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Two-sided Gaussian bounds for fundamental solutions of non-divergence form parabolic operators with Hölder continuous coefficients

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Abstract. We establish two-sided Gaussian bounds for fundamental solutions of general non-divergence form parabolic operators with Hölder continuous coefficients. The result we obtain is essentially based on parametrix method.

Keywords: Non-divergence form parabolic operator, fundamental solution, parametrix, two-sided Gaussian bounds, Dirichlet-Green function, Neumann-Green function.

MSC 2020 classification: 65M80

1 Introduction

1.1 Statement of the main result

The tremendous literature on Gaussian bounds for fundamental solutions of second order parabolic operators can be splitted into two classes: divergence or non-divergence operators. In the first class we only quote the deep results obtained by Aronson, following Nash's ideas, and we refer to [7] for a comprehensive treatment. The second class is more classical and can be found in the books [8, 9] where a fundamental solution is constructed, via the parametrix method, assuming Hölder continuity of the coefficients. By construction the fundamental solution satisfies precise upper bounds but, strangely enough, lower bounds are not proved. In this note we show that the parametrix method produces also lower bounds.

Gaussian bounds for the fundamental solution of parabolic operators in non-divergence form are known in more generality. We refer to [5, Theorem 1.2] which, combined with [6, Remark 3.3], gives two sided gaussian bounds for time independent parabolic operators with VMO coefficients and to [2], where the authors prove two-sided gaussian estimates local in time even for operators with a non-local part.

Nevertheless, we believe that the proof given below is worth mentioning, since it fits to the classical theory.

Let $P = \mathbb{R}^n_x \times \mathbb{R}_t$ and set

$$Q = \{(x,t,\xi,\tau);\ (x,t),(\xi,\tau) \in P,\ \tau < t\}.$$

The space of continuous and bounded functions $f : P \to \mathbb{R}$ is denoted by $C^0_b(P)$.

Let $f \in C^0_b(P)$. We say that f is Hölder continuous with exponent α, $0 < \alpha \le 1$, if

$$[f]_\alpha = \sup\left\{ \frac{|f(x,t)-f(x',t')|}{|(x-x',t-t')|_\alpha},\ (x,t),\ (x',t') \in P,\ (x,t) \neq (x',t') \right\} < \infty,$$

where

$$|(x-x',t-t')|_\alpha = \left(|x-x'|^2 + |t-t'|\right)^{\alpha/2}.$$

We define

$$C^\alpha(P) = \{f \in C^0_b(P);\ [f]_\alpha < \infty\}.$$

$C^\alpha(P)$ is a Banach space when it is endowed with its natural norm

$$\|f\|_\alpha = \|f\|_\infty + [f]_\alpha$$

and we also use the notation

$$\{f\}_\alpha = \sup\left\{ \frac{|f(x,t)-f(x',t)|}{|x-x'|^\alpha};\ x,x' \in \mathbb{R}^n,\ x \neq x' \text{ and } t \in \mathbb{R} \right\}.$$

We consider the second order parabolic operator

$$L = \sum_{i,j=1}^n a_{ij}(x,t)\partial^2_{ij} + \sum_{i=1}^n b_i(x,t)\partial_i + q(x,t) - \partial_t \tag{1.1}$$

with the following assumptions on its coefficients.

(a1) $a_{ij} \in C^\alpha(P)$, $1 \le i,j \le n$.

(a2) The matrix $\mathbf{a}(x,t) = (a_{ij}(x,t))$, $(x,t) \in P$, is symmetric, real-valued, and there exist constants $\kappa, M > 0$ so that

$$\kappa|\eta|^2 \le \langle \mathbf{a}(x,t)\eta,\eta\rangle \le M|\eta|^2,\ \ (x,t) \in P,\ \eta \in \mathbb{R}^n.$$

(a3) b_i, $q \in C_b^0(P)$, $1 \le i \le n$.
(a4) There exists a constant $N_1 > 0$ so that

$$\sum_{i,j=1}^{n} [a_{ij}]_\alpha \le N_1.$$

(a5) There exists a constant $N_2 > 0$ so that

$$\sum_{i=1}^{n} \|b_i\|_\infty + \|q\|_\infty \le N_2.$$

(a6) $\{b_i\}_\alpha < \infty$, $1 \le i \le n$, and $\{q\}_\alpha < \infty$.

Henceforth we use for notational convenience $\mathfrak{D}$ for $(n, \alpha, N_1, N_2, M, \kappa)$.

