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On the edge metric dimension and Wiener index of the blow up of graphs

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Abstract. Let $G = (V, E)$ be a connected graph. The distance between an edge $e = xy$ and a vertex v is defined as $\mathrm{d}(e, v) = \min\{\mathrm{d}(x, v), \mathrm{d}(y, v)\}$. A nonempty set $S \subseteq V(G)$ is an edge metric generator for G if for any two distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $s \in S$ such that $\mathrm{d}(e_1, s) \neq \mathrm{d}(e_2, s)$. An edge metric generating set with the smallest number of elements is called an edge metric basis of G, and the number of elements in an edge metric basis is called the edge metric dimension of G and it is denoted by $\mathrm{edim}(G)$. In this paper, we study the edge metric dimension of a blow up of a graph G, and also we study the edge metric dimension of the zero divisor graph of the ring of integers modulo n. Moreover, the Wiener index and the hyper-Wiener index of the blow up of certain graphs are computed.

Keywords: Edge metric dimension, Wiener index, Hyper-Wiener index, Blow up of a graph, Zero divisor graph

MSC 2020 classification: primary 05C12, 05C76, secondary 05C25

Introduction

1 Introduction

Let $G = (V, E)$ be a simple connected graph and $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. For a vertex v of G, $\mathrm{N}_G(v)$ denotes the set of vertices of G that are adjacent to v in G, and we denote $|\mathrm{N}_G(v)|$ by $\deg(v)$. Also, we denote the number of vertices of G by $|G|$. For distinct vertices u and v of G, we write $u \sim v$ if u and v are adjacent in G and the edge e between u and v will be denoted by $e = uv$. Also the *distance* between two distinct vertices u and v, denoted by $\mathrm{d}(u, v)$, is the length of the shortest path connecting u and v, if such a path exists; otherwise, we set $\mathrm{d}(u, v) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The distance between an edge $e = xy$ and a vertex v is defined as follows:

$$\mathrm{d}(e, v) = \min\{\mathrm{d}(x, v), \mathrm{d}(y, v)\}.$$

A vertex v *distinguishes* two edges e_1 and e_2 if $\mathrm{d}(e_1, v) \neq \mathrm{d}(e_2, v)$. A nonempty set $S \subseteq V(G)$ is an *edge metric generator* of a graph G if for any two distinct edges $e_1, e_2 \in E(G)$, there exists a vertex $s \in S$ such that s distinguishes e_1 and e_2. An edge metric generating set with the smallest number of elements is called an *edge metric basis* of G, and the number of elements in an edge metric basis is called the *edge metric dimension* of G and it is denoted by $\mathrm{edim}(G)$. For an ordered subset $S = \{v_1, \ldots, v_k\}$ of vertices in G and an edge e of $E(G)$, the *edge metric S-representation* of e is the vector $r(e|S) = (d(e, v_1), \ldots, d(e, v_k))$. Observe that S is an edge metric generator if and only if the edge metric S-representations are different for all edges of $E(G)$.

The concept of edge metric dimension was introduced in [14] in analogy with the classical metric dimension $\dim(G)$, which was introduced by Slater in 1975 in [20]. $\dim(G)$ is defined as follows: a vertex $v \in V(G)$ distinguishes $v_1, v_2 \in V(G)$ if $\mathrm{d}(v, v_1) \neq \mathrm{d}(v, v_2)$. A set $S \subseteq V(G)$ is a *vertex generating set* of G if for any two distinct $v_1, v_2 \in V(G)$, there exists a vertex $s \in S$ such that s distinguishes v_1 and v_2. A vertex generating set with the smallest number of elements is a *vertex basis* of G, and the number of elements in a vertex basis is its *metric dimension* (denoted by $\dim(G)$). See [6, 9, 12] and [24, 18] for more details on $\dim(G)$ and $\mathrm{edim}(G)$, respectively. Recently in [24], the edge metric dimension of some graph operations was investigated.

