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# Some properties of the mapping $T_\mu$ introduced by a representation in Banach and locally convex spaces

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**Abstract.** Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup $S$. We show that the mapping $T_\mu$ introduced by a mean on a subspace of $l^\infty(S)$ inherits some properties of $\mathcal{S}$ in Banach spaces and locally convex spaces. The notions of $Q$-$G$-nonexpansive mapping and $Q$-$G$-attractive point in locally convex spaces are introduced. We prove that $T_\mu$ is a $Q$-$G$-nonexpansive mapping when $T_s$ is $Q$-$G$-nonexpansive mapping for each $s \in S$ and a point in a locally convex space is $Q$-$G$-attractive point of $T_\mu$ if it is a $Q$-$G$-attractive point of $\mathcal{S}$.

**Keywords:** Representation, Nonexpansive, Attractive point, Directed graph, Mean.

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## Introduction and preliminaries

Let $C$ be a nonempty closed and convex subset of a Banach space $E$ and $E^*$ be the dual space of $E$. Let $\langle .,. \rangle$ denote the pairing between $E$ and $E^*$. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

for all $x \in E$. For more details, see [13].

The space of all bounded real-valued functions defined on $S$ with supremum norm is denoted by $l^\infty(S)$.

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$l_s$ and $r_s$ in $l^\infty(S)$ are defined as follows: $(l_t g)(s) = g(ts)$ and $(r_t g)(s) = g(st)$, for all $s \in S$, $t \in S$ and $g \in l^\infty(S)$.
Suppose that $X$ is a (linear) subspace of $l^\infty(S)$ containing 1 and let $X^*$ be its topological dual space. An element $m$ of $X^*$ is said to be a mean on $X$, provided $\|m\| = m(1) = 1$. For $m \in X^*$ and $g \in X$, $m_t(g(t))$ is often written instead of $m(g)$. Suppose that $X$ is left invariant (respectively, right invariant), i.e., $l_t(X) \subset X$ (respectively, $r_t(X) \subset X$) for each $t \in S$. A mean $m$ on $X$ is called left invariant (respectively, right invariant), provided $m(l_t g) = m(g)$ (respectively, $m(r_t g) = m(g)$) for each $t \in S$ and $g \in X$. $X$ is called left (respectively, right) amenable if $X$ possesses a left (respectively, right) invariant mean. $X$ is amenable, provided $X$ is both left and right amenable.
Let $D$ be a directed set in $X$. A net $\{m_\alpha : \alpha \in D\}$ of means on $X$ is called left regular, provided

$$\lim_{\alpha \in D} \|l_t^* m_\alpha - m_\alpha\| = 0,$$

for every $t \in S$, where $l_t^*$ is the adjoint operator of $l_t$.

Let $E$ a reflexive Banach space. Let $g$ be a function on $S$ into $E$ such that the weak closure of $\{g(s) : s \in S\}$ is weakly compact and suppose that $X$ is a subspace of $l^\infty(S)$ owning all the functions $s \to \langle g(s), x^* \rangle$ with $x^* \in E^*$. We know from [3] that, for any $m \in X^*$, there exists a unique element $g_m$ in $E$ such that $\langle g_m, x^* \rangle = m_s \langle g(s), x^* \rangle$ for all $x^* \in E^*$. We denote such $g_m$ by $\int g(s)m(s)$. Moreover, if $m$ is a mean on $X$, then from [5], $\int g(s)m(s) \in \overline{\text{co}}\,\{g(s) : s \in S\})$, where $\overline{\text{co}}\,\{g(s) : s \in S\})$ denotes the closure of the convex hull of $\{g(s) : s \in S\}$.

The following definitions and basic results are needed in the next section.