In this paper, the fundamental solution constructed by the parametrix method is denoted by $E = E(x, t; \xi, \tau)$, $(x, t, \xi, \tau) \in Q$. Recall that E is a fundamental solution if $E \in C^2(Q)$, $LE = 0$ and

$$\lim_{t \to \tau} \int_{\mathbb{R}^n} E(x, t; \xi, \tau) f(\xi) d\xi = f(x), \quad f \in C_0^\infty(\mathbb{R}^n).$$

Theorem 1.1. Let

$$c = \frac{1}{8M} \quad \text{and} \quad d = \frac{4 \ln \left[e 2^{3n} (M\kappa^{-1})^{n/2} \Gamma(n/2 + 1) \right]}{\kappa}.$$

Under assumptions (a1) to (a6), there exist four constants $\aleph_i = \aleph_i(\mathfrak{D})$, $i = 0, 1, 2, 3$, $\aleph_0 > 0$, $\aleph_1 \ge 0$, $\aleph_2 > 0$ and $\aleph_3 \ge 0$, such that

$$\aleph_0 e^{-\aleph_1 (t-\tau)} (t-\tau)^{-\frac{n}{2}} e^{-d \frac{|x-\xi|^2}{t-\tau}} \le E(x,t;\xi,\tau) \le \aleph_2 e^{\aleph_3 (t-\tau)} (t-\tau)^{-\frac{n}{2}} e^{-c \frac{|x-\xi|^2}{t-\tau}}, \tag{1.2}$$

for all $(x, t, \xi, \tau) \in Q$.

Remark 1.1. By inspecting the proof of Theorem 1.1 we see that, in the Gaussian upper bound, we can substitute c by $c^\epsilon = \frac{\epsilon}{4M}$, $0 < \epsilon < 1$, and $\aleph_i$ by $\aleph_i^\epsilon$, $i = 2, 3$, with an explicit dependence of $\aleph_2^\epsilon$ and $\aleph_3^\epsilon$ on ϵ.

1.2 Consequences

Let Ω be a $C^{1,1}$-bounded domain of $\mathbb{R}^n$. We denote the parabolic Dirichlet-Green (resp. Neumann-Green) function on Ω by G_Ω^D (resp. G_Ω^N).

It is well known that, according to the maximum principle, $0 \le G_\Omega^D \le E$. Therefore as a consequence of Theorem 1.1, we have

Corollary 1.1. Let the coefficients of L satisfy assumptions (a1) to (a6). Then the Dirichlet-Green function G_Ω^D satisfies

$$0 \le G_\Omega^D(x,t;\xi,\tau) \le \aleph_2 e^{\aleph_3(t-\tau)}(t-\tau)^{-\frac{n}{2}} e^{-c\frac{|x-\xi|^2}{t-\tau}}, \quad (x,t,\xi,\tau) \in Q,$$

where the constants in this inequality are the same as in Theorem 1.1.

We say that Ω satisfies the chain condition if there exists a constant $\varpi > 0$ such that for any two points x, $y \in \Omega$ and for any positive integer m there exists a sequence $(x_i)_{0\le i\le m}$ of points in Ω such that $x_0 = x$, $x_m = y$ and

$$|x_{i+1} - x_i| \le \frac{\varpi}{m}|x-y|, \ \ i = 0, \ldots, m-1.$$

The sequence $(x_i)_{0\le i\le m}$ is named a chain connecting x and y.

Since any bounded Lipschitz domain has the chain condition (see [12, Proposition A.1]), an adaptation of the proof of [3, Theorem 3.1] (see also [4]) and the reproducing property enable us to get the following result.

Corollary 1.2. If the coefficients of L satisfy assumptions (a1) to (a6) then there exist five constants $c_0 = c_0(\mathfrak{D})$ and $\aleph_i = \aleph_i(\mathfrak{D}) > 0$, $i = 0, 1, 2, 3$, such that

$$\aleph_0 e^{-\aleph_1(t-\tau)}(t-\tau)^{-\frac{n}{2}} e^{-c_0\frac{|x-\xi|^2}{t-\tau}} \le G_\Omega^N(x,t;\xi,\tau) \le \aleph_2 e^{\aleph_3(t-\tau)}(t-\tau)^{-\frac{n}{2}} e^{-c\frac{|x-\xi|^2}{t-\tau}},$$

for all $(x,t,\xi,\tau) \in Q$, where c is as in Theorem (1.1).

2 Preliminaries

In this section the coefficients of L satisfy assumptions (a1) to (a5).

2.1 Basic properties of generalized Gaussian kernels

In the sequel we frequently use

$$\int_{\mathbb{R}} e^{-\rho^2} d\rho = \sqrt{\pi}. \tag{2.1}$$

The Gaussian heat kernel is defined as follows

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \ t > 0. \tag{2.2}$$

We have, according to Fubini's theorem,

$$\int_{\mathbb{R}^n} G(x,t)dx = \left(\int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}} dy\right)^n, \quad t > 0.$$

Then the change of variable $\rho = \frac{y}{2\sqrt{t}}$ yields

$$\int_{\mathbb{R}^n} G(x,t)dx = 1, \quad t > 0, \tag{2.3}$$

where we used the value of the Gauss integral (2.1).

