X \otimes_\alpha Y$.*

We have $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous (resp exponentially equicontinuous) if and only if $(T(s) \otimes_\alpha I)_{s,t\geq 0}$ and $(I \otimes_\alpha S(t))_{s,t\geq 0}$ are locally equicontinuous (resp exponentially equicontinuous).

Proof. If $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous (resp exponentially equicontinuous) in particular for $s = 0$ and $t = 0$.

For the inverse, suppose that $(T(s) \otimes_\alpha I)_{s\geq 0}$ and $(I \otimes_\alpha S(t))_{t\geq 0}$ are locally equicontinuous. Let $s_0,\ t_0 \geq 0$, $z \in X \otimes_\alpha Y$ and $p \otimes_\alpha q \in \Gamma_{X\otimes_\alpha Y}$, then we have for $(s,t) \in [0,s_0] \times [0,t_0]$

$$p \otimes_\alpha q\,(T(s) \otimes_\alpha S(t)\, z) = p \otimes_\alpha q\,((T(s) \otimes_\alpha I)\,(I \otimes_\alpha S(t))\, z)$$

for all $z \in X \otimes_\alpha Y$. Since $(T(s) \otimes_\alpha I)_{s\geq 0}$ is locally continuous on $[0,s_0]$ then, there exist $p_1 \otimes_\alpha q_1 \in \Gamma_{X\otimes_\alpha Y}$ and $M_1 > 0$ such that

$$p \otimes_\alpha q\,(T(s) \otimes_\alpha S(t)\, z) \leq M_1 p_1 \otimes_\alpha q_1\,(I \otimes_\alpha S(t)\, z)$$

for all $z \in X \otimes_\alpha Y$ and $s \in [0,s_0]$, and we have that $(I \otimes_\alpha S(t))_{t\geq 0}$ is locally equicontinuous on $[0,t_0]$ then there exist $p_2 \otimes_\alpha q_2 \in \Gamma_{X\otimes_\alpha Y}$ and $M_2 > 0$ such that

$$p \otimes_\alpha q\,(T(s) \otimes_\alpha S(t)\, z) \leq M_1 M_2 p_2 \otimes_\alpha q_2\,(z)$$

for all $z \in X \otimes_\alpha Y$ and $(s,t) \in [0,s_0] \times [0,t_0]$. Hence $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous.

Next, we suppose that $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$ are exponentially equicontinuous.

In other words, there exist $w_1,\ w_2\geq 0$ such that $(e^{-w_1 s}T(s)\otimes_\alpha I)_{s\geq 0}$ and $\left(e^{-w_2 t}I\otimes_\alpha S(t)\right)_{t\geq 0}$ are equicontinuous. Let $w=\max(w_1,w_2)$, $t\geq 0$, since $(T(s)\otimes_\alpha I)_{s\geq 0}$ is exponentially equicontinuous, then for all $p\otimes_\alpha q\in\Gamma_{X\otimes_\alpha Y}$ there exist $p_1\otimes_\alpha q_1\in\Gamma_{X\otimes_\alpha Y}$ and $M_1>0$ such that

$$p\otimes_\alpha q\left(e^{-w(s+t)}T(s)\otimes_\alpha S(t)\,z\right)\leq M_1 p_1\otimes_\alpha q_1\left(e^{-wt}I\otimes_\alpha S(t)\,z\right)$$

for all $s\geq 0$ and $z\in X\otimes_\alpha Y$. Moreover, we have that $\left(e^{-wt}I\otimes_\alpha S(t)\right)_{t\geq 0}$ is equicontinuous, then there exist $p_2\otimes_\alpha q_2\in\Gamma_{X\otimes_\alpha Y}$ and $M_2>0$ such that

$$p\otimes_\alpha q\left(e^{-w(s+t)}T(s)\otimes_\alpha S(t)\,z\right)\leq M_1M_2p_2\otimes_\alpha q_2(z)$$

for all $s,\ t\geq 0$ and $z\in X\otimes_\alpha Y$. QED

Corollary 1. *If $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is a locally equicontinuous C_0-T.P.S on $X\otimes_\alpha Y$. Then $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous.*

Proof. According to the Lemma 4 and the Proposition 2 we have that $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$ are one-parameter locally equicontinuous C_0-semigroups on $X\otimes_\alpha Y$. Consequently, $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$ are strongly continuous. Once again, by Lemma 4 we get the result. QED

Proposition 3. *Let $(T(s,t))_{s,t\geq 0}$ be a two-parameter semigroup on a locally convex space (X,τ), we suppose that :*

(1) There exists a dense subspace $D\subseteq X$ such that :

$$\forall x\in D:\lim_{(s,t)\to(0^+,0^+)}T(s,t)\,x=x$$

(2) $(T(s,t))_{s,t\geq 0}$ is locally equicontinuous.

Then $(T(s,t))_{s,t\geq 0}$ is strongly continuous.

Proof. Let $x\in X$, then there exists an approximating net $(x_\alpha)_{\alpha\in I}\in D$ such that $\lim_\alpha x_\alpha=x$, let p a continuous seminorm on X and fix $\varepsilon>0$ and let $s_0,\ t_0\geq 0$, for all $(s,t)\in[0,s_0]\times[0,t_0]$, we have

$$p(T(s,t)\,x-x)\leq p(T(s,t)\,x-T(s,t)\,x_\alpha)+p(T(s,t)\,x_\alpha-x_\alpha)+p(x_\alpha-x).$$

By the hypothesis $(T(s,t))_{s,t\geq 0}$ is equicontinuous on $[0,s_0]\times[0,t_0]$, there exist a continuous seminorm q on X and $M>0$ such that

$$p(T(s,t)\,x-x)\leq Mq(x-x_\alpha)+p(T(s,t)\,x_\alpha-x_\alpha)+p(x_\alpha-x),$$

on the other hand, then there exists $\alpha_1 \in I$ such that if $\alpha \geq \alpha_1$

then $q(x - x_\alpha) \leq \frac{\varepsilon}{3M}$, similarly there exists $\alpha_2 \in I$ such that if $\alpha \geq \alpha_2$, then $p(x - x_\alpha) \leq \frac{\varepsilon}{3}$.

Let $\alpha_0 = \max(\alpha_1, \alpha_2)$, then for $\alpha \geq \alpha_0$ and for the middle term, there exists $\delta > 0$ such that if $\|(s,t)\| \leq \delta$ then $p(T(s,t)x_\alpha - x_\alpha) \leq \frac{\varepsilon}{3}$.

Finally, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|(s,t)\| \leq \delta$ then $p(T(s,t)x - x) \leq \varepsilon$, in other words, $\lim\limits_{(s,t)\to(0^+,0^+)} T(s,t)x = x$ for all $x \in X$.

We now prove the strong continuity of $(T(s,t))_{s,t\geq 0}$. If s_0, $t_0 \geq 0$ then for any $(s,t) \in]s_0, s_0+1] \times]t_0, t_0+1]$ and any $x \in X$, we have

$$T(s,t)x - T(s_0,t_0)x = T(s_0,t_0)(T(s-s_0, t-t_0)x - x).$$

The previous result clearly gives us $\lim\limits_{(s,t)\to(s_0,t_0)} T(s,t)x = T(s_0,t_0)x$.