The *Wiener index*, $W(G)$, is equal to the sum of all shortest distances in a graph (cf. [21]). In other words, $W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u, v)$. Wiener number was defined in 1947 by an American chemist H. Wiener. He used this index to estimate the boiling point of Alkans. There are many situations in communication, facility location, cryptology, architecture etc. where the Wiener index of the corresponding graph or the average distance is of great interest. One of these problems, for example, is to find a spanning tree with minimum average distance. The Wiener index is one of the most studied topological indices, both from a theoretical point of view and applications, see for details [8], [10] and [23].

The *hyper-Wiener index* of acyclic graphs was introduced by Milan Randic in 1993. Then Klein et al. [16], generalized Randic's definition for all connected graphs, as a generalization of the Wiener index. It is $WW(G) = \frac{1}{2} W(G) + \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d^2(u, v)$, where $d^2(u, v) = d(u, v)^2$. We encourage the reader to consult [4], [5], [11] and [15] for the mathematical properties of hyper-Wiener index and its applications in chemistry.

In Section 2 of this paper, we study the edge metric dimension, the Wiener index and the hyper-Wiener index of a blow up of a graph G. In Section 3, as an example of the blow up of graphs, we study the edge metric dimension, the Wiener index and the hyper-Wiener index of the zero divisor graph of the ring of integers modulo n.

All graphs considered in this paper are connected and simple. We say that G is an *empty graph* if $E(G) = \emptyset$. Also, K_n and $\overline{K_n}$ denote the complete graph with n vertices and its complement, respectively, and P_n denotes the path with n vertices.

2 The blow up of a graph

For two graphs H_1 and H_2 with disjoint vertex sets, the *join* $H_1 \vee H_2$ of the graphs H_1 and H_2 is the graph obtained from the union of H_1 and H_2 by adding new edges from each vertex of H_1 to every vertex of H_2. The concept of join graph is generalized (in [19], it is called as a generalized composition graph). Let G be a graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and let $H_1, H_2, \dots, H_k$ be k pairwise disjoint graphs. The *G-generalized join graph* $G[H_1, H_2, \dots, H_k]$ of $H_1, H_2, \dots, H_k$ is the graph formed by replacing each vertex v_i of G by the graph H_i and then joining each vertex of H_i to each vertex of H_j whenever $v_i \sim v_j$ in the graph G. Now, if the graph G consists of two adjacent vertices, then the G-generalized join graph $G[H_1, H_2]$ coincides with the join $H_1 \vee H_2$ of the graphs H_1 and H_2. If each graph H_i is a complete graph or empty graph, then $G[H_1, H_2, \dots, H_k]$ is called a *blow up* of G.

In the rest of the paper, we consider $G[H_1, H_2, \dots, H_k]$ as a blow up of G, and we always assume that there exists $1 \leqslant i \leqslant k$ such that $|H_i| > 1$, in $G[H_1, H_2, \dots, H_k]$. In the following proposition, we study the edge metric dimension of $G[H_1, H_2, \dots, H_k]$.

Proposition 2.1. Assume that G is a connected graph on k vertices with $V(G) = \{v_1, v_2, \dots, v_k\}$, and $G[H_1, H_2, \dots, H_k]$ be the blow up of G. Then

$$\sum_{i=1}^{k} |H_i| - k \leqslant \operatorname{edim}(G[H_1, H_2, \dots, H_k]) \leqslant \operatorname{edim}(G) + \sum_{i=1}^{k} |H_i| - k.$$

Proof. First we show that at most one vertex from each graph H_i can be removed from any edge metric generator of $G[H_1, H_2, \dots, H_k]$). To do this, assume that S is an edge metric generator of $G[H_1, H_2, \dots, H_k]$ and let $h_1, \dots, h_k$ be arbitrary vertices in $H_1, \dots, H_k$, respectively, such that $h_1, \dots, h_k \notin S$. For each $1 \leqslant i \leqslant k$, we show that $H_i - \{h_i\} \subseteq S$. Suppose on the contrary that there exists $h_i' \in H_i$ with $h_i' \neq h_i$ such that $h_i' \notin S$. Since $G[H_1, H_2, \dots, H_k]$ is connected, we have the edges $e_1 = h_i x$ and $e_2 = h_i' x$, for some vertex $x \in V(G)$. We have $\mathrm{d}(h_i, s) = \mathrm{d}(h_i', s)$, for each $s \in S$. So we have $r(e_1|S) = r(e_2|S)$, which implies that S is not an edge metric generator of $G[H_1, H_2, \dots, H_k]$ and this is impossible. Therefore we have