If $\mathbf{a} = (a^{ij})$ is $n \times n$ symmetric positive definite matrix, we define the generalized Gaussian heat kernel by

$$G_{\mathbf{a}}(x,t) = \frac{\sqrt{\det \mathbf{a}}}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\langle \mathbf{a}x, x\rangle}{4t}}, \ \ x \in \mathbb{R}^n, \ t > 0. \tag{2.4}$$

Let $\mathbf{d} = \operatorname{diag}(d_1, \ldots, d_n)$ be a diagonal matrix and $\mathbf{u}$ an orthogonal matrix, that is $\mathbf{u}^t\mathbf{u} = I$, so that $\mathbf{u}\mathbf{a}\mathbf{u}^t = \mathbf{d}$. Then

$$\langle \mathbf{a}x, x\rangle = \langle \mathbf{d}\mathbf{u}x, \mathbf{u}x\rangle, \quad \det \mathbf{a} = \prod_{i=1}^n d_i$$

and

$$\int_{\mathbb{R}^n} G_{\mathbf{a}}(x,t)dx = \int_{\mathbb{R}^n} \frac{\sqrt{\det \mathbf{a}}}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\langle \mathbf{d}\mathbf{u}x, \mathbf{u}x\rangle}{4t}} dx, \quad t > 0.$$

Since $|\det \mathbf{u}| = 1$, the change of variable $y = \mathbf{u}x$ gives

$$\int_{\mathbb{R}^n} G_{\mathbf{a}}(x,t)dx = \int_{\mathbb{R}^n} \frac{\sqrt{\det \mathbf{a}}}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{\langle \mathbf{d}x, x\rangle}{4t}} dx, \quad t > 0.$$

Applying again Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} G_{\mathbf{a}}(x,t)dx &= \sqrt{\det \mathbf{a}} \prod_{j=1}^n \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{d_i\rho^2}{4t}} d\rho \\ &= \sqrt{\det \mathbf{a}} \prod_{j=1}^n \int_{\mathbb{R}} \frac{1}{2\sqrt{d_i\pi t}} e^{-\frac{\rho^2}{4t}} d\rho \\ &= \prod_{j=1}^n \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi t}} e^{-\frac{\rho^2}{4t}} d\rho = 1, \quad t > 0. \end{aligned} \tag{2.5}$$

It is straightforward to check that $G_{\mathbf{a}} \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and, since

$$\partial_k \langle \mathbf{a}x, x\rangle = 2\sum_{j=1}^n a^{kj}x_j = 2(\mathbf{a}x)_k, \quad x \in \mathbb{R}^n,$$

we have

$$\partial_k G_{\mathbf{a}}(x,t) = -\frac{1}{2t}G_{\mathbf{a}}(x,t)(\mathbf{a}x)_k, \quad x \in \mathbb{R}^n,\ t>0. \tag{2.6}$$

We easily derive from (2.6)

$$\partial^2_{k\ell} G_{\mathbf{a}}(x,t) = \frac{1}{4t^2}G_{\mathbf{a}}(x,t)(\mathbf{a}x)_k(\mathbf{a}x)_\ell - \frac{1}{2t}G_{\mathbf{a}}(x,t)a^{k\ell}, \quad x \in \mathbb{R}^n,\ t>0. \tag{2.7}$$

Let $\mathbf{a}^{-1} = (a_{ij})$. Inserting the identity

$$\sum_{k,\ell=1}^n a_{k\ell}(\mathbf{a}x)_k(\mathbf{a}x)_\ell = \langle \mathbf{a}^{-1}\mathbf{a}x, x\rangle = \langle \mathbf{a}x, x\rangle$$

in (2.7) we obtain

$$\sum_{k,\ell=1}^n a_{k\ell}\partial^2_{k\ell}G_{\mathbf{a}}(x,t) = \left(\frac{1}{4t^2}\langle \mathbf{a}x,x\rangle - \frac{n}{2t}\right)G_{\mathbf{a}}(x,t), \quad x \in \mathbb{R}^n,\ t>0. \tag{2.8}$$

On the other hand, it is straightforward to check that

$$\partial_t G_{\mathbf{a}}(x,t) = \left(\frac{1}{4t^2}\langle \mathbf{a}x,x\rangle - \frac{n}{2t}\right)G_{\mathbf{a}}(x,t), \quad x \in \mathbb{R}^n,\ t>0. \tag{2.9}$$

We define the parabolic operator $L_{\mathbf{a}^{-1}}$ by

$$L_{\mathbf{a}^{-1}} = \sum_{i,j=1}^n a_{ij}\partial^2_{ij} - \partial_t.$$

Comparing (2.8) and (2.9) we see that $G_{\mathbf{a}}$ satisfies

$$L_{\mathbf{a}^{-1}}G_{\mathbf{a}}(x,t) = 0, \quad x \in \mathbb{R}^n,\ t>0. \tag{2.10}$$

2.2 The parametrix

Let $\mathbf{a}^{-1}(x,t) = (a^{ij}(x,t))$, $(x,t) \in P$, where $(a^{ij}(x,t))$ is the inverse of the matrix $(a_{ij}(x,t))$, and define

$$Z(x,t;\xi,\tau) = G_{\mathbf{a}^{-1}(\xi,\tau)}(x-\xi, t-\tau), \quad (x,t,\xi,\tau) \in Q,$$

that is

$$Z(x,t;\xi,\tau)=\frac{\sqrt{\det\mathbf{a}^{-1}(\xi,\tau)}}{(4\pi(t-\tau))^{\frac{n}{2}}}e^{-\frac{\langle\mathbf{a}^{-1}(\xi,\tau)(x-\xi),(x-\xi)\rangle}{4(t-\tau)}},\quad (x,t,\xi,\tau)\in Q. \tag{2.11}$$