Now, for any $(s,t) \in [0, s_0[\times [0, t_0[$ and any

$$T(s,t)x - T(s_0,t_0)x = T(s,t)(x - T(s_0 - s, t_0 - t)x),$$

so the result follows by the local equicontinuity of the semigroup $(T(s,t))_{s,t\geq 0}$ and that $\lim\limits_{(s,t)\to(0^+,0^+)} T(s,t)x = x$ for all $x \in X$. QED

Lemma 5. *Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a T.P.S. on the locally convex space $X\hat{\otimes}_\alpha Y$ then we have*

(1) $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous (resp. exponentially equicontinuous) if and only if $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous (resp. exponentially equicontinuous).

(2) $\left(T(s)\hat{\otimes}_\alpha I\right)_{s,t\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ are locally equicontinuous (resp exponentially equicontinuous) if and only if $(T(s)\otimes_\alpha I)_{s,t\geq 0}$ and $(I\otimes_\alpha S(t))_{s,t\geq 0}$ are locally equicontinuous (resp exponentially equicontinuous).

Proof. Suppose that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous, let s_0, $t_0 \geq 0$ and $p\otimes_\alpha q \in \Gamma_{X\otimes_\alpha Y}$. Observe that for every s, $t \geq 0$ and $z \in X\otimes_\alpha Y$

$$p\otimes_\alpha q(T(s)\otimes_\alpha S(t)z) = p\hat{\otimes}_\alpha q\left(T(s)\hat{\otimes}_\alpha S(t)z\right).$$

Moreover, $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous, then there exist $p_1\hat{\otimes}_\alpha q_1 \in \Gamma_{X\hat{\otimes}_\alpha Y}$ and $M > 0$ such that

$$p\otimes_\alpha q(T(s)\otimes_\alpha S(t)z) \leq Mp_1\hat{\otimes}_\alpha q_1(z) = Mp_1\otimes_\alpha q_1(z),$$

for all $z \in X\otimes_\alpha Y$ and $(s,t) \in [0, s_0] \times [0, t_0]$.

For the inverse, let s_0, $t_0 \geq 0$ and $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$, then if $z \in X\hat{\otimes}_\alpha Y$ by density there exists a net $(z_\alpha)_{\alpha\in I} \in X \otimes_\alpha Y$ such that $\lim_\alpha z_\alpha = z$. We have for every s, $t \geq 0$

$$p\hat{\otimes}_\alpha q \left(T(s)\,\hat{\otimes}_\alpha S(t)\, z_\alpha\right) = p \otimes_\alpha q \left(T(s) \otimes_\alpha S(t)\, z_\alpha\right),$$

and $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous, then there exists $p_1 \otimes_\alpha q_1 \in \Gamma_{X\otimes_\alpha Y}$ and $M > 0$ such that

$$p\hat{\otimes}_\alpha q \left(T(s)\,\hat{\otimes}_\alpha S(t)\, z_\alpha\right) \leq M p_1 \otimes_\alpha q_1 (z_\alpha) = M p_1 \hat{\otimes}_\alpha q_1 (z_\alpha)$$

for all $z \in X \otimes_\alpha Y$ and $(s,t) \in [0,s_0] \times [0,t_0]$. Therefore

$$\begin{aligned} p\hat{\otimes}_\alpha q \left(T(s)\,\hat{\otimes}_\alpha S(t)\, z\right) &= \lim_\alpha p\hat{\otimes}_\alpha q \left(T(s)\,\hat{\otimes}_\alpha S(t)\, z_\alpha\right) \\ &\leq M \lim_\alpha p_1 \hat{\otimes}_\alpha q_1 (z_\alpha) = M p_1 \hat{\otimes}_\alpha q_1 (z) \end{aligned}$$

for all $z \in X\hat{\otimes}_\alpha Y$ and $(s,t) \in [0,s_0] \times [0,t_0]$. Hence $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is locally equicontinuous.

Suppose that $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is exponentially equicontinuous, then there exists $a \geq 0$ such that $\left(e^{-a(s+t)} T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is equicontinuous. The previous result clearly gives us that $\left(e^{-a(s+t)} T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is equicontinuous if and only if $\left(e^{-a(s+t)} T(s) \otimes_\alpha S(t)\right)_{s,t\geq 0}$ is equicontinuous. QED

Proposition 4. *Let $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ be a T.P.S. on the locally convex space $X \otimes_\alpha Y$ then we have :*

$(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous (resp exponentially equicontinuous) if and only if $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are locally equicontinuous (resp exponentially equicontinuous).

Proof. If $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous, then by Proposition 2, $(T(s) \otimes_\alpha I)_{s,t\geq 0}$ is locally equicontinuous. Let $p \in \Gamma_X$, $x \in X$ and $y \in Y \setminus \{0\}$, then there exists $q \in \Gamma_Y$ such that $q(y) \neq 0$, therefore for every $s \geq 0$ we have

$$p(T(s)\, x) = \frac{1}{q(y)} p \otimes_a q \left(T(s)\, x \otimes y\right),$$

let $s_0 \geq 0$ so there exists $p_1 \otimes_\alpha q_1 \in \Gamma_{X\otimes_\alpha Y}$ and $M_1 > 0$ such that

$$p(T(s)\, x) \leq \frac{M_1}{q(y)} p_1 \otimes_\alpha q_1 (x \otimes y) = M_1 \frac{q_1(y)}{q(y)} p_1(x),$$

for all $s \in [0, s_0]$ and $x \in X$. Let $M = M_1 \frac{q_1(y)}{q(y)} > 0$, then we have shown that $(T(s))_{s\geq 0}$ is locally equicontinuous. With the same procedure we get that $(S(t))_{t\geq 0}$ is locally equicontinuous.

If $(T(s) \otimes_\alpha S(t))_{s,t\geq 0}$ is exponentially equicontinuous, then by Proposition 2, $(T(s) \otimes_\alpha I)_{s\geq 0}$ is exponentially equicontinuous, in other words, there exists $a \geq 0$ such that $(e^{-as}T(s) \otimes_\alpha I)_{s\geq 0}$ is equicontinuous. Therefore by the previous result $(e^{-as}T(s))_{s\geq 0}$ is equicontinuous. With the same method, we prove that there exists $b \geq 0$ such that $\left(e^{-bt}S(t)\right)_{t\geq 0}$ is equicontinuous. Hence $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially equicontinuous.

For the converse, suppose that $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are locally equicontinuous.