(1) Let $E$ be a Banach space or a locally convex space, $C$ be a nonempty closed and convex subset of $E$ and $S$ be a semigroup. Then, a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from $C$ into itself is called a representation of $S$ as mappings on $C$ into itself provided $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$. Note that, Fix($\mathcal{S}$) is the set of common fixed points of $\mathcal{S}$, that is

$$Fix(\mathcal{S}) = \bigcap_{s \in S} \{x \in C : T_s x = x\}.$$

(2) Let $E$ be a real Banach space and $C$ be a subset of $E$. The mapping $T : C \to C$ is called:

a. nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;

b. quasi nonexpansive [10] if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in$ Fix($T$), the fixed point set of $T$;

c. strongly quasi nonexpansive [10] if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C \setminus \mathrm{Fix}(T)$ and $f \in \mathrm{Fix}(T)$;

d. $F$-quasi nonexpansive (for a subset $F \subseteq \mathrm{Fix}(T)$) if $\|Tx - f\| \leq \|x - f\|$ for all $x \in C$ and $f \in F$;

e. strongly $F$-quasi nonexpansive [10] (for a subset $F \subseteq \mathrm{Fix}(T)$) if

$$\|Tx - f\| \leq \|x - f\|,$$

for all $x \in C \setminus \mathrm{Fix}(T)$ and $f \in F$,

f. retraction [10] if $T^2 = T$,

g. asymptotically nonexpansive [6] if for all $x, y \in C$ the following inequality holds:

$$\limsup_{n\to\infty} \|T^n x - T^n y\| \leq \|x - y\|. \tag{0.1}$$

(3) Suppose that $\mathcal{S} = \{T_s : s \in S\}$ is a representation of a semigroup $S$ on a set $C$ in a Banach space $E$. An element $a \in E$ is called:

a. asymptotically attractive point of $S$ for $C$ provided

$$\limsup_{n\to\infty} \|a - T_t^n x\| \leq \|a - x\|, \tag{0.2}$$

for all $t \in S$ and $x \in C$,

b. uniformly asymptotically nonexpansive representation, if for each $x, y \in C$,

$$\limsup_{n\to\infty} \sup_t \|T_t^n x - T_t^n y\| \leq \|x - y\|, \tag{0.3}$$

c. uniformly asymptotically attractive point, if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_t \|a - T_t^n x\| \leq \|a - x\|. \tag{0.4}$$

(4) Let $X$ be a locally convex topological vector space ( for short, locally convex space) generated by a family of seminorms $Q$, $C$ be a nonempty closed and convex subset of $X$ and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ ( for more details refer to [4]). A mapping $T$ of $C$ into itself is called $Q$-$G$-nonexpansive if $q(Tx - Ty) \leq q(x - y)$, whenever $(x, y) \in E(G)$ for any $x, y \in C$ and $q \in Q$, and a mapping $f$ is a $Q$-contraction on $E$ if $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in E$ such that $0 \leq \beta < 1$.

It is easy to see that the locally convex space $X$ generated by a family of seminorms $Q$ is separated (Hausdorff) if and only if the family of seminorms $Q$ possesses the following property:

for each $x \in X \backslash \{0\}$ there exists $q \in Q$ such that $q(x) \neq 0$ or equivalently

$$\bigcap_{q \in Q} \{x \in X : q(x) = 0\} = \{0\},$$

( see [1]).

The following results play crucial role in the next section.

**Lemma 1.** *[12, 3] Suppose that $g$ is a function of $S$ into $E$ such that the weak closure of $\{g(t) : t \in S\}$ is weakly compact and let $X$ be a subspace of $l^\infty(S)$ containing all the functions $t \to \langle g(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element $g_\mu$ in $E$ such that*

$$\langle g_\mu, x^* \rangle = \mu_t \langle g(t), x^* \rangle,$$

*for all $x^* \in E^*$. Moreover, if $\mu$ is a mean on $X$ then*

$$\int g(t)\, d\mu(t) \in \overline{co}\, \{g(t) : t \in S\}.$$

*We can write $g_\mu$ by*

$$\int g(t)\, d\mu(t).$$

Next, we will need some concepts in locally convex spaces.
Consider a family of seminorms $Q$ on the locally convex space $X$ which determines the topology of $X$ and the seminorm $q \in Q$. Let $Y$ be a subset of $X$, we put

$$q_Y^*(f) = \sup\{|f(y)| : y \in Y, q(y) \leq 1\}$$

and

$$q^*(f) = \sup\{|f(x)| : x \in X, q(x) \leq 1\},$$

for every linear functional $f$ on $X$. Observe that, for each $x \in X$ that $q(x) \neq 0$ and $f \in X^*$, then $|\langle x, f \rangle| \leq q(x) q^*(f)$. We will make use of the following Theorems.

