Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **40** (2020) no. 1, 81–86. doi:10.1285/i15900932v40n1p81

Cones in PG$(3, q)$

Vito Napolitano[i]
Dipartimento di Matematica e Fisica, Università degli Studi della Campania "Luigi Vanvitelli"
Viale Lincoln 5, 81100 Caserta (Italy)
vito.napolitano@unicampania.it

Received: 22.3.2019; accepted: 24.2.2020.

Abstract. In this paper, sets of points of PG$(3, q)$ of size $q^2 + q + 1$ and intersecting every plane in 1, m or n points are studied.

Keywords: projective space, ovals, quadratic cones, intersection number

MSC 2000 classification: primary 05B25, secondary 51E20

1 Introduction

In finite geometry, there are many papers devoted to the characterizaterization of special subsets of points of projective spaces, such as e.g. quadrics, hermitian varieties and subgeometries, having few intersection sizes with respect to all the subspaces of a prescribed family of subspaces,

Recently, we can find some papers containing characterizations of quadratic cones as subsets of points of PG$(3, q)$ of size q^2+q+1, q odd, with exactly three intersections sizes with respect to the planes and satisfying some extra condition (cf [2, 4, 6, 3]). In [5], within this approach, the authors give new characterizations of (ovoids and) cones projecting an oval of a plane π from a point V not in π in PG$(3, q)$, $q = p^h$, with an assumption on the integer h and they conclude pointing out that certain extra assumptions in such characterizations are essential.

Starting from these conclusions, in this paper we study subsets of points of PG$(3, q)$ of size q^2+q+1 having exactly three intersection sizes with the planes of PG$(3, q)$ and containing at least one line.

A quadratic cone of PG$(3, q)$ has size q^2+q+1, intersects every plane of the projective space in 1, $q+1$ or $2q+1$ points and contains $q+1$ lines. Examples 1 and 2 show that there are sets of points of PG$(3, q)$ containing at least one line, with the same size as a quadratic cone and intersecting every plane in 1, $q+1$ or $2q+1$ wich are not cones.

[i]This work is partially supported by G.N.S.A.G.A. of INdAM.

Let $\mathbb{P}$ a projective space of dimension r and let $h \le r$ and $0 \le m_1 < \cdots < m_s$ be $s+1$ non–negative integers. We recall [7] that a subset $\mathcal{K}$ of points of $\mathbb{P}$ is of *class* $[m_1, \ldots, m_s]_h$ with respect to the dimension h if $|\pi \cap \mathcal{K}| \in \{m_1, \ldots, m_s\}$ for every h–dimensional subspace π of $\mathbb{P}$. The set $\mathcal{K}$ is of *type* $(m_1, \ldots, m_s)_h$ with respect to the dimension h if for every $m_i, i = 1, \ldots, s$ (*intersection numbers*) there exists at least one h–dimensional subspace π such that $|\pi \cap \mathcal{K}| = m_i$. When $h = 1, 2$ the set $\mathcal{K}$ is of *line*(respectively *plane*)*–class* (or *type*). As costumary, a k–set is a set of size k, and a plane (line) intersecting $\mathcal{K}$ in i points is called *i–plane* (*i–line*). For $i = 1$ the line or the plane are called *tangent*. A *secant* line is an i–line with $2 \le i \le q$. Throughout the paper, let c_i (b_i) denote the number of i–planes (i–lines).

The next examples show that there are (q^2+q+1)–sets of plane type $(1, q+1, 2q+1)_2$ and containing exactly one line wich are not cones projecting an oval of a plane of the space.

Example 1. Let $C(V, \Gamma)$ a cone projecting an oval Γ from a point V not in Γ, and let ℓ be a generator of $C(V, \Gamma)$. Let $\mathcal{K}' = C(V, \Gamma) \setminus \ell$. It is a set of points of $\mathrm{PG}(3, q)$ of size q^2 and of plane–type $(0, q, 2q)$. Let L be an external line to $C(V, \Gamma)$ and let $\mathcal{K} = \mathcal{K}' \cup L$. The set $\mathcal{K}$ has size $q^2 + q + 1$ contains exactly one line and is of plane–type $(1, q+1, 2q+1)_2$.

Example 2. Let $C(V, \Gamma)$, ℓ and $\mathcal{K}'$ as in the above example. Let L be a tangent line at V to $C(V, \Gamma)$ and let $\mathcal{K} = \mathcal{K}' \cup L$. The set $\mathcal{K}$ has size $q^2 + q + 1$ contains exactly one line and is of plane–type $(1, q+1, 2q+1)_2$.

In this paper we prove the following two theorems.