If $\alpha = \pi$ the projective topology on the tensor product. Let $p \otimes_\pi q \in \Gamma_{X\otimes_\pi Y}$ and s_0, $t_0 \geq 0$ and $z \in X \otimes_\pi Y$. Then for every $(s,t) \in [0,s_0] \times [0,t_0]$ we have

$$\begin{aligned} p \otimes_\pi q\left(T(s) \otimes_\pi S(t) z\right) &\leq \sum_{i=1}^{n} p \otimes_\pi q\left(T(s) x_i \otimes_\pi S(t) y_i\right) \\ &= \sum_{i=1}^{n} p\left(T(s) x_i\right) q\left(S(t) y_i\right), \end{aligned}$$

and there exist $(p_1, q_1) \in \Gamma_X \times \Gamma_Y$ and M_1, $M_2 > 0$ such that

$$p \otimes_\pi q\left(T(s) \otimes_\pi S(t) z\right) \leq M_1 M_2 \sum_{i=1}^{n} p_1(x_i)\, q_1(y_i)$$

for all representation of $z = \sum\limits_{i=1}^{n} x_i \otimes_\pi y_i \in X \otimes_\pi Y$. Hence

$$p \otimes_\pi q\left(T(s) \otimes_\pi S(t) z\right) \leq M_1 M_2 p_1 \otimes_\pi q_1(z)$$

for all $z \in X \otimes_\pi Y$ and $(s,t) \in [0,s_0] \times [0,t_0]$.

If $\alpha = \varepsilon$ the injective topology on the tensor product. Let $p \otimes_\varepsilon q \in \Gamma_{X\otimes_\varepsilon Y}$ and $z \in X \otimes_\varepsilon Y$, if $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are locally equicontinuous, then for all $\varepsilon > 0$ there exist δ_1, $\delta_1 > 0$, $p_1 \in \Gamma_X$ and $q_1 \in \Gamma_Y$ such that $T(s)(\delta_1 B_{p_1}) \subseteq \varepsilon B_p$ and $S(t)(\delta_2 B_{p_2}) \subseteq \varepsilon B_q$ for all $(s,t) \in [0,s_0] \times [0,t_0]$.

Let $z = \sum\limits_{i=1}^{n} x_i \otimes y_i \in X \otimes_\varepsilon Y$ and $\varphi \in B_p^\circ$, $\psi \in B_q^\circ$ we have for all $(s,t) \in [0,s_0] \times [0,t_0]$

$$\begin{aligned} \left|\langle \varphi \otimes \psi, T(s) \otimes S(t) z\rangle\right| &= \left|\sum_{i=1}^{n} \varphi\left(T(s) x_i\right) \psi\left(S(t) y_i\right)\right| \\ &= \left|\langle \varphi \circ T(s) \otimes \psi \circ S(t), z\rangle\right|. \end{aligned}$$

Moreover, we have that $\varphi \circ T(s) \in \frac{\varepsilon}{\delta_1} B^{\circ}_{p_1}$ and $\psi \circ S(t) \in \frac{\varepsilon}{\delta_2} B^{\circ}_{q_1}$, then there exist $\varphi_1 \in B^{\circ}_{p_1}$ and $\psi_1 \in B^{\circ}_{q_1}$ such that $\varphi \circ T(s) = \frac{\varepsilon}{\delta_1}\varphi_1$ and $\psi \circ S(t) = \frac{\varepsilon}{\delta_2}\psi_1$, we obtain

$$\begin{aligned} |\langle \varphi \otimes \psi, T(s) \otimes S(t) z\rangle| &= \frac{\varepsilon^2}{\delta_1\delta_2} |\langle \varphi_1 \otimes \psi_1, z\rangle| \\ &\leq \frac{\varepsilon^2}{\delta_1\delta_2} \sup_{\substack{\varphi_1 \in B^{\circ}_{p_1} \\ \psi_1 \in B^{\circ}_{q_1}}} |\langle \varphi_1 \otimes \psi_1, z\rangle| \end{aligned}$$

for all $\varphi \in B^{\circ}_p$ and $\psi \in B^{\circ}_q$, then it follows that

$$p \otimes_{\varepsilon} q\,(T(s) \otimes S(t) z) \leq \frac{\varepsilon^2}{\delta_1\delta_2} p_1 \otimes_{\varepsilon} q_1(z)$$

for all $(s,t) \in [0,s_0] \times [0,t_0]$.

If $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are exponentially equicontinuous, then there exists $a, b \geq 0$ such that $(e^{-as}T(s))_{s\geq 0}$ and $\left(e^{-bt}S(t)\right)_{t\geq 0}$ are equicontinuous, by the previous results we conclude that $\left(e^{-as}T(s) \otimes_{\alpha} e^{-bt}S(t)\right)_{s,t\geq 0}$ is equicontinuous. Take $\omega = \max(a,b)$, then $\left(e^{-\omega(s+t)}T(s) \otimes_{\alpha} S(t)\right)_{s,t\geq 0}$ is equicontinuous. Hence $(T(s) \otimes_{\alpha} S(t))_{s,t\geq 0}$ is exponentially equicontinuous. QED

The following two Theorems are the principal results of this section.

Theorem 2. *Let $\left(T(s)\hat{\otimes}_{\alpha}S(t)\right)_{s,t\geq 0}$ be a T.P.S. on $X\hat{\otimes}_{\alpha}Y$ then the following are equivalent :*

(1) $\left(T(s)\hat{\otimes}_{\alpha}S(t)\right)_{s,t\geq 0}$ is locally equicontinuous (resp. exponentially equicontinuous).

(2) $\left(T(s)\hat{\otimes}_{\alpha}I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_{\alpha}S(t)\right)_{t\geq 0}$ are locally equicontinuous (resp. exponentially equicontinuous).

(3) $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are locally equicontinuous (resp. exponentially equicontinuous).

Proof. We have that 1. is equivalent to 2. by Proposition 2 and Lemma 5.

Moreover, by Proposition 4 and Lemma 5 we get that 3. is equivalent to 1. QED

Theorem 3. *If $\left(T(s)\hat{\otimes}_{\alpha}S(t)\right)_{s,t\geq 0}$ is a locally equicontinuous T.P.S. on $X\hat{\otimes}_{\alpha}Y$ then the following are equivalent*

(1) $\left(T(s)\hat{\otimes}_{\alpha}S(t)\right)_{s,t\geq 0}$ is strongly continuous.

(2) $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ *and* $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ *are strongly continuous.*

(3) $(T(s))_{s\geq 0}$ *and* $(S(t))_{t\geq 0}$ *are strongly continuous.*

PROOF. Suppose that $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is strongly continuous.

Then $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous, by Proposition 2 and Lemma 4, we have $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$ are strongly continuous, locally equicontinuous one-parameter semigroups on $X\otimes_\alpha Y$, therefore by Proposition 3, $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ are strongly continuous.

If $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ are strongly continuous, then the same for $(T(s)\otimes_\alpha I)_{s\geq 0}$ and $(I\otimes_\alpha S(t))_{t\geq 0}$, again by Lemma 4 we have that $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are strongly continuous.

If $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are strongly continuous, then by Lemma 4 $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is strongly continuous, moreover $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ is locally equicontinuous, hence by Proposition 3, $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ is strongly continuous. QED

5 The Infinitesimal Generator of a T.P.S.

Let X and Y be a locally convex sequentially complete Hausdorff spaces.

Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a locally equicontinuous C_0 tensor product of semigroups on $X\hat{\otimes}_\alpha Y$, and let A_1, A_2 be the infinitesimal generators of the one-parameter locally equicontinous C_0-semigroups $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ respectively.

Remark 2. Let's recall the following

(1) Let $(A,\mathcal{D}(A))$ a linear operator on X, we define the graph of A by the set $G(A)=\{(x,Ax)\mid x\in\mathcal{D}(A)\}$. Let $x\in\mathcal{D}(A)$, we equip $G(A)$ with the graph topology τ_A generated by the family of seminorms

$\{p(x)+q(Ax)\mid p,\,q\in\Gamma_X\}$, [10].