**Theorem 1.** *[2] Suppose that $Q$ is a family of seminorms on a real locally convex space $X$ which determines the topology of $X$ and $q \in Q$ is a continuous seminorm and $Y$ is a vector subspace of $X$ such that*

$$Y \cap \{x \in X : q(x) = 0\} = \{0\}.$$

*Let $f$ be a real linear functional on $Y$ such that $q_Y^*(f) < \infty$. Then there exists a continuous linear functional $h$ on $X$ that extends $f$ such that $q_Y^*(f) = q^*(h)$.*

**Theorem 2.** *[2] Suppose that $Q$ is a family of seminorms on a real locally convex space $X$ which determines the topology of $X$ and $q \in Q$ a nonzero continuous seminorm. Let $x_0$ be a point in $X$. Then there exists a continuous linear functional $f$ on $X$ such that $q^*(f) = 1$ and $f(x_0) = q(x_0)$.*

Consider a reflexive Banach space $E$, a nonempty closed convex subset $C$ of $E$, a semigroup $S$ and a representation $\mathcal{S} = \{T_s : s \in S\}$ of $S$ and let $X$ be a subspace of $l^\infty(S)$ and $\mu$ be a mean on $X$. We write $T_\mu x$ instead of $\int T_t x\, d\mu(t)$. The relations between the representation $\mathcal{S}$ and the mapping $T_\mu$ have been studied by many authors, for instance see [6, 7, 10, 11].

In this paper, we establish some relations between the representation $\mathcal{S}$ and $T_\mu$ in Banach and locally convex spaces.

# 1 Main results

In the following theorem, we prove that $T_\mu$ inherits some properties of representation $\mathcal{S}$ in Banach spaces.

**Theorem 3.** *Suppose that $C$ is a nonempty closed, convex subset of a reflexive Banach space $E$, $S$ a semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a representation of $S$ as self mappings on $C$ such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$. If $X$ is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, x^*\rangle$ is an element of $X$ for each $x \in C$ and $x^* \in E^*$, then the following assertions hold:*

*(a) Let the mapping $t \to \langle T_t^n x - T_t^n y, x^*\rangle$ be an element of $X$ for each $x, y \in C$, $n \in \mathbb{N}$ and $x^* \in E$. Let $\mu$ be a mean on $X$ and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as uniformly asymptotically nonexpansive self mappings on $C$, then $T_\mu$ is an asymptotically nonexpansive self mapping on $C$,*

*(b) $T_\mu x = x$ for each $x \in Fix(\mathcal{S})$,*

*(c) $T_\mu x \in \overline{co}\,\{T_t x : t \in S\}$ for each $x \in C$,*

*(d) if $X$ is $r_s$-invariant for each $s \in S$ and $\mu$ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$,*

*(e) let $a \in C$ be a uniformly asymptotically attractive point of $\mathcal{S}$ and the mapping $t \to \langle a - T_t^n x, x^*\rangle$ be an element of $X$ for each $x \in C$, $n \in \mathbb{N}$ and $x^* \in E$. Then $a$ is an asymptotically attractive point of $T_\mu$,*

*(f) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as the affine self mappings on $C$, then $T_\mu$ is an affine self mapping on $C$,*

*(g) let $P$ be a self mappings on $C$ that commutes with $T_s \in \mathcal{S} = \{T_s : s \in S\}$ for each $s \in S$. Let the mapping $t \to \langle PT_t x, x^* \rangle$ be an element of $X$ for each $x \in C$ and $x^* \in E$. Then $T_\mu$ commutes with $P$,*