This function is usually called the parametrix associated to the parabolic operator L. According to the results of the previous subsection, for any $(\xi,\tau)\in P$, $Z(\cdot,\cdot;\xi,\tau)\in C^\infty(P_\tau)$ with $P_\tau=\{(x,t)\in\mathbb{R}^n;\ t>\tau\}$, and

$$\sum_{i,j=1}^{n}a_{ij}(\xi,\tau)\partial^2_{ij}Z(\cdot,\cdot;\xi,\tau)-\partial_t Z(\cdot,\cdot;\xi,\tau)=0 \ \text{ in } P_\tau. \tag{2.12}$$

Let us define

$$d_i(x,t;\xi,\tau)=-\frac{1}{2(t-\tau)}\sum_{j=1}^{n}a^{ij}(\xi,\tau)(x_j-\xi_j),$$
$$d_{ij}(x,t;\xi,\tau)=-\frac{a^{ij}(\xi,\tau)}{2(t-\tau)}+d_i(x,t;\xi,\tau)d_j(x,t;\xi,\tau).$$

From (2.6) and (2.7) we have

$$\partial_i Z=d_i Z \ \text{ and } \ \partial^2_{ij}Z=d_{ij}Z.$$

Therefore, taking into account (2.12), we have

$$LZ=\left[\sum_{i,j=1}^{n}\left(a_{ij}(x,t)-a_{ij}(\xi,\tau)\right)d_{ij}+\sum_{i=1}^{n}d_ib_i+q\right]Z=\Psi Z, \tag{2.13}$$

where

$$\Psi=\sum_{i,j=1}^{n}\left(a_{ij}(x,t)-a_{ij}(\xi,\tau)\right)d_{ij}+\sum_{i=1}^{n}d_ib_i+q.$$

We need a pointwise estimate for LZ. To this end, we start with the following lemma

Lemma 2.1. We have

$$|\mathbf{a}^{-1}(x,t)\eta|\le\frac{1}{\kappa}|\eta|,\quad (x,t)\in P,\ \eta\in\mathbb{R}^n, \tag{2.14}$$

$$\sup_{1\le i,j\le n}\|a^{ij}\|_\infty\le\frac{1}{\kappa}. \tag{2.15}$$

and

$$\frac{\langle\mathbf{a}^{-1}(x,\tau)(x-\xi),x-\xi\rangle}{4(t-\tau)}\ge\frac{1}{4M}\frac{|x-\xi]^2}{t-\tau}. \tag{2.16}$$

Proof. From assumption (a2), we have

$$\langle \mathbf{a}(x,t)\eta,\eta\rangle \geq \kappa|\eta|^2, \ \ (x,t)\in P, \ \eta\in\mathbb{R}^n.$$

In this inequality we get by substituting η by $\mathbf{a}^{-1}(x,t)\eta$

$$|\mathbf{a}^{-1}(x,t)\eta||\eta| \geq \langle \mathbf{a}^{-1}(x,t)\eta,\eta\rangle \geq \kappa|\mathbf{a}^{-1}(x,t)\eta|^2, \quad (x,t)\in P, \ \eta\in\mathbb{R}^n$$

and (2.14) follows.

Since $a^{ij} = \langle \mathbf{a}^{-1}\mathbf{e}_i, \mathbf{e}_j\rangle$, where $(\mathbf{e}_1,\ldots,\mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$, (2.15) follows from (2.14).

Finally, (2.16) is equivalent to $\langle \mathbf{a}^{-1}(x,\tau)\eta,\eta\rangle \geq \frac{1}{M}|\eta|^2$ or $\langle \mathbf{a}(x,\tau)\eta,\eta\rangle \leq M|\eta|^2$, which holds by assumption. QED

From (2.14), we get

$$\|d_i\|_\infty \leq \frac{|x-\xi|}{2\kappa(t-\tau)} \quad \text{or} \quad \|d_i\|_\infty \leq \frac{\varrho}{2\kappa\sqrt{t-\tau}}, \tag{2.17}$$

where

$$\varrho = \frac{|x-\xi|}{\sqrt{t-\tau}}.$$

It is easy to see that (2.15) and (2.17) entail

$$\|d_{ij}\|_\infty \leq \left(\frac{1}{2\kappa}+\frac{\varrho^2}{4\kappa^2}\right)\frac{1}{t-\tau}. \tag{2.18}$$

Hence

$$\left|\sum_{i,j=1}^n \left(a_{ij}(x,t)-a_{ij}(\xi,\tau)\right) d_{ij}\right| \leq N_1\left(\frac{1}{2\kappa}+\frac{\varrho^2}{4\kappa^2}\right)\frac{(1+\varrho^2)^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{2}}}. \tag{2.19}$$