(2) A subspace D of the domain $\mathcal{D}(A)$ is called a core for A if D is dense in $\mathcal{D}(A)$ for the graph topology τ_A.

(3) An operator $(B,\mathcal{D}(B))$ on X is called an extension of $(A,\mathcal{D}(A))$ if $\mathcal{D}(A)\subseteq\mathcal{D}(B)$ and $B_{/\mathcal{D}(A)}=A$. The operator $(A,\mathcal{D}(A))$ is called closable if it admits a closed extension. The smallest closed extension of a closable operator $(A,\mathcal{D}(A))$ is called the closure of $(A,\mathcal{D}(A))$ and it is denoted by $\left(\overline{A},\mathcal{D}\left(\overline{A}\right)\right)$.

(4) If $(A, \mathcal{D}(A))$ is a closed linear operator and D is a subspace of $\mathcal{D}(A)$ then D is a core for A if and only if $\overline{A_{/D}} = A$, where $\overline{A_{/D}}$ is the closed extension of $A_{/D}$ [12], [14].

(5) Let $(T(s))_{s\geq 0}$ a one-parameter C_0-semigroup and A its generator, suppose that D is a dense subspace of $\mathcal{D}(A)$. If $T(s)D \subseteq D$ then D is a core for A, [12].

Lemma 6. *$A_1 \otimes I$ and $I \otimes A_2$ are the infinitesimal generators of the one-parameter locally equicontinous C_0-semigroups $(T(s) \otimes I)_{s\geq 0}$ and $(I \otimes S(t))_{t\geq 0}$ respectively.*

Proof. Let A be the infinitesimal generator of the one-parameter locally equicontinous C_0-semigroup $(T(s) \otimes I)_{s\geq 0}$ and let $\mathcal{D}(A)$ its domain.

Let $x \otimes y \in \mathcal{D}(A)$, then we have

$$\begin{aligned} A(x \otimes y) &= \lim_{h\to 0^+} \frac{(T(h) \otimes I)(x \otimes y) - x \otimes y}{h} \\ &= \lim_{h\to 0^+} \frac{(T(h)x) \otimes y - x \otimes y}{h} \\ &= \lim_{h\to 0^+} \frac{(T(h)x - x) \otimes y}{h} \\ &= \lim_{h\to 0^+} \left[\left(\frac{T(h)x - x}{h} \right) \otimes y \right] \\ &= \left(\lim_{h\to 0^+} \frac{T(h)x - x}{h} \right) \otimes y \\ &= (A_1 x) \otimes y \end{aligned}$$

Hence

$$\mathcal{D}(A) = \mathcal{D}(A_1) \otimes Y \text{ and } A = A_1 \otimes I.$$

Similarly, if B is the infinitesimal generator of the one-parameter locally equicontinous C_0-semigroup $(I \otimes S(t))_{t\geq 0}$ and let $\mathcal{D}(B)$ its domain, then $\mathcal{D}(B) = X \otimes \mathcal{D}(A_2)$ and $B = I \otimes A_2$. QED

Proposition 5. *If A and B are the infinitesimal generators of the one-parameter locally equicontinous C_0-semigroups $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ respectively then :*

(1) $\mathcal{D}(A_1) \otimes Y$ and $X \otimes \mathcal{D}(A_2)$ are subspaces of $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively.

(2) $\mathcal{D}(A_1) \otimes Y$ and $X \otimes \mathcal{D}(A_2)$ are dense in $X\hat{\otimes}_\alpha Y$.

(3) $\mathcal{D}(A_1) \otimes Y$ *and* $X \otimes \mathcal{D}(A_2)$ *are invariant under* $T(s)\hat{\otimes}_\alpha I$ *and* $I\hat{\otimes}_\alpha S(t)$ *respectively.*

(4) $\mathcal{D}(A_1) \otimes Y$ *and* $X \otimes \mathcal{D}(A_2)$ *are cores for* A *and* B *respectively.*

Proof. (1) From Lemma 6, we have that $A_1 \otimes I$ is the infinitesimal generator of the one-parameter locally equicontinous C_0-semigroup $(T(s) \otimes_\alpha I)_{s\geq 0}$. If $x \otimes y \in \mathcal{D}(A_1 \otimes I) = \mathcal{D}(A_1) \otimes Y$, then

$$\lim_{s\to 0^+} \frac{\left(T(h)\hat{\otimes}_\alpha I\right)(x\otimes y) - x\otimes y}{h} = \lim_{s\to 0^+} \frac{\left(T(h)\otimes_\alpha I\right)(x\otimes y) - x\otimes y}{h}$$

then $\mathcal{D}(A_1) \otimes Y \subset \mathcal{D}(A)$.

Likewise we show that $X \otimes \mathcal{D}(A_2)$ is a subspace of $\mathcal{D}(B)$.

(2) It suffices to show that $\mathcal{D}(A_1) \otimes Y$ and $X \otimes \mathcal{D}(A_2)$ are dense in $X \otimes_\alpha Y$. But this is obvious since $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ are dense in X and Y respectively, because $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are one-parameter locally equicontinous C_0-semigroups on the locally convex sequentially complete Hausdorff spaces X and Y respectively.

(3) If $x \otimes y \in \mathcal{D}(A_1) \otimes Y$, then

$$\left(T(s)\hat{\otimes}_\alpha I\right)(x\otimes y) = \left(T(s)\otimes_\alpha I\right)(x\otimes y) = T(s)x\otimes y.$$

but $(T(s))_{s\geq 0}$ is a locally equicontinous one-parameter C_0-semigroup on X and A_1 its generator, then $T(s)D(A_1) \subseteq D(A_1)$. Finally $\mathcal{D}(A_1) \otimes Y$ is invariant under $T(s)\hat{\otimes}_\alpha I$.

In a similar way, we show that $X \otimes \mathcal{D}(A_2)$ is invariant under $I\hat{\otimes}_\alpha S(t)$.

(4) This is clear from (Remark 2, 5) since $\mathcal{D}(A_1) \otimes Y$ and $X \otimes \mathcal{D}(A_2)$ are dense in $X\hat{\otimes}_\alpha Y$ and $\mathcal{D}(A_1) \otimes Y$, $X \otimes \mathcal{D}(A_2)$ are invariant under $T(s)\hat{\otimes}_\alpha I$ and $I\hat{\otimes}_\alpha S(t)$ respectively.

QED

Lemma 7. $\overline{A_1 \otimes I}$ *and* $\overline{I \otimes A_2}$ *are the infinitesimal generators of the one-parameter locally equicontinous* C_0*-semigroups* $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ *and* $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ *respectively on* $X\hat{\otimes}_\alpha Y$*. where* $\overline{A_1 \otimes I}$ *and* $\overline{I \otimes A_2}$ *are the closure of the linear operators* $A_1 \otimes I$ *and* $I \otimes A_2$ *respectively.*

Proof. Let $z = x \otimes y \in \mathcal{D}(A_1) \otimes Y$. If A is the infinitesimal generator of $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$, then by Proposition 5, we have that $\mathcal{D}(A_1 \otimes I) \subset \mathcal{D}(A)$. In other words, A is an extension of $A_1 \otimes I$. Since A is the infinitesimal generator of a one-parameter locally equicontinous C_0-semigroup on $X\hat{\otimes}_\alpha Y$, A is closed.