$$\sum_{i=1}^{k} |H_i| - k \leqslant \mathrm{edim}(G[H_1, H_2, \ldots, H_k]).$$

Clearly the induced subgraph on vertex set $\{h_1, \ldots, h_k\}$ is isomorphic to G. So we may assume that $V(G) = \{h_1, \ldots, h_k\}$. Now, let $S = S' \cup \cup_{i=1}^{k} (H_i - \{h_i\})$, where S' is an edge metric generator for G of smallest order. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two distinct edges of $G[H_1, H_2, \ldots, H_k]$. Clearly if $x_1, y_1 \in S$ or $x_2, y_2 \in S$, then e_1 and e_2 have distinct edge metric S-representations. Also if $x_1, y_1 \notin S$ and $x_2, y_2 \notin S$, then since $e_1, e_2 \in E(G)$, e_1 and e_2 have distinct edge metric S'-representations and so they have distinct edge metric S-representations. Now, without loss of generality, assume that $x_1 \in S$ and $y_1, y_2 \notin S$. Hence $y_1, y_2 \in V(G)$. If $x_1 \neq x_2$, then the component corresponding to x_1 in $r(e_1|S)$ is zero which is not zero in $r(e_2|S)$, and so $r(e_1|S) \neq r(e_2|S)$. So let $x_1 = x_2$. If $x_1 \in V(G)$, then, since $e_1, e_2 \in E(G)$, we have $r(e_1|S) \neq r(e_2|S)$. Now, let $x_1 \notin V(G)$. Then there exists $h_i \in V(G)$ such that $\mathrm{d}(u, x_1) = \mathrm{d}(u, h_i)$, for each $u \neq x_1, h_i$. So $\mathrm{d}(e_1, u) = \mathrm{d}(y_1h_i, u)$ and $\mathrm{d}(e_2, u) = \mathrm{d}(y_2h_i, u)$, for each $u \neq x_1, h_i$. Since $y_1h_i, y_2h_i \in E(G)$, there exists $s' \in S'$ such that $\mathrm{d}(y_1h_i, s') \neq \mathrm{d}(y_2h_i, s')$. Thus $\mathrm{d}(e_1, s') \neq \mathrm{d}(e_2, s')$ because $s' \neq x_1, h_i$. Hence $r(e_1|S) \neq r(e_2|S)$. So

$$\mathrm{edim}(G[H_1, H_2, \ldots, H_k]) \leqslant \mathrm{edim}(G) + \sum_{i=1}^{k} |H_i| - k.$$

QED

In the following theorem, we determine the edge metric dimension of $G[H_1, H_2, \ldots, H_n]$, in the case that G is the path P_n. Note that $\mathrm{edim}(G) = 1$ if and only if G is a path.

Theorem 2.1. Assume that $G[H_1, H_2, \ldots, H_n]$ is a blow up of a graph G, where G is the path on $n \geqslant 2$ vertices. Then $\mathrm{edim}(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |H_i| - n + 1$, if one of the following conditions holds.

- (i) $n = 2$ and there exists $1 \leqslant i \leqslant 2$ such that H_i is a complete graph with at least two vertices.
- (ii) $n = 3$.
- (iii) $n \geqslant 4$ and there is only one H_i with at least two vertices, or there are only two H_i's with at least two vertices of the form H_i, H_{i+1} or H_i, H_{i+2}, or there are only three consecutive H_i's with at least two vertices.

Otherwise, $\mathrm{edim}(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |H_i| - n$.