On the other hand, we get from (2.17)

$$\left|\sum_{i=1}^n b_i d_i + q\right| \leq N_2\left(\frac{\varrho}{2k\sqrt{t-\tau}}+1\right) \leq N_2\frac{1+\frac{\varrho}{2\kappa}}{(t-\tau)^{1-\frac{\alpha}{2}}}, \quad t-\tau\leq 1. \tag{2.20}$$

In light of (2.19) and (2.20), we obtain

$$\|\Psi\|_\infty \leq N_1\left(\frac{1}{2\kappa}+\frac{\varrho^2}{4\kappa^2}\right)\frac{(1+\varrho^2)^{\frac{\alpha}{2}}}{(t-\tau)^{1-\frac{\alpha}{2}}} + N_2\frac{1+\frac{\varrho}{2\kappa}}{(t-\tau)^{1-\frac{\alpha}{2}}}, \quad t-\tau\leq 1. \tag{2.21}$$

Now (2.16) implies

$$|Z(x,t)| \leq \frac{1}{(4\kappa\pi(t-\tau))^{\frac{n}{2}}} e^{-\frac{1}{4M}\varrho^2}. \tag{2.22}$$

Recall that $c = \frac{1}{8M}$ and let

$$C = \frac{1}{(4\kappa\pi)^{\frac{n}{2}}} \max_{\lambda>0} \left[N_1 \left(\frac{1}{2\kappa} + \frac{\lambda^2}{4\kappa^2} \right) (1+\lambda^2)^{\alpha/2} + N_2 \left(\frac{\lambda}{\kappa} + 1 \right) \right] e^{-c\lambda^2}. \tag{2.23}$$

If $\Phi_1 = LZ = \Psi Z$, then a combination of (2.21) and (2.22) gives

$$|LZ| = |\Psi Z| \leq C(t-\tau)^{-\frac{n}{2}-1+\beta} e^{-c\varrho^2}, \quad t-\tau \leq 1, \tag{2.24}$$

with $\beta = \frac{\alpha}{2}$.

3 Two-sided Gaussian bounds

In this section the coefficients of L satisfy (a1) to (a6). Let $\Phi_1 = LZ$,

$$\Phi_{\ell+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^n} \Phi_1(x,t;\eta,\sigma)\Phi_\ell(\eta,\sigma,\xi,\tau)d\eta d\sigma, \quad \ell \geq 1,$$

and define

$$\Phi = \sum_{\ell \geq 1} \Phi_\ell.$$

Let E be the fundamental solution, associated to L, constructed by the parametrix method. According to [8, 9], E is given by

$$E(x,t;\xi,\tau) = Z(x,t;\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma)\Phi(\eta,\sigma;\xi,\eta)d\eta d\sigma, \tag{3.1}$$

for all $(x,t,\xi,\tau) \in Q$.

We refer to [8, Chapter 1] or to [9, Chapter IV] for more details.

3.1 Preliminary estimate

The following lemma will be useful in the sequel.

Lemma 3.1. ([8, Chapter 1, Section 4]) Let $\lambda > 0$ and $-\infty < \gamma, \delta < 1$. Then

$$\int_\tau^t \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n}{2}-\gamma} e^{-\frac{\lambda|x-\eta|^2}{t-\sigma}} (\sigma-\tau)^{-\frac{n}{2}-\delta} e^{-\frac{\lambda|\eta-\xi|^2}{\sigma-\tau}} d\eta d\sigma$$
$$= \left(\frac{4\pi}{\lambda}\right)^{\frac{n}{2}} B(1-\gamma, 1-\delta)(t-\tau)^{-\frac{n}{2}+1-\gamma-\delta} e^{-\frac{\lambda|x-\xi|^2}{t-\tau}},$$

where B is the Euler beta function.

Let C be the constant given by (2.23) and assume that $t-\tau \le 1$. We deduce from (2.24)

$$|\Phi_1| \le C(t-\tau)^{-\frac{n}{2}-1+\beta}e^{-c\varrho^2}. \tag{3.2}$$

Let $\widetilde{C} = \left(\frac{4\pi}{c}\right)^{\frac{n}{2}}$. We have by applying Lemma 3.1

$$|\Phi_2| \le \widetilde{C}C^2B(\beta,\beta)(t-\tau)^{-\frac{n}{2}-1+2\beta}e^{-c\varrho^2}.$$

By induction in ℓ, wo obtain

$$|\Phi_\ell| \le \widetilde{C}^{\ell-1}C^\ell \prod_{j=1}^{\ell-1} B(\beta,j\beta)(t-\tau)^{-\frac{n}{2}-1+\ell\beta}e^{-c\varrho^2}, \quad \ell \ge 2.$$