*(h) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as quasi nonexpansive self mappings on $C$, then $T_\mu$ is a $Fix(\mathcal{S})$-quasi nonexpansive self mapping on $C$,*

*(i) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as $F$-quasi nonexpansive self mappings on $C$ (for a subset $F \subseteq Fix(\mathcal{S})$), then $T_\mu$ is an $F$-quasi nonexpansive self mapping on $C$,*

*(j) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as strongly $F$-quasi nonexpansive self mappings on $C$ (for a subset $F \subseteq Fix(\mathcal{S})$), then $T_\mu$ is an strongly $F$-quasi nonexpansive self mapping on $C$,*

*(k) let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as retraction self mappings on $C$, then $T_\mu$ is a retraction self mapping on $C$,*

*(l) let $E = H$ be a Hilbert space and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as monotone self mappings on $H$, then $T_\mu$ is a monotone self mapping on $H$.*

*Proof.* (a) Since $\mathcal{S}$ is a representation as uniformly asymptotically nonexpansive self mappings on $C$, hence, from (0.3) and the part (b) of Theorem 3. 1. 7 in [8], there exists an integer $m_0 \in \mathbb{N}$ such that

$$\sup_t \|T_t^n x - T_t^n y\| \leq \|x - y\|,$$

for all $n \geq m_0$, $x, y \in C$. Suppose that $x_1^* \in J(T_\mu^n x - T_\mu^n y)$ and $x, y \in C$, where $J$ is the normalized duality mapping on $E$. We know from [3], see Lemma 1.1, that for any $\mu \in X^*$, there exists a unique element $f_\mu$ in $E$ such that

$$\langle f_\mu, x^* \rangle = \mu_s \langle f(s), x^* \rangle, \tag{1.1}$$

for all $x^* \in E^*$, where $f$ is a function of $S$ into $E$ such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact. Then from (1.1) we have

$$\begin{aligned}\|T_\mu^n x - T_\mu^n y\|^2 =& \langle T_\mu^n x - T_\mu^n y, x_1^* \rangle = \mu_t \langle T_t^n x - T_t^n y, x_1^* \rangle \\ \leq& \sup_t \|T_t^n x - T_t^n y\| \|T_\mu^n x - T_\mu^n y\| \\ \leq& \|x - y\| \|T_\mu^n x - T_\mu^n y\|,\end{aligned}$$

and

$$\|T_\mu^n x - T_\mu^n y\| \leq \|x - y\|,$$

for all $n \geq m_0$, $x, y \in C$. Therefore, we get

$$\limsup_{n\to\infty} \|T_\mu^n x - T_\mu^n y\| \leq \|x - y\|.$$

(b) Suppose that $x \in Fix(\mathcal{S})$ and $x^* \in E^*$. Hence

$$\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle.$$

(c) The assertion follows from Lemma 1.

(d) It follows from

$$\langle T_\mu(T_s x), x^* \rangle = \mu_t \langle T_{ts} x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle.$$

(e) Since $a$ is a uniformly attractive point, hence, from (0.4) and from part (b) of Theorem 3. 1. 7 in [8], for each $x \in C$ there exists an integer $m_0 \in \mathbb{N}$ such that

$$\sup_t \|a - T_t^n x\| \leq \|a - x\|,$$

for all $n \geq m_0$. Suppose that $x_2^* \in J(a - T_\mu^n x)$, therefore from (1.1) we have,

$$\begin{aligned} \|a - T_\mu^n x\|^2 =& \langle a - T_\mu^n x, x_2^* \rangle = \mu_t \langle a - T_t^n x, x_2^* \rangle \\ \leq& \sup_t \|a - T_t^n x\| \|a - T_\mu^n x\| \\ \leq& \|a - x\| \|a - T_\mu^n x\|. \end{aligned}$$

Hence,

$$\|a - T_\mu^n x\| \leq \|a - x\|,$$

for all $n \geq m_0$. Thus, we get

$$\limsup_{n\to\infty} \|a - T_\mu^n x\| \leq \|a - x\|,$$

for each $x \in C$.