Thus A is a closed extension of $A_1 \otimes I$. From Proposition 5, we have that $\mathcal{D}(A_1 \otimes I)$ is a core for $\mathcal{D}(A)$, hence

$$A = \overline{A_1 \otimes I}$$

Similarly, we show that $B = \overline{I \otimes A_2}$ is the infinitesimal generator of the one-parameter locally equicontinous C_0-semigroup $\left(I \hat{\otimes}_\alpha S(t)\right)_{t \geq 0}$. QED

Next, we will give some reminder on the semi-differentiability of a vector valued function of two variable, for more details about the semi-diffrentiability see [2].

Definition 3. (1) A subset C of a real vector space X is called a cone (with vertex 0) if $tC := \{tx \mid x \in C\} \subset C$ for all $t \geq 0$. If f is a map from a cone C of a vector space X to a cone C' of a vector space Y, we shall say that f is (positively) homogeneous (of degree 1) if $f(ty) = tf(y)$, for all $t > 0$ and $y \in C$. We say that the cone C is pointed if $C \cap (-C) = \{0\}$, where $-C := \{-x \mid x \in C\}$.

(2) Let $(T(s,t))_{s,t \geq 0}$ be a two-parameters semigroup on a locally convex Hausdorff space X and let $x \in X$. $T(.,.)x : \mathbb{R}^{+^2} \to X$ is said to be semi-differentiable at $(0,0)$ with respect to the cone $\mathbb{R}^{+^2}$, if there exists a (positively) homogeneous map $L : \mathbb{R}^{+^2} \to X$ such that for all $p \in \Gamma_X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all h, $k \geq 0$ if $0 < \|(h,k)\| \leq \delta$ then

$$p(T(h,k)x - T(0,0)x - L(h,k)) \leq \varepsilon \|(h,k)\|$$

where $\|(h,k)\| = \max(|h|, |k|)$. In other word

$$\lim_{(h,k) \to (0^+,0^+)} \frac{p(T(h,k)x - T(0,0)x - L(h,k))}{\|(h,k)\|} = 0$$

which is equivalent to

$$T(h,k)x - T(0,0)x = L(h,k) + R(h,k)$$

where

$$\lim_{(h,k) \to (0^+,0^+)} \frac{p(R(h,k))}{\|(h,k)\|} = 0$$

(3) The (positively) homogeneous map L if it exists, is unique, and it is called the semi-differential of $T(.,.)x$ at $(0,0)$ with respect to the cone $\mathbb{R}^{+^2}$.

(4) Let D the set of all $x \in X$ such that $T(.,.)x$ is semi-differentiable at $(0,0)$.

(5) Let $x \in D$ and let $D\left(T\left(s,t\right)x\right)_{|(s,t)=(0,0)}$ the semi-derivative of $T\left(.,.\right)x$ as a vector valued function of two variable at $(0,0)$. Then

$$L\left(h,k\right) = D\left(T\left(s,t\right)x\right)_{|(s,t)=(0,0)} \begin{pmatrix} h \\ k \end{pmatrix} = \left(\frac{\partial^{+}T\left(s,0\right)x}{\partial s}_{|s=0}, \frac{\partial^{+}T\left(0,t\right)x}{\partial t}_{|t=0} \right) \cdot \begin{pmatrix} h \\ k \end{pmatrix}$$

where $\frac{\partial^{+}T(s,0)x}{\partial s}_{|s=0}$ and $\frac{\partial^{+}T(0,t)x}{\partial t}_{|t=0}$ are the semi-derivative of $T\left(.,.\right)x$ with respect to $s=0$ and $t=0$ respectively, defined by

$$\frac{\partial^{+}T\left(s,0\right)x}{\partial s}_{|s=0} = \lim_{s\to 0^{+}} \frac{T\left(s,0\right)x - T\left(0,0\right)x}{s}$$

and

$$\frac{\partial^{+}T\left(0,t\right)x}{\partial t}_{|t=0} = \lim_{t\to 0^{+}} \frac{T\left(0,t\right)x - T\left(0,0\right)x}{t}$$

(6) Let $x \in D$. If there is a (positively) homogeneous map $\widehat{L} : \mathbb{R}^{+^2} \to \mathfrak{L}\left(D,X\right)$ with $(h,k) \mapsto \widehat{L}\left(h,k\right)$ such that

$$L\left(h,k\right) = \widehat{L}\left(h,k\right)x$$

for all $(h,k) \in \mathbb{R}^{+^2}$. Then one can consider the semi-differential as a (positively) homogeneous map $\widehat{L} : \mathbb{R}^{+^2} \to \mathfrak{L}\left(D,X\right)$ where $\mathfrak{L}\left(D,X\right)$ is the space of linear (not necessarily bounded) operators from D in X.

(7) Hence in this case the semi-differential is $\widehat{L}$. Next we will denote $\widehat{L}$ by L .

(8) Let $x \in D$. If there exist $L_1 : \mathcal{D}\left(L_1\right) \subseteq X \to X$ and $L_2 : \mathcal{D}\left(L_2\right) \subseteq X \to X$ such that

$$L\left(h,k\right)x = hL_1x + kL_2x$$

for all $(h,k) \in \mathbb{R}^{+^2}$, then we will denote L by (L_1, L_2) and write $(L_1, L_2)\binom{h}{k} := L\left(h,k\right)$, and we have

$$\mathcal{D}\left(L\left(h,k\right)\right) = \mathcal{D}\left(\left(L_1, L_2\right)\begin{pmatrix} h \\ k \end{pmatrix}\right) = \mathcal{D}\left(hL_1 + kL_2\right) = \mathcal{D}\left(L_1\right) \cap \mathcal{D}\left(L_2\right).$$

Definition 4. Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a T.P.S. on $X\hat{\otimes}_\alpha Y$.

We consider the operator $(A,\mathcal{D}(A))$ defined on its domain:

$$\mathcal{D}(A)=\left\{z\in X\hat{\otimes}_\alpha Y \mid \left(T(.)\hat{\otimes}_\alpha S(.)\right)z \text{ is semi-differentiable at } (0,0)\right\}$$

by setting

$$Az=D\left(\left(T(s)\hat{\otimes}_\alpha S(t)\right)z\right)_{|(s,t)=(0,0)} \text{ for all } z\in D(A)$$

where $D\left(\left(T(s)\hat{\otimes}_\alpha S(t)\right)z\right)_{|(s,t)=(0,0)}$ is the semi-derivative of $\left(T(.)\hat{\otimes}_\alpha S(.)\right)z$ at $(0,0)$.

The operator A will be called the **infinitesimal generator** of the T.P.S. $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$.