Proof. Let $h_1, \ldots, h_n$ be arbitrary vertices in $H_1, \ldots, H_n$, respectively. Set $S = \bigcup_{i=1}^{n}(H_i - \{h_i\})$. By Proposition 2.1, every edge metric generator of $G[H_1, H_2, \ldots, H_n]$, contains S. If $n = 2$ and H_1 and H_2 are empty graphs, then clearly $\mathrm{edim}(G[H_1, H_2]) = \sum_{i=1}^{2} |H_i| - 2$. If H_1 is a complete graph and $h_1, h_1' \in V(H_1)$, then the edges $h_1'h_1$ and $h_1'h_2$ have the same edge metric S-representations. So in this situation $S \cup \{h_1\}$ is an edge metric generator of $G[H_1, H_2]$ of smallest order, and hence $\mathrm{edim}(G[H_1, H_2]) = \sum_{i=1}^{2} |H_i| - 1$. If $n = 3$, then the edges h_1h_2 and h_2h_3 have the same edge metric S-representations. So in this situation we also have $\mathrm{edim}(G[H_1, H_2, H_3]) = \sum_{i=1}^{3} |H_i| - 2$. Now let $n \geqslant 4$. If there is only one H_i with at least two vertices, then either the edges $h_ih_{i+1}, h_{i+1}h_{i+2}$ or $h_{i-2}h_{i-1}, h_{i-1}h_i$ have the same edge metric S-representations, where $1 \leqslant i \leqslant n$. If there are only two H_i's with at least two vertices of the form H_i, H_{i+1} or H_i, H_{i+2}, then either the edges $h_ih_{i+1}, h_{i+1}h_{i+2}$ or $h_{i-1}h_i, h_ih_{i+1}$ have the same edge metric S-representations, where $1 \leqslant i \leqslant n$. If there are only three consecutive H_i's with at least two vertices, say H_i, H_{i+1} and H_{i+2}, then the edges h_ih_{i+1} and $h_{i+1}h_{i+2}$ have the same edge metric S-representations. Therefore in these situations $S \cup \{h_1\}$ is an edge metric generator of $G[H_1, H_2, \ldots, H_n]$ of smallest order, and hence we have $\mathrm{edim}(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |H_i| - n + 1$.

Otherwise, S is an edge metric basis of $G[H_1, H_2, \ldots, H_n]$, and so

$$\mathrm{edim}(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |H_i| - n.$$

QED

In the following proposition, we investigate the edge metric dimension of a blow up of the complete graph K_n. Note that the edge metric dimension of the complete graph K_n is $n - 1$.

Proposition 2.2. Let $G[H_1, H_2, \ldots, H_n]$ be a blow up of a graph G, where $G \cong K_n$ and $n > 2$. Then $\mathrm{edim}(G[H_1, H_2, \ldots, H_n]) = \sum_{i=1}^{n} |H_i| - 1$.

Proof. Assume that $h_1, \ldots, h_n$ are arbitrary vertices in $H_1, \ldots, H_n$, respectively. Let $S = \bigcup_{i=1}^{n}(H_i - \{h_i\})$ and S' be an edge metric generator of $G[H_1, H_2, \ldots, H_n]$. By Proposition 2.1, we have $S \subseteq S'$. If there exist distinct h_i and h_j with $1 \leqslant i, j \leqslant n$ such that $h_i, h_j \notin S'$, then the edges h_ih_t and h_jh_t have the same edge metric S'-representations, where $t \neq i, j$. Thus S' must contain all of the h_i's except one of them. Hence the result holds. QED

In the following theorem, we study the Wiener index and hyper-Wiener index of $P_n[H_1, H_2, \ldots, H_n]$.

Theorem 2.2. Let $G = P_n[H_1, H_2, \ldots, H_n]$, where $n > 1$. Then

$$W(G) = \frac{1}{2}\sum_{i=1}^{n} |H_i| D_{H_i}$$

and

$$WW(G) = \frac{1}{4}\sum_{i=1}^{n} |H_i| D_{H_i} + \frac{1}{4}\sum_{i=1}^{n} |H_i| D'_{H_i},$$

where for $1 \leqslant i \leqslant n$,

$$D_{H_i} = \begin{cases} t - 1 + \sum_{j=1}^{n} |i-j||H_j| & H_i \cong K_t \\ 2(t-1) + \sum_{j=1}^{n} |i-j||H_j| & H_i \cong \overline{K_t} \end{cases}$$

and

$$D'_{H_i} = \begin{cases} t - 1 + \sum_{j=1}^{n} (i-j)^2|H_j| & H_i \cong K_t \\ 4(t-1) + \sum_{j=1}^{n} (i-j)^2|H_j| & H_i \cong \overline{K_t}. \end{cases}$$