(f) If $x_1^* \in E^*$, then for all positive integers $\alpha, \beta$ and $x, y \in C$ with $\alpha + \beta = 1$, we have

$$\begin{aligned} \langle T_\mu(\alpha x + \beta y), x_1^* \rangle =& \mu_t \langle T_t(\alpha x + \beta y), x_1^* \rangle \\ =& \mu_t \langle \alpha T_t x + \beta T_t y, x_1^* \rangle \\ =& \alpha \mu_t \langle T_t x, x_1^* \rangle + \beta \mu_t \langle T_t y, x_1^* \rangle \\ =& \alpha \langle T_\mu x, x_1^* \rangle + \beta \langle T_\mu y, x_1^* \rangle \\ =& \langle \alpha T_\mu x + \beta T_\mu y, x_1^* \rangle, \end{aligned}$$

and so

$$T_\mu(\alpha x + \beta y) = \alpha T_\mu x + \beta T_\mu y.$$

(g) Let $x_1^* \in E^*$. Then considering the functions $f_1, f_2 : S \longrightarrow E$, by $f_1(t) = T_t P x$ and $f_2(t) = P T_t x$ and applying them in (1.1), then we have

$$\mu_t \langle T_t P x, x_1^* \rangle = \langle f_1(t), x_1^* \rangle = \mu_t \langle (f_1)_\mu, x_1^* \rangle = \langle T_\mu P x, x_1^* \rangle$$

and

$$\mu_t \langle P T_t x, x_1^* \rangle = \langle f_2(t), x_1^* \rangle = \mu_t \langle (f_2)_\mu, x_1^* \rangle = \langle P T_\mu x, x_1^* \rangle,$$

for each $x \in C$. Since $P$ commutes with $T_t \in \mathcal{S} = \{T_t : t \in S\}$ for each $s \in S$, we conclude that

$$\begin{aligned} \langle T_\mu P x, x_1^* \rangle =& \mu_t \langle T_t P x, x_1^* \rangle \\ =& \mu_t \langle P T_t x, x_1^* \rangle \\ =& \langle P T_\mu x, x_1^* \rangle, \end{aligned}$$

therefore $T_\mu P = P T_\mu$.

(h)Since $X$ is a subspace of $l^\infty(S)$, $1 \in X$ and the mapping $t \to \langle T_t x, x^* \rangle$ is an element of $X$ for each $x \in C$ and $x^* \in E$, hence, the mapping $t \to \langle T_t x - f, x^* \rangle$ is an element of $X$ for each $x \in C$, $x^* \in E$ and $f \in \text{Fix}(\mathcal{S})$. For each $t \in S$, we have

$$\|T_t x - f\| \leq \|x - f\|,$$

for each $f \in \text{Fix}(T_t)$ and $x \in C$.

Suppose $f \in \text{Fix}(\mathcal{S})$ and $x_2^* \in J(T_\mu x - f)$, then from (1.1), we have

$$\begin{aligned} \|T_\mu x - f\|^2 =& \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \\ \leq& \sup_t \|T_t x - f\| \|T_\mu x - f\| \\ \leq& \|x - f\| \|T_\mu x - f\|. \end{aligned}$$

Then

$$\|T_\mu x - f\| \leq \|x - f\|,$$

and so $T_\mu$ is a $\text{Fix}(\mathcal{S})$-quasi nonexpansive self mapping on $C$.