If Γ is the Euler gamma function, we recall that

$$B(\beta,j\beta) = \frac{\Gamma(\beta)\Gamma(j\beta)}{\Gamma((j+1)\beta)}.$$

Therefore

$$\prod_{j=1}^{\ell-1} B(\beta,j\beta) = \frac{\Gamma(\beta)^\ell}{\Gamma(\ell\beta)}$$

and hence

$$|\Phi_\ell| \le \widetilde{C}^{-1}\frac{\Lambda^\ell}{\Gamma(\ell\beta)}(t-\tau)^{-\frac{n}{2}-1+\ell\beta}e^{-c\varrho^2}, \quad \ell \ge 2,$$

where $\Lambda = C\widetilde{C}\Gamma(\beta)$. Since $t-\tau \le 1$, we obtain

$$|\Phi_\ell| \le \widetilde{C}^{-1}\frac{\Lambda^\ell}{\Gamma(\ell\beta)}(t-\tau)^{-\frac{n}{2}-1+\beta}e^{-c\varrho^2}, \quad \ell \ge 2, \tag{3.3}$$

If $\overline{C} = \widetilde{C}^{-1}$, then (3.3) takes the form

$$|\Phi_\ell| \le \overline{C}\frac{\Lambda^\ell}{\Gamma(\ell\beta)}(t-\tau)^{-\frac{n}{2}-1+\beta}e^{-c\varrho^2}, \quad \ell \ge 2. \tag{3.4}$$

From Stirling's formula for the Γ function (see for instance [10, Chapter V, Section 3]) we have

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{2\pi x}, \quad x \to \infty.$$

Therefore, the series

$$S = C + \overline{C}\sum_{\ell\ge 2}\frac{\Lambda^\ell}{\Gamma(\ell\beta)} \tag{3.5}$$

is convergent.

We get from (2.24) and (3.4)

$$|\Phi| \le S(t-\tau)^{-\frac{n}{2}-1+\beta}e^{-c\varrho^2}. \tag{3.6}$$

3.2 The upper bound

In light of (2.24) and (3.6), Lemma 3.1 yields

$$\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma)\Phi(\eta,\sigma;\xi,\tau)d\eta d\sigma \right| \le \frac{SB(1,\beta)}{(\kappa c)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}+\beta}e^{-c\varrho^2}, \tag{3.7}$$

for all $(x,t,\xi,\tau) \in Q$ and $t-\tau \le 1$.

Let

$$\widehat{C} = \frac{1}{(4\kappa\pi)^{\frac{n}{2}}} + \frac{SB(1,\beta)}{(\kappa c)^{\frac{n}{2}}}.$$

As an immediate consequence of (2.24) and (3.7), we have

$$|E(x,t;\xi,\tau)| \le \widehat{C}(t-\tau)^{-\frac{n}{2}}e^{-c\varrho^2}, \quad (x,t,\xi,\tau) \in Q,\ t-\tau \le 1. \tag{3.8}$$

We recall that E possesses the so-called reproducing property

$$E(x,t;\xi,\tau) = \int_{\mathbb{R}^n} E(x,t;\eta,\sigma)E(\eta,\sigma;\xi,\tau)\, d\eta, \quad \tau < \sigma < t. \tag{3.9}$$

Applying (3.8), we get

$$|E(x,t,\xi,\tau)| \le \widehat{C}^2 \int_{\mathbb{R}^n} (t-\sigma)^{-\frac{n}{2}} e^{-c\frac{|x-\eta|^2}{4(t-\sigma)}} (\sigma-\tau)^{-\frac{n}{2}} e^{-c\frac{|\eta-\xi|^2}{4(\sigma-\tau)}} d\eta, \tag{3.10}$$

for all $t-\tau \le 2$, where $\sigma = \frac{t+\tau}{2}$.

We introduce a variable z so that

$$c\frac{|x-\eta|^2}{4(t-\sigma)} + c\frac{|\eta-\xi|^2}{4(\sigma-\tau)} = c\frac{|x-\xi|^2}{4(t-\tau)} + |z|^2.$$

Using the identity $|x-\eta|^2 = |x-\xi|^2 + |\xi-\eta|^2 + \langle x-\xi, \xi-\eta\rangle$, we get

$$\begin{aligned}
&\frac{|x-\eta|^2}{t-\sigma} + \frac{|\eta-\xi|^2}{\sigma-\tau} - \frac{|x-\xi|^2}{t-\tau} \\
&\quad = \frac{(\sigma-\tau)|x-\xi|^2}{(t-\sigma)(t-\tau)} + \frac{(t-\tau)|\eta-\xi|^2}{(t-\sigma)(\sigma-\tau)} + \frac{2\langle x-\xi, \xi-\eta\rangle}{(t-\sigma)^2}. \\
&\quad = \left| \left(\frac{\sigma-\tau}{(t-\sigma)(t-\tau)}\right)^{\frac{1}{2}} (x-\xi) + \left(\frac{t-\tau}{(t-\sigma)(\sigma-\tau)}\right)^{\frac{1}{2}} (\xi-\eta) \right|^2.
\end{aligned}$$