Theorem 4. *Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be a locally equicontinous C_0-T.P.S. on $X\hat{\otimes}_\alpha Y$ and let A be its infinitesimal generator, then we have*

$$\mathcal{D}(A)=\mathcal{D}\left(\overline{A_1\otimes I}\right)\cap\mathcal{D}\left(\overline{I\otimes A_2}\right),$$

and we can consider the infinitesimal generator A as a (positively) homogeneous map

$$L:\mathbb{R}^{+^2}\to\mathfrak{L}\left(\mathcal{D}\left(\overline{A_1\otimes I}\right)\cap\mathcal{D}\left(\overline{I\otimes A_2}\right),X\hat{\otimes}_\alpha Y\right)$$

defined by :

$$L(h,k)=h\left(\overline{A_1\otimes I}\right)+k\left(\overline{I\otimes A_2}\right):=\left(\overline{A_1\otimes I},\overline{I\otimes A_2}\right)\begin{pmatrix}h\\k\end{pmatrix}$$

where A_1 and A_2 are the infinitesimal generators of the one-parameter locally equicontinuous C_0-semigroups $(T(s))_{s\geq 0}$and $(S(t))_{t\geq 0}$ respectively.

Proof. Let $z\in\mathcal{D}(A)$, set $F(s,t)z=T(s)\hat{\otimes}_\alpha S(t)z$ for s, $t\geq 0$. then we have

$$\begin{aligned}Az&=D(F(s,t)z)_{|(s,t)=(0,0)}\\&=\left(\frac{\partial^+F(s,0)z}{\partial s}_{|s=0},\frac{\partial^+F(0,t)z}{\partial t}_{|t=0}\right)\end{aligned}$$

where $D(F(s,t)z)_{|(s,t)=(0,0)}$ is the semi-derivative of $F(.,.)z$ at the point $(0,0)$, and we have :

$$\begin{aligned}\frac{\partial^+F(s,0)z}{\partial s}_{|s=0}&=\lim_{s\to 0^+}\frac{F(s,0)z-F(0,0)z}{s}\\&=\lim_{s\to 0^+}\frac{\left(T(s)\hat{\otimes}_\alpha S(0)\right)z-\left(T(0)\hat{\otimes}_\alpha S(0)\right)z}{s}\\&=\lim_{s\to 0^+}\frac{\left(T(s)\hat{\otimes}_\alpha I\right)z-z}{s}.\end{aligned}$$

Since $(T(s))_{s\geq 0}$ is a one-parameter locally equicontinous C_0-semigroup with the infinitesimal generator A_1. Then Lemma 7 gives :

$$\frac{\partial^+ F(s,0)z}{\partial s}_{|s=0} = \left(\overline{A_1 \otimes I}\right) z \text{ and } z \in \mathcal{D}\left(\left(\overline{A_1 \otimes I}\right)\right).$$

Similarly, we show that

$$\frac{\partial^+ F(0,t)z}{\partial t}_{|t=0} = \left(\overline{I \otimes A_2}\right) z \text{ and } z \in \mathcal{D}\left(\left(\overline{I \otimes A_2}\right)\right).$$

Hence,

$$\mathcal{D}(A) \subseteq \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right).$$

Let $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$. To prove that $z \in \mathcal{D}(A)$, it is sufficient to prove that for any $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$

$$\lim_{(h,k)\to(0^+,0^+)} \frac{p\hat{\otimes}_\alpha q\left(\left(T(h)\hat{\otimes}_\alpha S(k)\right)z - z - h\left(\overline{A_1 \otimes I}\right)z - k\left(\overline{I \otimes A_2}\right)z\right)}{\|(h,k)\|} = 0.$$

Let $h_0,\ k_0 > 0$ and let $(h,k) \in\,]0,h_0] \times]0,k_0]$, $p\hat{\otimes}_\alpha q \in \Gamma_{X\hat{\otimes}_\alpha Y}$, we set

$$R_z(h,k) = \left(T(h)\hat{\otimes}_\alpha S(k)\right)z - z - h\left(\overline{A_1 \otimes I}\right) - k\left(\overline{I \otimes A_2}\right)z.$$

We have

$$\begin{aligned} R_z(h,k) = \left(T(h)\hat{\otimes}_\alpha I\right)&\left[\left(I\hat{\otimes}_\alpha S(k)\right)z - z - k\left(\overline{I \otimes A_2}\right)z\right] \\ &+ k\left[\left(T(h)\hat{\otimes}_\alpha I\right)\left(\overline{I \otimes A_2}\right)z - \left(\overline{I \otimes A_2}\right)z\right] \\ &+ \left[\left(T(h)\hat{\otimes}_\alpha I\right)z - z - h\left(\overline{A_1 \otimes I}\right)z\right]. \end{aligned}$$

Given the fact that we have $\frac{h}{\|(h,k)\|} \leq 1$ and $\frac{h}{\|(h,k)\|} \leq 1$ for all $h,\ k > 0$, we obtain the following inequalities

$$\begin{aligned} &\frac{p\hat{\otimes}_\alpha q\left(R_z(h,k)\right)}{\|(h,k)\|} \\ &\leq \frac{k}{\|(h,k)\|} p\hat{\otimes}_\alpha q\left(\left(T(h)\hat{\otimes}_\alpha I\right)\left[\frac{\left(I\hat{\otimes}_\alpha S(k)\right)z - z - k\left(\overline{I \otimes A_2}\right)z}{k}\right]\right) \\ &\quad + \frac{k}{\|(h,k)\|} p\hat{\otimes}_\alpha q\left(\left(T(h)\hat{\otimes}_\alpha I\right)\left(\overline{I \otimes A_2}\right)z - \left(\overline{I \otimes A_2}\right)z\right) \\ &\quad + \frac{h}{\|(h,k)\|} p\hat{\otimes}_\alpha q\left(\frac{\left(T(h)\hat{\otimes}_\alpha I\right)z - z - h\left(\overline{A_1 \otimes I}\right)z}{h}\right) \end{aligned}$$

$$\leq p\hat{\otimes}_{\alpha}q\left(\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)\left[\frac{\left(I\hat{\otimes}_{\alpha}S\left(k\right)\right)z-z-k\left(\overline{I\otimes A_2}\right)z}{k}\right]\right)$$
$$+\,p\hat{\otimes}_{\alpha}q\left(\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)\left(\overline{I\otimes A_2}\right)z-\left(\overline{I\otimes A_2}\right)z\right)$$
$$+\,p\hat{\otimes}_{\alpha}q\left(\frac{\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)z-z-h\left(\overline{A_1\otimes I}\right)z}{h}\right).$$

We have $\left(T\left(s\right)\hat{\otimes}_{\alpha}S\left(t\right)\right)_{s,t\geq 0}$ is locally equicontinous, so $\left(T\left(s\right)\hat{\otimes}_{\alpha}I\right)_{s\geq 0}$ is locally equicontinous, then there exists $p_1\hat{\otimes}_{\alpha}q_1\in\Gamma_{X\hat{\otimes}_{\alpha}Y}$ and $M>0$ such that

$$\frac{p\hat{\otimes}_{\alpha}q\left(R_z\left(h,k\right)\right)}{\left\|\left(h,k\right)\right\|}\leq Mp_1\hat{\otimes}_{\alpha}q_1\left(\frac{\left(I\hat{\otimes}_{\alpha}S\left(k\right)\right)z-z}{k}-\left(\overline{I\otimes A_2}\right)z\right)$$
$$+\,p\hat{\otimes}_{\alpha}q\left(\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)\left(\overline{I\otimes A_2}\right)z-\left(\overline{I\otimes A_2}\right)z\right)$$
$$+\,p\hat{\otimes}_{\alpha}q\left(\frac{\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)z-z}{h}-\left(\overline{A_1\otimes I}\right)z\right).$$

for all $(h,k)\in\,]0,h_0]\times\,]0,k_0]$.