Proof. Assume that $h \in H_i$, for some $1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned} \sum_{x \in V(G)} \mathrm{d}(h,x) &= \sum_{x \in H_i} \mathrm{d}(h,x) + \sum_{x \in V(G)\setminus H_i} \mathrm{d}(h,x) \\ &= \sum_{x \in H_i} \mathrm{d}(h,x) + \sum_{j=1}^{n} |i-j||H_j| \\ &= \begin{cases} t - 1 + \sum_{j=1}^{n} |i-j||H_j| & H_i \cong K_t \\ 2(t-1) + \sum_{j=1}^{n} |i-j||H_j| & H_i \cong \overline{K_t}. \end{cases} \end{aligned}$$

It is easy to see that for each $h, h' \in H_i$, $\sum_{x\in V(G)} \mathrm{d}(h,x) = \sum_{x \in V(G)} \mathrm{d}(h',x)$. Let $D_{H_i} = \sum_{x \in V(G)} \mathrm{d}(h,x)$, for some $h \in H_i$. Thus we have

$$\sum_{h \in H_i} \sum_{x \in V(G)} \mathrm{d}(h,x) = |H_i| D_{H_i}.$$

Therefore

$$W(G) = \frac{1}{2}\sum_{i=1}^{n} |H_i| D_{H_i}.$$

Now let $h \in H_i$, for some $1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned}\sum_{x\in V(G)} \mathrm{d}^2(h,x) &= \sum_{x\in H_i} \mathrm{d}^2(h,x) + \sum_{x\in V(G)\setminus H_i} \mathrm{d}^2(h,x)\\ &= \sum_{x\in H_i} \mathrm{d}^2(h,x) + \sum_{j=1}^{n}(i-j)^2|H_j|\\ &= \begin{cases} t-1+\sum_{j=1}^{n}(i-j)^2|H_j| & H_i\cong K_t\\ 4(t-1)+\sum_{j=1}^{n}(i-j)^2|H_j| & H_i\cong \overline{K_t}.\end{cases}\end{aligned}$$

Clearly for each $h,h'\in H_i$, $\sum_{x\in V(G)}\mathrm{d}^2(h,x)=\sum_{x\in V(G)}\mathrm{d}^2(h',x)$. Let $D'_{H_i}=\sum_{x\in V(G)}\mathrm{d}^2(h,x)$, for some $h\in H_i$. Thus we have

$$\sum_{h\in H_i}\sum_{x\in V(G)} \mathrm{d}^2(h,x) = |H_i|D'_{H_i}.$$

Therefore

$$WW(G)=\frac{1}{4}\sum_{i=1}^{n}|H_i|D_{H_i}+\frac{1}{4}\sum_{i=1}^{n}|H_i|D'_{H_i}.$$

QED

We end this section with the following theorem which determines the Wiener index and hyper-Wiener index of $K_n[H_1,H_2,\ldots,H_n]$.

Theorem 2.3. Let $G=K_n[H_1,H_2,\ldots,H_n]$, where $n>1$. Then

$$W(G)=\frac{1}{2}\sum_{i=1}^{n}|H_i|D_{H_i}$$

and

$$WW(G)=\frac{1}{4}\sum_{i=1}^{n}|H_i|D_{H_i}+\frac{1}{4}\sum_{i=1}^{n}|H_i|D'_{H_i},$$

where for $1\leqslant i\leqslant n$,

$$D_{H_i}=\begin{cases} t-1+\sum_{j=1,j\neq i}^{n}|H_j| & H_i\cong K_t\\ 2(t-1)+\sum_{j=1,j\neq i}^{n}|H_j| & H_i\cong \overline{K_t}\end{cases}$$

and

$$D'_{H_i}=\begin{cases} t-1+\sum_{j=1,j\neq i}^{n}|H_j| & H_i\cong K_t\\ 4(t-1)+\sum_{j=1,j\neq i}^{n}|H_j| & H_i\cong \overline{K_t}.\end{cases}$$