Therefore, we can for instance take

$$z = \left(c\frac{t-\tau}{t-\sigma}\right)^{\frac{1}{2}} \frac{\eta-\xi}{2(\sigma-\tau)^{\frac{1}{2}}} + \left(c\frac{\sigma-\tau}{t-\sigma}\right)^{\frac{1}{2}} \frac{\xi-x}{2(t-\tau)^{\frac{1}{2}}}.$$

Passing to the variable z in (3.10), we deduce

$$|E(x,t,\xi,\tau)| \le \widetilde{C}\widehat{C}^2(t-\tau)^{-\frac{n}{2}}e^{-c\varrho^2}, \quad t-\tau \le 2.$$

Next assume that $t-\tau > 2$ and let m be the smallest integer so that $t-\tau \le m$. Define

$$\sigma_0 = \tau, \quad \sigma_1 = \tau + \frac{t-\tau}{m}, \dots, \sigma_{m-1} = \tau + (m-1)\frac{t-\tau}{m}, \quad \sigma_m = t.$$

Iterating the reproducing property (3.9), we get

$$E(x,t;\xi,\tau) = \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^m} E(x,\sigma_m,\eta_m,\sigma_{m-1})E(\eta_m,\sigma_{m-1},\eta_{m-1},\sigma_{m-2}) \\ \dots E(\eta_1,\sigma_1,\xi,\sigma_0)d\eta_1 \dots d\eta_m.$$

Repeating inductively the case $m=2$, we find

$$|E(x,t,\xi,\tau)| \le \widetilde{C}^{m-1}\widehat{C}^m(t-\tau)^{-\frac{n}{2}}e^{-c\varrho^2}.$$

This and the fact that $m < t-\tau+1$ entail

$$|E(x,t,\xi,\tau)| \le \widetilde{C}^{-1}e^{\max\left(0,\ln(\widetilde{C}\widehat{C})\right)}e^{\max\left(0,\ln(\widetilde{C}\widehat{C})\right)(t-\tau)}(t-\tau)^{-\frac{n}{2}}e^{-c\varrho^2}.$$

This is the expected Gaussian upper bound.

A more precise upper bound can be obtained by optimizing the constants appearing in the previous computations. We do it in the special case $b_i = q = 0$, where the iteration procedure based on (3.9) is not needed.

Corollary 3.1. If $b_i = q = 0$, then

$$E(x,t;\xi,\tau) \le \frac{1}{(4\kappa\pi)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}}e^{-\frac{\varrho^2}{4M}}\left(1 + c_1(t-\tau)^{\frac{\alpha}{2}}e^{c_2((t-\tau)+\varrho^\gamma)}\right),$$

for all $(x,t,\xi,\tau) \in Q$, where $\varrho = \frac{|x-\xi|}{\sqrt{t-\tau}}$ and $\gamma = \frac{4\alpha+8}{3\alpha+4} < 2$.

Proof. First we note that the restriction $t-\tau \le 1$ is not needed in (2.21), since it comes from (2.20) only. Then we define C_ϵ as in (2.23) with $c = \frac{\epsilon}{4M}$, $N_2 = 0$. It is easy to see that $C_\epsilon \le A\epsilon^{-2-\alpha}$ with $A > 0$ and this leads to (2.24) with this C_ϵ and $c = \frac{(1-\epsilon)}{4M}$. Next we write (3.4) with $\ell\beta$ instead of β, since we no longer assume that $t-\tau \le 1$.

Entering this estimate in the constants C, Λ defining S (see (3.5)), using [1, Theorem 2, Section 15, Chapter V] and Stirling's formula again, we deduce that

$$\sum_{\ell \ge 2} \frac{\Lambda^\ell(t-\tau)^{\ell\beta}}{\Gamma(\ell\beta)} \le c_1(t-\tau)^{2\beta}e^{c_2((t-\tau)+\Lambda^{\frac{1}{\beta}})}$$

and $S \leq c_1 e^{c_2((t-\tau)+\epsilon^{-(2+\frac{4}{\alpha})})}$. Then we use this estimate in (3.7) with $c = \frac{(1-\epsilon)}{4M}$ to get

$$\left| \int_\tau^t \int_{\mathbb{R}^n} Z(x,t;\eta,\sigma)\Phi(\eta,\sigma;\xi,\eta)d\eta d\sigma \right| \\ \leq c_1(t-\tau)^{-\frac{n}{2}+\beta} e^{-\frac{(1-\epsilon)}{4M}\varrho^2 + c_2\epsilon^{-(2+\frac{4}{\alpha})}+c_2(t-\tau)}.$$

Optimizing over ϵ and using (3.1), the corollary follows. QED

3.3 The lower bound

From the previous analysis, we easily get

$$Z(x,t;\xi,\tau) \geq \frac{1}{(4\pi M)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}} e^{-\frac{1}{\kappa}\rho^2}.$$

Hence,

$$Z(x,t;\xi,\tau) \geq \frac{e^{-1}}{(4\pi M)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}}, \quad |x-\xi|^2 \leq \kappa(t-\tau). \tag{3.11}$$

A combination of (3.7) and (3.11) yields

$$E(x,t;\xi,\tau) \geq \frac{e^{-1}}{(4\pi M)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}} - \frac{SB(1,\beta)}{(\kappa c)^{\frac{n}{2}}}(t-\tau)^{-\frac{n}{2}+\beta},$$

for all $|x-\xi|^2 \leq \kappa(t-\tau)$ and $t-\tau \leq 1$.