Since $z\in\mathcal{D}\left(\overline{A_1\otimes I}\right)\cap\mathcal{D}\left(\overline{I\otimes A_2}\right)$, we have

$$\lim_{h\to 0^+}p\hat{\otimes}_{\alpha}q\left(\frac{\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)z-z}{h}-\left(\overline{A_1\otimes I}\right)z\right)=0$$

and

$$\lim_{k\to 0^+}p_1\hat{\otimes}_{\alpha}q_1\left(\frac{\left(I\hat{\otimes}_{\alpha}S\left(k\right)\right)z-z}{k}-\left(\overline{I\otimes A_2}\right)z\right)=0.$$

We have $\left(T\left(s\right)\hat{\otimes}_{\alpha}S\left(t\right)\right)_{s,t\geq 0}$ is locally equicontinous C_0-T.P.S,
then $\left(T\left(s\right)\hat{\otimes}_{\alpha}I\right)_{s\geq 0}$ is strongly continuous. Therefore

$$\lim_{h\to 0^+}p\hat{\otimes}_{\alpha}q\left(\left(T\left(h\right)\hat{\otimes}_{\alpha}I\right)\left(\overline{I\otimes A_2}\right)z-\left(\overline{I\otimes A_2}\right)z\right)=0.$$

Finally

$$\lim_{(h,k)\to(0^+,0^+)}\frac{\left\|R_z\left(h,k\right)\right\|}{\left\|\left(h,k\right)\right\|}=0,$$

which means that $\left(T\left(.\right)\hat{\otimes}_{\alpha}S\left(.\right)\right)z$ is semi-differentiable at $(0,0)$ for all $z\in\mathcal{D}\left(\overline{A_1\otimes I}\right)\cap\mathcal{D}\left(\overline{I\otimes A_2}\right)$, so $z\in\mathcal{D}\left(A\right)$.

Now, let $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$, then $D\left(\left(T(s)\,\hat{\otimes}_\alpha S(t)\right) z\right)_{|(s,t)=(0,0)}$ exists as a (positively) homogeneous map $L(.,.) : \mathbb{R}^{+^2} \to X \hat{\otimes}_\alpha Y$ defined by

$$L(h,k) = D\left(\left(T(s)\,\hat{\otimes}_\alpha S(t)\right) z\right)_{|(s,t)=(0,0)} \begin{pmatrix} h \\ k \end{pmatrix} = h\left(\overline{A_1 \otimes I}\right) z + k\left(\overline{I \otimes A_2}\right) z.$$

Let $\hat{L}(.,.) : \mathbb{R}^{+^2} \to \mathfrak{L}\left(\mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right), X \hat{\otimes}_\alpha Y\right)$ defined by

$$\hat{L}(h,k) = h\left(\overline{A_1 \otimes I}\right) + k\left(\overline{I \otimes A_2}\right),$$

then $\hat{L}(.,.)$ is a (positively) homogeneous map and we have for any $(h,k) \in \mathbb{R}^{+^2}$ and $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$

$$L(h,k) = \hat{L}(h,k)\, z,$$

then we have for all $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$

$$L(.,.) = \hat{L}(.,.)\, z$$

therefore, for any $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$ we have

$$\begin{aligned} Az &= D\left(\left(T(s)\,\hat{\otimes}_\alpha S(t)\right) z\right)_{|(s,t)=(0,0)} \\ &= L(.,.) \\ &= \hat{L}(.,.)\, z. \end{aligned}$$

Thus,

$$A = \hat{L}(.,.).$$

Therefore, we can consider the infinitesimal generator A as (positively) homogeneous map as follows

$$A : \mathbb{R}^{+^2} \to \mathfrak{L}\left(\mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right), X \hat{\otimes}_\alpha Y\right)$$

defined by

$$A(h,k) = h\left(\overline{A_1 \otimes I}\right) + k\left(\overline{I \otimes A_2}\right).$$

Next, we will denote the infinitesimal generator of $\left(T(s)\,\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ by $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)$, and we will write for all $(h,k) \in \mathbb{R}^{+^2}$ and $z \in \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)$

$$\left(\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right) \begin{pmatrix} h \\ k \end{pmatrix}\right) z = h\left(\overline{A_1 \otimes I}\right) z + k\left(\overline{I \otimes A_2}\right) z.$$

QED

The following Theorem is proved in the same way by following the same steps as the Theorem 4.

Theorem 5. *Let* $(T(s)\otimes_{\alpha} S(t))_{s,t\geq 0}$ *be a locally equicontinuous* C_0*-T.P.S. on* $X\otimes_{\alpha} Y$*. The infinitesimal generator of the* C_0*-T.P.S.* $(T(s)\otimes_{\alpha} S(t))_{s,t\geq 0}$ *is the (positively) homogeneous map*

$$L:\mathbb{R}^{+^2}\rightarrow \mathfrak{L}\left(\mathcal{D}\left(A_1\right)\otimes \mathcal{D}\left(A_2\right)\right)$$

defined by

$$L\left(a,b\right)=\left(A_1\otimes I, I\otimes A_2\right)\begin{pmatrix} a\\ b\end{pmatrix}=aA_1\otimes I+bI\otimes A_2$$

with A_1 *and* A_2 *are the infinitesimal generators of the* C_0*-semigroups* $(T(s))_{s\geq 0}$ *and* $(S(t))_{t\geq 0}$ *respectively.*

Proposition 6. *Let* $\left(T(s)\,\hat{\otimes}_{\alpha}S(t)\right)_{s,t\geq 0}$ *be a locally equicontinuous* C_0*-T.P.S. on* $X\hat{\otimes}_{\alpha}Y$*. For* $(a,b)\in\mathbb{R}^{+^2}$ *with* $(a,b)\neq(0,0)$*,* $\left(T(as)\,\hat{\otimes}_{\alpha}S(bs)\right)_{s\geq 0}$ *is a one-parameter locally equicontinuous* C_0*-semigroup with the infinitesimal generator*

$$A=a\left(\overline{A_1\otimes I}\right)+b\left(\overline{I\otimes A_2}\right)$$

Proof. First of all, it's clear that $\left(T(as)\,\hat{\otimes}_{\alpha}S(bs)\right)_{s\geq 0}$ is a one-parameter locally equicontinuous C_0-semigroup.