Theorem 1. *Let $\mathcal{K}$ be a (q^2+q+1)–set of plane type $(1, m, n)_2$ containing at least one line. Then $m = q+1$, and there exists an integer $r \ge 2$ such that $n = rq+1$, $(r-1)|q$ and $r|q(q+1)$. The number of tangent planes is $c_1 \le \dfrac{q(q-1)}{2}$ and the equality occurs if and only if $r = 2$ and either $\mathcal{K}$ is a cone projecting an oval of a plane π from a point V not in π or $\mathcal{K}$ contains exactly one line ℓ and $\mathcal{K} = \ell \cup \mathcal{K}'$, with $\mathcal{K}'$ a q^2–set of plane–type $(0, q, 2q)_2$ (examples of such sets are described in Examples 1 and 2) .*

Theorem 2. *Let $\mathcal{K}$ be a (q^2+q+1)–set of plane type $(1, q+1, rq+1)_2$, $r \ge 2$, containing at least two lines. Then $\mathcal{K}$ is a cone projecting a $(q+1)$–set X of line–type $(0, 1, r)_1$ of a plane π from a point V not in π.*

Thus, when $r = 2$ by the results in [3] the set $\mathcal{K}$ is a cone projecting an oval of a plane π from a point V not in π .

Note that, the set X of the above Theorem 2 together with the r–secant lines is a Steiner system and so the existence of such sets $\mathcal{K}$, for $r \geq 3$, is related to the embeddability of a Steiner system in a projective plane (cf e.g. [1]).

We end this section, by remarking that we have only considered (q^2+q+1)–sets of points of $\mathrm{PG}(3,q)$ of plane type $(1,m,n)_2$ and not those of plane–class $[1,m,n]_2$, since as remarked in [6] a (q^2+q+1)–set of class $[1,m,n]_2$ is either a plane or of type $(1,m,n)_2$.

2 Proof of Theorems 1 and 2

Let $\mathcal{K}$ be a set of points of $\mathrm{PG}(3,q)$ of size q^2+q+1 of plane–type $(1,m,n)_2$ and containing at least one line, say ℓ.

If on ℓ there are only n–planes then $q^2+q+1 = q+1+(q+1)(n-q-1)$ and so $(q+1)|q^2$, a contradiction. Thus on ℓ there are m–planes, it follows that $m \geq q+1$.

Similarly, if on ℓ there are only m–planes then $m \geq q+2$ and $q^2+q+1 = q+1+(q+1)(m-q-1)$, again a contradiction.

Moreover, from $q^2+q+1 \leq (q+1)+(q+1)(n-q-1)$ it follows that $n \geq 2q+1$.

Proposition 1. *If $\mathcal{K}$ is a (q^2+q+1)–set of plane–type $(1,m,n)_2$ containing at least one line, then $m = q+1$ and there exists an integer r such that $n = rq+1$, $(r-1)|q$ and every line contained in $\mathcal{K}$ belongs to exactly $\dfrac{q}{r-1}$ n–planes.*

PROOF. Let ℓ be a line contained in $\mathcal{K}$ and let p be a point of $\mathcal{K}$ not in ℓ. Since $k = q^2+q+1$ we have that on p there is at least one tangent line.

Let t be a tangent line (skew with ℓ), counting the number of points $\mathcal{K}$ via the planes on t gives

$$q^2+q+1 = k \geq 1+(q+1)(m-1)$$

and so $m = q+1$.

Let x be the number of n–planes on ℓ, counting the number of points of $\mathcal{K}$ via the planes on ℓ we get

$$q^2+q+1 = q+1+x(n-(q+1))$$

and so $n-(q+1)$ divides q^2.

Let $q = p^h$ then $n-(q+1) = p^s$ with $e \leq 2h$. But $p^h+p^s+1 = n \geq 2q+1 = 2p^h+1$ implies $s \geq h$.

Write $n = p^h(p^{s-h}+1)+1$ and put $r := p^{s-h}+1$, thus $n = rq+1$ and $x = \dfrac{q^2}{n-q-1} = \dfrac{q}{r-1}$.

So, any line contained in $\mathcal{K}$ belongs to exactly $\dfrac{q}{r-1}$ n–planes. QED

Proposition 2. *Let $\mathcal{K}$ be a (q^2+q+1)–set of plane–type $(1, q+1, rq+1)_2$ containing at least one line , with $r \geq 2$ defined as above, then*

$$c_1 = \frac{q(q+1)}{r} - q, c_{q+1} = q^3+q^2+2q+1-\frac{q(q+1)}{r-1} \text{ and } c_n = \frac{q(q+1)}{r(r-1)}.$$

PROOF. Let p be a point of $\mathcal{K}$ not in ℓ, on p there is no tangent plane. Let σ_p and τ_p denote the number of m–planes and n–planes on p, respectively. Then, $\sigma_p + \tau_p = q^2+q+1$.