Proof. Let $h, h' \in H_i$, for some $1 \leqslant i \leqslant n$. Then we have

$$\begin{aligned}
\sum_{x\in V(G)} \mathrm{d}(h,x) &= \sum_{x\in H_i} \mathrm{d}(h,x) + \sum_{x\in V(G)\setminus H_i} 1 \\
&= \sum_{x\in H_i} \mathrm{d}(h,x) + \sum_{j=1,j\neq i}^{n} |H_j| \\
&= \begin{cases} t-1+\sum_{j=1,j\neq i}^{n} |H_j| & H_i \cong K_t \\ 2(t-1)+\sum_{j=1,j\neq i}^{n} |H_j| & H_i \cong \overline{K_t} \end{cases} \\
&= \sum_{x\in V(G)} \mathrm{d}(h',x)
\end{aligned}$$

Let $D_{H_i} = \sum_{x\in V(G)} \mathrm{d}(h,x)$, for some $h \in H_i$. Thus we have

$$\sum_{h\in H_i} \sum_{x\in V(G)} \mathrm{d}(h,x) = |H_i| D_{H_i}.$$

Therefore

$$W(G) = \frac{1}{2} \sum_{i=1}^{n} |H_i| D_{H_i}.$$

Now, for $h, h' \in H_i$, where $1 \leqslant i \leqslant n$, we have

$$\begin{aligned}
\sum_{x\in V(G)} \mathrm{d}^2(h,x) &= \sum_{x\in H_i} \mathrm{d}^2(h,x) + \sum_{x\in V(G)\setminus H_i} 1 \\
&= \sum_{x\in H_i} \mathrm{d}^2(h,x) + \sum_{j=1,j\neq i}^{n} |H_j| \\
&= \begin{cases} t-1+\sum_{j=1,j\neq i}^{n} |H_j| & H_i \cong K_t \\ 4(t-1)+\sum_{j=1,j\neq i}^{n} |H_j| & H_i \cong \overline{K_t} \end{cases} \\
&= \sum_{x\in V(G)} \mathrm{d}^2(h',x)
\end{aligned}$$

Let $D'_{H_i} = \sum_{x\in V(G)} \mathrm{d}^2(h,x)$, for some $h \in H_i$. Thus we have

$$\sum_{h\in H_i} \sum_{x\in V(G)} \mathrm{d}^2(h,x) = |H_i| D'_{H_i}.$$

Therefore

$$WW(G) = \frac{1}{4} \sum_{i=1}^{n} |H_i| D_{H_i} + \frac{1}{4} \sum_{i=1}^{n} |H_i| D'_{H_i}.$$

QED

3 The zero divisor graph of $\mathbb{Z}_n$

Let R be a commutative ring with nonzero identity. We denote the set of all zero divisors of R by $Z(R)$, and by $Z^*(R)$ we denote the set $Z(R) - \{0\}$. Also, for $n > 1$, $\mathbb{Z}_n$ denotes the ring of integers modulo n.

The concept of the *zero-divisor graph* of a commutative ring was introduced by Beck [3], whose work was mostly concerned with coloring of rings. The zero-divisor graph of various algebraic structures has been studied by several authors; see [1], [2], [7], [13] and [17].

The zero-divisor graph $\Gamma(R)$ is a graph with vertex set $Z^*(R)$ and two distinct vertices a and b are adjacent if and only if $ab = 0$. Clearly if R is an integral domain, then $Z^*(R)$ is an empty set and so $\Gamma(R)$ has no vertices. Hence, in the following, we assume that R is not an integral domain.

We begin this section with the following proposition.

Proposition 3.1. Let G be a connected graph with $\operatorname{diam}(G) = m < \infty$. If $\operatorname{edim}(G) = k < \infty$, then $|E(G)| \leqslant (m+1)^k$.

Proof. Let S be an edge metric basis for G with $|S| = k$. Since $\operatorname{diam}(G) = m$, for each edge e and for each $x \in S$, we have $d(e, x) \in \{0, 1, \ldots, m\}$. So for each edge e, $r(e|S)$ is a k-coordinate vector where each coordinate belongs to the set $\{0, 1, \ldots, m\}$. Thus there are only $(m+1)^k$ possibilities for $r(e|S)$. Since $r(e|S)$ is unique for each $e \in E(G)$, we have $|E(G)| \leqslant (m+1)^k$. QED

Corollary 3.1. Let G be a connected graph with finite diameter. Then $\operatorname{edim}(G)$ is finite if and only if G is finite.