(i) Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as $F$-quasi nonexpansive self mappings on $C$ that $F \subseteq \mathrm{Fix}(\mathcal{S})$. Then for each $t \in S$, we have

$$\|T_t x - f\| \leq \|x - f\|,$$

for each $f \in F$ and $x \in C$. Suppose that $f \in F$, $x \in C$ and $x_2^* \in J(T_\mu x - f)$, then, as in the proof of (h), from (1.1), we have

$$\begin{aligned}\|T_\mu x - f\|^2 =& \langle T_\mu x - f, x_2^*\rangle = \mu_t\langle T_t x - f, x_2^*\rangle \\ \leq& \sup_t \|T_t x - f\|\|T_\mu x - f\| \\ \leq& \|x - f\|\|T_\mu x - f\|,\end{aligned}$$

thus

$$\|T_\mu x - f\| \leq \|x - f\|.$$

This means that $T_\mu$ is an $F$-quasi nonexpansive self mapping on $C$.

(j) Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as strongly $F$-quasi nonexpansive self mappings on $C$ such that $F \subseteq Fix(\mathcal{S})$, then for each $t \in S$ we have

$$\|T_t x - f\| < \|x - f\|, \quad \forall (x, f) \in C \backslash F \times F.$$

Suppose that $f \in F$, $x \in C\backslash F$ and $x_2^* \in J(T_\mu x - f)$, then from (1.1), we have

$$\begin{aligned}\|T_\mu x - f\|^2 =& \langle T_\mu x - f, x_2^*\rangle = \mu_t\langle T_t x - f, x_2^*\rangle \\ \leq& \sup_t \|T_t x - f\|\|T_\mu x - f\| \\ <& \|x - f\|\|T_\mu x - f\|,\end{aligned}$$

then we have

$$\|T_\mu x - f\| < \|x - f\|,$$

therefore $T_\mu$ is a strongly $F$-quasi nonexpansive self mapping on $C$.

(k) Since $T_t^2 = T_t$ and the mapping $t \to \langle T_t x, x^*\rangle$ is an element of $X$ for each $x \in C$ and $x^* \in E$, hence the mapping $t \to \left\langle T_t^2 x, x^*\right\rangle$ is an element of $X$ for each $x \in C$ and $x^* \in E$. Suppose that $x \in C$ and $x_1^* \in E^*$, then from (1.1), we have

$$\begin{aligned}\langle T_\mu^2 x, x_1^*\rangle =& \mu_t\langle T_t^2 x, x_1^*\rangle \\ =& \mu_t\langle T_t x, x_1^*\rangle \\ =& \langle T_\mu x, x_1^*\rangle,\end{aligned}$$

hence $T_\mu^2 = T_\mu$.

(l) Since $T_s$ is monotone for every $s \in S$, then we have $\langle T_s x - T_s y, x - y\rangle \geq 0$ for every $x, y \in H$ and $s \in S$. As in the proof of Theorem 1.4.1 in [13] we know that $\mu$ is positive i.e., $\langle \mu, f\rangle \geq 0$ for each $f \in X$ that $f \geq 0$ . Then for each $x, y \in H$, from (1.1) we have

$$\langle T_\mu x - T_\mu y, x - y\rangle = \mu_t \langle T_t x - T_t y, x - y\rangle \geq 0,$$

then $T_\mu$ is a monotone self mapping on $H$. QED

Now we present some properties of $T_\mu$ in locally convex spaces.

**Theorem 4.** *Let $S$ be a semigroup, $E$ a locally convex space with predual locally convex space $D$, $U$ a convex neighbourhood of 0 in $D$ and $p_U$ be the Minkowski functional. Let $f : S \to E$ be a function such that*

$$\langle x, f(t)\rangle \leq 1,$$

*for all $t \in S$ and $x \in U$. Let $X$ be a subspace of $l^\infty(S)$ such that the mapping $t \to \langle x, f(t)\rangle$ is an element of $X$, for each $x \in D$. Then, for any $\mu \in X^*$, there exists a unique element $F_\mu \in E$ such that*

$$\langle x, F_\mu\rangle = \mu_t \langle x, f(t)\rangle,$$

*for each $x \in D$. Furthermore, if $1 \in X$ and $\mu$ is a mean on $X$, then $F_\mu$ is contained in $\overline{co\{f(t) : t \in S\}}^{w^*}$.*