Let $z\in\mathcal{D}(A)$, then

$$Az=\lim_{h\rightarrow 0^+}\frac{T\left(ah\right)\hat{\otimes}_{\alpha}S\left(bh\right)z-z}{h}$$

exists, especially if $v=(a,b)=(a,0)$ then

$$Az=\lim_{h\rightarrow 0^+}\frac{\left(T\left(ah\right)\hat{\otimes}_{\alpha}I\right)z-z}{h},$$

so $z\in\mathcal{D}\left(\overline{A_1\otimes I}\right)$. Similarly, if $v=(a,b)=(0,b)$, we get $z\in\mathcal{D}\left(\overline{I\otimes A_2}\right)$, finally

$$\mathcal{D}\left(A\right)\subseteq\mathcal{D}\left(\overline{A_1\otimes I}\right)\cap\mathcal{D}\left(\overline{I\otimes A_2}\right).$$

Now, let $z \in D\left(\overline{A_1 \otimes I}\right) \cap D\left(\overline{I \otimes A_2}\right)$, for $v = (a, b) \in \mathbb{R}^{+^2}$ we have

$$\begin{aligned}
Az &= \lim_{h \to 0^+} \frac{\left(T(ah) \hat{\otimes}_\alpha S(bh)\right) z - z}{h} \\
&= \lim_{h \to 0^+} \frac{\left(T(ah) \hat{\otimes}_\alpha S(0)\right)\left(T(0) \hat{\otimes}_\alpha S(bh)\right) z - z}{h} \\
&= \lim_{h \to 0^+} \frac{\left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right)\left(I \hat{\otimes}_\alpha \tilde{S}(bh)\right) z - z}{h} \\
&= \lim_{h \to 0^+} \frac{\left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right)\left(I \hat{\otimes}_\alpha \tilde{S}(bh)\right) z - \left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right) z + \left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right) z - z}{h} \\
&= \lim_{h \to 0^+} b\left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right)\left(\frac{\left(I \hat{\otimes}_\alpha \tilde{S}(bh)\right) z - z}{bh}\right) + a\left(\frac{\left(\tilde{T}(ah) \hat{\otimes}_\alpha I\right) z - z}{ah}\right) \\
&= a\left(\overline{A_1 \otimes I}\right) z + b\left(\overline{I \otimes A_2}\right) z.
\end{aligned}$$

Finally

$$A = \left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\begin{pmatrix} a \\ b \end{pmatrix}.$$

QED

Corollary 2. *Let $\left(T(s) \hat{\otimes}_\alpha S(t)\right)_{s,t \geq 0}$ be a locally equicontinuous C_0-T.P.S. on $X \hat{\otimes}_\alpha Y$, and $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)$ its generator with A_1 and A_2 are the infinitesimal generators of the locally equicontinuous C_0-semigroups $(T(s))_{s \geq 0}$ and $(S(t))_{t \geq 0}$, let $(a, b) \in \mathbb{R}^{+^2}$ with $(a, b) \neq (0, 0)$ then*

(1) $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\binom{a}{b}$ is a densely defined closed operator in $X \hat{\otimes}_\alpha Y$.

(2) $\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$ is a core of $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\binom{a}{b}$.

Proof. (1) It follows from the fact that $\left(\overline{A_1 \otimes I}, \overline{I \otimes A_2}\right)\binom{a}{b}$ is the infinitesimal generator of the one-parameter locally equicontinuous C_0-semigroup $\left(T(as) \hat{\otimes}_\alpha S(bs)\right)_{s \geq 0}$ on $X \hat{\otimes}_\alpha Y$.

(2) We have

$$\begin{aligned}
\mathcal{D}(A_1) \otimes \mathcal{D}(A_2) &= \mathcal{D}(A_1 \otimes I) \cap \mathcal{D}(I \otimes A_2) \\
&\subseteq \mathcal{D}\left(\overline{A_1 \otimes I}\right) \cap \mathcal{D}\left(\overline{I \otimes A_2}\right)
\end{aligned}$$

but $\mathcal{D}(A_1) \otimes \mathcal{D}(A_2)$ is dense in $X \hat{\otimes}_\alpha Y$ as a tensor product of two dense spaces.

And we have for all s, $t \geq 0$

$$T(s)\hat{\otimes}_\alpha S(t)(\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)) \subseteq \mathcal{D}(A_1)\otimes\mathcal{D}(A_2),$$

then by applying (Remark 2, 5) we get the result. QED

Proposition 7. *Let $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ be a locally equicontinuous C_0-T.P.S on $X\otimes_\alpha Y$, with infinitesimal generator $(A_1\otimes I, I\otimes A_2)$ where A_1, A_2 are the infinitesimal generators of the one-parameter semigroups $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ respectively, then $(A_1\otimes I, I\otimes A_2)\binom{a}{b}$ is closable in $X\hat{\otimes}_\alpha Y$ with a dense domain $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$.*

In addition we have for all a, $b\geq 0$

$$\overline{aA_1\otimes I + bI\otimes A_2} = a\left(\overline{A_1\otimes I}\right) + b\left(\overline{I\otimes A_2}\right).$$

Proof. According to Corollary 2, we have $D(A_1)\otimes D(A_2)$ is a core of $\left(\overline{A_1\otimes I}, \overline{I\otimes A_2}\right)\binom{a}{b}$, and we have :

$$\left[a\left(\overline{A_1\otimes I}\right) + b\left(\overline{I\otimes A_2}\right)\right]_{|\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)} = a(A_1\otimes I) + b(I\otimes A_2)$$

Again by Corollary 2, we have $a\left(\overline{A_1\otimes I}\right) + b\left(\overline{I\otimes A_2}\right)$ is closed, so it is a closed extension of $A = a(A_1\otimes I) + b(I\otimes A_2)$, then

$$\overline{aA_1\otimes I + bI\otimes A_2} = a\left(\overline{A_1\otimes I}\right) + b\left(\overline{I\otimes A_2}\right)$$

QED

Theorem 6. *Let $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ be a locally equicontinuous C_0-TPS on $X\otimes_\alpha Y$, then $(A_1\otimes I, I\otimes A_2)$ is the infinitesimal generator of $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$ if and only if $\left(\overline{A_1\otimes I}, \overline{I\otimes A_2}\right)$ is the infinitesimal generator of the locally equicontinuous C_0-TPS $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ on $X\hat{\otimes}_\alpha Y$.*

Proof. Let $v = (a,b)\in {\mathbb{R}^+}^2$.

Suppose that the application $(a,b)\mapsto (A_1\otimes I, I\otimes A_2)\binom{a}{b}$ is the infinitesimal generator of $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$.

Then the application $(a,b)\mapsto \overline{(A_1\otimes I, I\otimes A_2)\binom{a}{b}} = \left(\overline{A_1\otimes I}, \overline{I\otimes A_2}\right)\binom{a}{b}$ is the infinitesimal generator of the C_0-T.P.S $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ on $X\hat{\otimes}_\alpha Y$.

Conversely, if $(a,b)\mapsto \left(\overline{A_1\otimes I}, \overline{I\otimes A_2}\right)\binom{a}{b}$ is the infinitesimal generator of $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$, then we have for a, $b\geq 0$

$$\left[\left(\overline{A_1\otimes I}, \overline{I\otimes A_2}\right)\binom{a}{b}\right]_{|\mathcal{D}(A_1\otimes I)\cap\mathcal{D}(I\otimes A_2)} = (A_1\otimes I, I\otimes A_2)\binom{a}{b}$$

but the application $(a,b)\mapsto (A_1\otimes I, I\otimes A_2)\binom{a}{b}$ is the infinitesimal generator of the C_0-TPS $(T(s)\otimes_\alpha S(t))_{s,t\geq 0}$. QED

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