Theorem 3.1. Let R be a commutative ring which is not an integral domain. Then $\operatorname{edim}(\Gamma(R))$ is finite if and only if R is finite.

Proof. By [2, Theorem 2.3], $\Gamma(R)$ is a connected graph with diameter less than four. Now Corrollary 3.1 implies that $\operatorname{edim}(\Gamma(R))$ is finite if and only if $\Gamma(R)$ is finite. By [2, Theorem 2.2], we have that R is finite. QED

In the following, for two integers r and s, the notation (r, s) stands for the greatest common divisor of r and s. Also we denote the elements of the ring $\mathbb{Z}_n$, where $n > 1$, by $0, 1, 2, \ldots, n-1$. For every nonzero element a in $\mathbb{Z}_n$, if $(a, n) = 1$, then a is a unit element; otherwise, $(a, n) \neq 1$, and so a is a zero divisor. Therefore, $|U(\mathbb{Z}_n)| = \phi(n)$ and $|Z(\mathbb{Z}_n)| = n - \phi(n)$, where ϕ is Euler's totient function.

An integer d is said to be a *proper divisor* of n if $1 < d < n$ and $d \mid n$. Now let $d_1, d_2, \ldots, d_k$ be the distinct proper divisors of n. For $1 \leqslant i \leqslant k$, set

$$A_{d_i} := \{x \in \mathbb{Z}_n \mid (x, n) = d_i\}.$$

Clearly, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_k}$ are pairwise disjoint and it follows that

$$Z^*(\mathbb{Z}_n) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}$$

and

$$V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_k}.$$

The following lemma is stated from [22].

Lemma 3.1. [22, Proposition 2.1] Let $1 \leqslant i \leqslant k$. Then $|A_{d_i}| = \phi(\frac{n}{d_i})$.

In the following lemma, which is from [7], the adjacency of vertices in $\Gamma(\mathbb{Z}_n)$ is described.

Lemma 3.2. For $i, j \in \{1, 2, \ldots, k\}$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $d_i d_j$.

In the rest of the paper, the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the set A_{d_i} is denoted by $\Gamma(A_{d_i})$, where $1 \leqslant i \leqslant k$.

By Lemma 3.2, it is easy to see that for $i \in \{1, 2, \ldots, k\}$, the induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is either the complete graph $K_{\phi(\frac{n}{d_i})}$ or its complement graph $\overline{K}_{\phi(\frac{n}{d_i})}$. In fact, $\Gamma(A_{d_i})$ is $K_{\phi(\frac{n}{d_i})}$ if and only if n divides d_i^2. Moreover, for $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, a vertex of A_{d_i} is adjacent to either all or none of the vertices of A_{d_j} in $\Gamma(\mathbb{Z}_n)$.

In the following, we denote by G_n the simple graph whose vertices are the proper divisors $\{d_1, d_2, \ldots, d_k\}$ of n, and two distinct vertices d_i and d_j are adjacent if and only if n divides $d_i d_j$. By [7, Lemma 2.6], G_n is a connected graph, and also $\Gamma(\mathbb{Z}_n)$ is connected. Now, we have

$$\Gamma(\mathbb{Z}_n) = G_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \ldots, \Gamma(A_{d_k})],$$

which means that $\Gamma(\mathbb{Z}_n)$ is a blow up of the graph G_n.

In the following theorem, we study the edge metric dimension of $\Gamma(\mathbb{Z}_n)$.