Fix $\delta \leq 1$ sufficiently small in such a way that

$$\frac{e^{-1}}{(4\pi M)^{\frac{n}{2}}} - \frac{SB(1,\beta)}{(\kappa c)^{\frac{n}{2}}}\delta^\beta \geq \frac{e^{-1}}{2(4\pi M)^{\frac{n}{2}}}.$$

Then, with $\mu = \frac{e^{-1}}{2(4\pi M)^{\frac{n}{2}}}$,

$$E(x,t;\xi,\tau) \geq \mu(t-\tau)^{-\frac{n}{2}}, \quad |x-\xi|^2 \leq \kappa(t-\tau),\ t-\tau \leq \delta. \tag{3.12}$$

Let x and ξ be given so that $2|x-\xi| > \sqrt{\kappa(t-\tau)}$ and let $m \geq 2$ be the smallest integer so that

$$\frac{4|x-\xi|^2}{m} \leq \kappa(t-\tau). \tag{3.13}$$

Define the sequence $(x_k)_{0\leq k\leq m}$

$$x_k = x + \frac{k}{m}(\xi - x), \quad 0 \leq k \leq m.$$

Set

$$r=\frac{1}{4}\frac{\sqrt{\kappa(t-\tau)}}{\sqrt{m}}$$

and

$$\sigma_k=\tau+\frac{k}{m}(t-\tau),\quad 0\le k\le m.$$

Using (3.12), the positivity of E and the reproducing property, we get

$$E(x,t;\xi,\tau) \ge \mu^m\int_{B(x_1,r)}\dots\int_{B(x_{m-1},r)}(\sigma_1-\sigma_0)^{-\frac{n}{2}}\dots(\sigma_m-\sigma_{m-1})^{-\frac{n}{2}}\,d\eta_1\dots d\eta_{m-1},$$

where we used

$$|x_{i+1}-x_i|=\frac{1}{\sqrt{m}}\frac{|x-\xi|}{\sqrt{m}}\le\frac{1}{2}\frac{\sqrt{\kappa(t-\tau)}}{\sqrt{m}}=2r,$$

and

$$\begin{aligned}|\eta_{i+1}-\eta_i|&\le|\eta_{i+1}-x_{i+1}|+|x_{i+1}-x_i|+|x_i-\eta_i|\\&<2r+|x_{i+1}-x_i|\le 4r=\frac{\sqrt{\kappa(t-\tau)}}{\sqrt{m}}=\sqrt{\kappa(\sigma_{i+1}-\sigma_i)}.\end{aligned}$$

Whence

$$E(x,t;\xi,\tau)\ge\kappa^{-\frac{n}{2}}\nu^m(t-\tau)^{-\frac{n}{2}},$$

with

$$\nu=\frac{\kappa^{\frac{n}{2}}}{eM^{\frac{n}{2}}2^{3n}\Gamma(n/2+1)}<1.$$

Noting that

$$m<\frac{4|x-\xi|^2}{\kappa(t-\tau)}+1,$$

we obtain

$$\begin{aligned}E(x,t;\xi,\tau)&\ge\kappa^{-\frac{n}{2}}e^{-|\ln\nu|m}(t-\tau)^{-\frac{n}{2}}\\&\ge\kappa^{-\frac{n}{2}}e^{-|\ln\nu|}(t-\tau)^{-\frac{n}{2}}e^{-\frac{4|\ln\nu|}{\kappa}\frac{|x-\xi|^2}{t-\tau}},\quad t-\tau\le\delta.\end{aligned}$$

If $C_0=\min\left(\mu,\kappa^{-\frac{n}{2}}e^{-|\ln\nu|}\right)$ and $d=\frac{4|\ln\nu|}{\kappa}$, then the last inequality and (3.12) yield

$$E(x,t;\xi,\tau)\ge C_0(t-\tau)^{-\frac{n}{2}}e^{-d\frac{|x-\xi|^2}{t-\tau}},\quad t-\tau\le\delta.$$

We now proceed similarly to the case of the upper bound to remove the condition $t-\tau \le \delta$. If m is the smallest integer so that $t-\tau \le m\delta$, we get

$$E(x,t;\xi,\tau) \ge \widetilde{C}^{-1}\left(\widetilde{C}C_0\right)^m (t-\tau)^{-\frac{n}{2}}e^{-d\varrho^2},$$

from which we deduce

$$E(x,t;\xi,\tau) \ge \widetilde{C}^{-1}e^{\min\left(0,\ln(\widetilde{C}C_0)\right)}e^{\min\left(0,\frac{\ln(\widetilde{C}C_0)}{\delta}\right)(t-\tau)}(t-\tau)^{-\frac{n}{2}}e^{-d\varrho^2}.$$

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