*Proof.* We define $F_\mu$ by

$$\langle x, F_\mu\rangle = \mu_t \langle x, f(t)\rangle,$$

for each $x \in D$. Obviously, $F_\mu$ is linear in $x$. Moreover it follows from Proposition 3.8 in [9] that

$$|\langle x, F_\mu\rangle| = |\mu_t \langle x, f(t)\rangle| \leq \sup_t |\langle x, f(t)\rangle|.\|\mu\| \leq p_U(x).\|\mu\|, \tag{1.2}$$

for all $x \in D$. Assume that $(x_\alpha)$ is a net in $D$ that converges to $x_0$. Then by (1.2) we have

$$|\langle x_\alpha, F_\mu\rangle - \langle x_0, F_\mu\rangle| = |\langle x_\alpha - x_0, F_\mu\rangle| \leq p_U(x_\alpha - x_0).\|\mu\|,$$

taking limit and using the continuity (see Theorem 3.7 in [9]) of $p_U$, we get $F_\mu$ is continuous on $D$ and so $F_\mu \in E$.

Now, let $1 \in X$ and $\mu$ be a mean on $X$. Then, there exists a net $\{\mu_\alpha\}_I$ of finite means on $X$ such that $\{\mu_\alpha\}_I$ converges to $\mu$ with the weak$^*$ topology on $X^*$. For each $\alpha$, we may consider that

$$\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \delta_{t_{\alpha,i}},$$

such that $\lambda_{\alpha,i} \geq 0$ for each $i = 1, \cdots, n_\alpha$ and $\sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} = 1$. Therefore,

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle = \langle x, \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \rangle,$$

for each $x \in D$ and $\alpha \in I$. Then we have

$$F_{\mu_\alpha} = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \in \mathrm{co}\{f(t) : t \in S\}.$$

Also

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle \to \mu_t \langle x, f(t) \rangle = \langle x, F_\mu \rangle,$$

for each $x \in D$, therefore $\{F_{\mu_\alpha}\}$ converges to $F_\mu$ in the weak* topology and

$$F_\mu \in \overline{\mathrm{co}\{f(t) : t \in S\}}^{w^*},$$

we can write $F_\mu$ by $\int f(t) d\mu(t)$. QED

In the next we show that $T_\mu$ inherits some properties of the representation $\mathcal{S}$ in locally convex spaces.

**Theorem 5.** *Let $S$ be a semigroup, $C$ a closed convex subset of the locally convex space $E$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $\mathcal{B}$ a base at 0 for the topology $E$ which consists of convex and balanced sets. Let $Q = \{q_V : V \in \mathcal{B}\}$ where $q_V$ is the associated Minkowski functional with $V$. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as Q-G-nonexpansive mappings from $C$ into itself and $X$ be a subspace of $B(S)$ with $1 \in X$ and $\mu$ be a mean on $X$ such that the mapping $t \to \langle T_t x, x^* \rangle$ is an element of $X$ for each $x \in C$ and $x^* \in E^*$. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$, then the following facts hold:*

*(i) $T_\mu$ is a Q-G-nonexpansive mapping from $C$ into $C$.*

*(ii) $T_\mu x = x$ for each $x \in Fix(\mathcal{S})$.*

*(iii) If the dual of $E$ is a locally convex space with predual locally convex space $D$ and $C$ a $w^*$-closed convex subset of $E$ and $U$ a convex neighbourhood of 0 in $D$ and $p_U$ is the associated Minkowski functional. Let the mapping $t \to \langle z, T_t x \rangle$ be an element of $X$ for each $x \in C$ and $z \in D$, then*

$$T_\mu x \in \overline{co\{T_t x : t \in S\}}^{w^*}.$$

*(iv) if $X$ is $r_s$-invariant for each $s \in S$ and $\mu$ is right invariant, then*

$$T_\mu T_t = T_\mu,$$

*for each $t \in S$.*

*(v) let $a \in E$ be a $Q$-$G$-attractive point of $\mathcal{S}$ and the mapping $t \to \langle a - T_t x, x^* \rangle$ be an element of $X$ for each $x \in C$ and $x^* \in E$, then $a$ is a $Q$-$G$-attractive point of $T_\mu$.*