Theorem 3.2. Let $d_1, d_2, \ldots, d_k$ be the proper divisors of n. Then the edge metric dimension of $\Gamma(\mathbb{Z}_n)$ satisfies the following inequality:

$$\sum_{i=1}^{k} \phi(\frac{n}{d_i}) - k \leqslant \mathrm{edim}(\Gamma(\mathbb{Z}_n)) \leqslant \mathrm{edim}(G_n) + \sum_{i=1}^{k} \phi(\frac{n}{d_i}) - k.$$

Proof. Since $\Gamma(\mathbb{Z}_n)$ is the blow up of the graph G_n, the result follows from Proposition 2.1 and Lemma 3.1. QED

Example 3.3. Consider the ring $\mathbb{Z}_{12}$. We have $d_1 = 2, d_2 = 3, d_3 = 4$, and $d_4 = 6$. Then G_{12} is the graph $2 \sim 6 \sim 4 \sim 3$, which is isomorphic to P_4. Now we have

$$\Gamma(\mathbb{Z}_{12}) = G_{12}[\Gamma(A_2), \Gamma(A_3), \Gamma(A_4), \Gamma(A_6)],$$

where $\Gamma(A_2) = \overline{K}_2$, $\Gamma(A_3) = \overline{K}_2$, $\Gamma(A_4) = \overline{K}_2$, and $\Gamma(A_6) = K_1$. By Theorem 2.1 we have $\text{edim}(\Gamma(\mathbb{Z}_{12})) = 4$, and by Theorem 2.2 we have $W(\Gamma(\mathbb{Z}_{12})) = 34$ and $WW(\Gamma(\mathbb{Z}_{12})) = 49$.

Example 3.4. Let p and q be distinct prime numbers. We discuss the edge metric dimension, Wiener index and hyper-Wiener index of $\Gamma(\mathbb{Z}_n)$ for (i) $n = pq$ and (ii) $n = p^2q$.

(i) Let $n = pq$, where p and q are distinct prime numbers with $p < q$. It follows that G_{pq} is $p \sim q$ and that $\Gamma(\mathbb{Z}_{pq}) = G_{pq}[\Gamma(A(p)), \Gamma(A(q))]$, where $\Gamma(A(p)) = \overline{K}_{\phi(q)}$ and $\Gamma(A(q)) = \overline{K}_{\phi(p)}$. Now, by Theorem 2.1 we have $\text{edim}(\Gamma(\mathbb{Z}_{pq})) = p+q-3$. Also by Theorem 2.3, we have $W(\Gamma(\mathbb{Z}_{pq})) = p^2+q^2+pq-4p-4q+5$ and $WW(\Gamma(\mathbb{Z}_{pq})) = \frac{1}{2}W(\Gamma(\mathbb{Z}_{pq}))+\frac{2p^2+2q^2+pq-7q-7p+9}{2} = \frac{3p^2+3q^2+2pq-11q-11p+14}{2}$.

(ii) Let $n = p^2q$. We know that p, q, p^2, and pq are the proper divisors of p^2q. So, the graph G_{p^2q} is $p \sim pq \sim p^2 \sim q$ and

$$\Gamma(\mathbb{Z}_{p^2q}) = G_{p^2q}[\Gamma(A(p)), \Gamma(A(pq)), \Gamma(A(p^2)), \Gamma(A(q))].$$

The graphs $\Gamma(A(p)) = \overline{K}_{\phi(pq)} = \overline{K}_{pq-p-q+1}$, $\Gamma(A(p^2)) = \overline{K}_{\phi(q)} = \overline{K}_{q-1}$, $\Gamma(A(q)) = \overline{K}_{\phi(p^2)} = \overline{K}_{p^2-p}$ and $\Gamma(A(pq)) = K_{\phi(p)} = K_{p-1}$. Now, by Theorem 2.1 we have $\text{edim}(\Gamma(\mathbb{Z}_{p^2q})) = pq + p^2 - p - 5$ and, by Theorem 2.2, we have

$$\begin{aligned}2W(\Gamma(\mathbb{Z}_{p^2q})) = {} & \phi(pq)(2(\phi(pq) - 1) + \phi(p) + 2\phi(q) + 3\phi(p^2)) \\ & + \phi(p)(\phi(p) - 1 + \phi(pq) + \phi(q) + 2\phi(p^2)) \\ & + \phi(q)(2(\phi(q) - 1) + 2\phi(pq) + \phi(q) + \phi(p^2)) \\ & + \phi(p^2)(2(\phi(p^2) - 1) + 3\phi(pq) + 2\phi(p) + \phi(q)).\end{aligned}$$

Also by Theorem 2.2, the hyper-Wiener index of $\Gamma(\mathbb{Z}_{p^2q})$ can be determined.

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