*Proof.* (i) Let $x, y \in C$ and $V \in \mathcal{B}$. By Proposition 3.33 in [9], the topology on $E$ induced by $Q$ is the original topology on $E$. By Theorem 3.7 in [9], $q_V$ is a continuous seminorm and from Theorem 1.36 in [8], $q_V$ is a nonzero seminorm because if $x \notin V$ then $q_V(x) \geq 1$, hence from Theorem 2, there exists a functional $x_V^* \in X^*$ such that

$$q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x_V^* \rangle,$$

and $q_V^*(x_V^*) = 1$. Also from Theorem 3.7 in [9], $q_V(z) \leq 1$ for each $z \in V$.

We conclude that $\langle z, x_V^* \rangle \leq 1$ for all $z \in V$. Therefore from Theorem 3.8 in [9], $\langle z, x_V^* \rangle \leq q_V(z)$ for all $z \in E$. Hence for each $t \in S$, $x, y \in C$ that $(x, y) \in E(G)$ and $x^* \in E^*$, from (1.1), we have

$$\begin{aligned} q_V(T_\mu x - T_\mu y) =& \langle T_\mu x - T_\mu y, x_V^* \rangle = \mu_t \langle T_t x - T_t y, x_V^* \rangle \\ \leq& \|\mu\| \sup_t |\langle T_t x - T_t y, x_V^* \rangle| \\ \leq& \sup_t q_V(T_t x - T_t y) \\ \leq& q_V(x - y), \end{aligned}$$

then we have

$$q_V(T_\mu x - T_\mu y) \leq q_V(x - y),$$

for all $V \in \mathcal{B}$.

(ii) Let $x \in Fix(\mathcal{S})$ and $x^* \in E^*$. Then we have

$$\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle.$$

(iii) The assertion follows from Theorem 4.

(iv) This part obtains from the following equalities:

$$\langle T_\mu (T_s x), x^* \rangle = \mu_t \langle T_{ts} x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle.$$

(v) Let $x \in C$ and $V \in \mathcal{B}$. From Theorem 2, there exists a linear functional $x_V^* \in X^*$ such that

$$q_V(a - T_\mu x) = \langle a - T_\mu x, x_V^* \rangle,$$

and $q_V^*(x_V^*) = 1$. It follows from [9, Theorem 3.7] that $q_V(z) \leq 1$ and $\langle z, x_V^* \rangle \leq 1$, for each $z \in V$. Therefore Theorem 3.8 in [9] implies

$$\langle z, x_V^* \rangle \leq q_V(z),$$

for each $z \in E$. Then by applying (1.1) and for each $t \in S$ and $x, y \in C$ that $(x, y) \in E(G)$ and $x^* \in E^*$, we have

$$\begin{aligned} q_V(a - T_\mu x) =& \langle a - T_\mu x, x_V^* \rangle = \mu_t \langle a - T_t x, x_V^* \rangle \\ \leq& \|\mu\| \sup_t |\langle a - T_t x, x_V^* \rangle| \\ \leq& \sup_t q_V(a - T_t x) \\ \leq& q_V(a - x), \end{aligned}$$

and

$$q_V(a - T_\mu x) \leq q_V(a - x),$$

for all $V \in \mathcal{B}$. QED

## 2 Conclusion

In this paper, we prove that some properties of the mapping in the representation $\mathcal{S} = \{T_s : s \in S\}$ can be transferred to the mapping $T_\mu$ introduced by a mean on a subspace of $B(S)$, for example nonexpansiveness, quasi-nonexpansiveness, strongly quasi-nonexpansiveness, monotonicity, retraction property and another properties in Banach spaces, and $Q$-$G$-nonexpansiveness using a directed graph in locally convex spaces.

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