Consider the set of pairs $\{((p,x),\pi), x \neq p, x \in \mathcal{K}, x, p \in \pi\}$. Double counting give

$$q(q+1)^2 = \sigma_p \cdot q + \tau_p \cdot (rq+1-1) =$$
$$= \sigma_p \cdot q + (q^2+q+1-\sigma_p)rq = -\sigma_p \cdot q(r-1) + rq(q^2+q+1).$$

So,

$$q^2+2q+1 = -\sigma_p \cdot (r-1) + r(q^2+q+1)$$
$$\sigma_p \cdot (r-1) = (r-1)q^2 + (r-1)q + r - 1 - q$$
$$\sigma_p = q^2+q+1-\frac{q}{r-1}$$

It follows that any point $p \in \mathcal{K} \setminus \ell$ belongs to exactly $\dfrac{q}{r-1}$ n–planes.

Again double counting give

$$q^2\frac{q}{r-1} = \frac{q}{r-1}(r-1)q + (c_n - \frac{q}{r-1})(rq)$$

$$\frac{q^2}{r-1} = q + c_n r - \frac{rq}{r-1}$$

$$\frac{q^2-rq+q+rq}{r-1} = c_n r$$

$$c_n = \frac{q(q+1)}{r(r-1)}.$$

A similar counting argument shows that

$$c_{q+1} = q^3 + q^2 + 2q + 1 - \frac{q(q+1)}{r-1}.$$

And so, $c_1 = \dfrac{q(q+1)}{r} - q.$ $\boxed{QED}$

In order to complete the proofs of theorems 1 and 2 we distinguish two cases.

2.1 $\mathcal{K}$ has the maximum number of tangent planes.

Being $r \geq 2$ one gets

$$c_1 = \frac{q(q+1)}{r} - q \leq \frac{q(q-1)}{2}$$

and the equality holds if and only if $r = 2$ and so $\mathcal{K}$ is a (q^2+q+1)–set of plane–type $(1, q+1, 2q+1)_2$.

By the results in [3] it follows that if $\mathcal{K}$ contains at least two lines it is a cone projecting an oval of a plane π from a point V not in π.

Thus, we may assume that ℓ is the unique line contained in $\mathcal{K}$. Let $\mathcal{K}' = \mathcal{K} \backslash \ell$. Planes through ℓ are either $(q+1)$–planes or $(2q+1)$–planes, so they intersect $\mathcal{K}'$ in either 0 or q points. Planes not through ℓ intersect $\mathcal{K}$ in either 1 or $q+1$ or $2q+1$ points and so they intersect $\mathcal{K}'$ in either 0 or q or $2q$ points. It follows that $\mathcal{K}'$ is a set of q^2 points of plane–type $(0, q, 2q)_2$. A set of this type is described in Example 1

2.2 $\mathcal{K}$ contains at least two lines

Throughout this section, let ℓ and m denote two lines contained in $\mathcal{K}$. Since through a line contained in $\mathcal{K}$ there is at least one $(q+1)$–plane, it follows that the two lines ℓ and m interset each other in one point, say V.

Since there are tangent planes it follows that external lines to $\mathcal{K}$ do exist.

Proposition 3. *Let $\mathcal{K}$ be a (q^2+q+1)–set of plane–type $(1, q+1, rq+1)_2$ containing at least two lines. Any line external to $\mathcal{K}$ belongs to exactly one tangent plane and q $(q+1)$–planes.*

PROOF. Let ℓ_0 be a line external to $\mathcal{K}$. Being ℓ_0 skew with both ℓ and m, it belongs to at most one tangent plane (which has to contain V) and so the assertion follows by counting as usual the size of $\mathcal{K}$ via the planes on ℓ_0. $\boxed{QED}$

On each point of $\mathcal{K}$ there is at least one tangent line, let t be a tangent line not through V. Since through t there is no tangent plane, it follows that all the planes containing t are $(q+1)$–planes.

Let π be a $(q+1)$–plane not containing V, let p be a point of $\pi \cap \mathcal{K}$ and t be a tangent line through p contained in π. The plane $\langle V, t\rangle$ is a $(q+1)$–plane and it does not contain an external line. Namely this is true if $\langle V, t\rangle$ contains ℓ or m. So we may assume that it does contain neither ℓ nor m. Since any line of $\langle V, t\rangle$ belongs to planes intersecting both ℓ and m by Proposition 3 it follows that no line in $\langle V, t\rangle$ may be an external one. So, the set $\langle V, t\rangle \cap \mathcal{K}$ is a $(q+1)$–set intersected by every line of $\langle V, t\rangle$ and so it is necessarily a line.

It follows that for any point $p \in \pi \cap \mathcal{K}$ the line Vp is a line contained in $\mathcal{K}$ and so $\mathcal{K}$ is the set of points contained in the set union of the lines pV with $p \in \pi \cap \mathcal{K}$.

Thus, there is no $(rq+1)$–plane outside V, each one of these planes contains r lines contained in $\mathcal{K}$ and passing through V and so the lines of $\mathrm{PG}(3, q)$ intersect $\mathcal{K}$ in $0, 1, r$ and $q+1$ points. This proves Theorem 2.

References

[1] L.M. Batten, Minimally Projectively Embeddable Steiner Systems, Discrete Math. **77** (1989), 21–27.

[2] E. De Bernardinis, A note on a characterization of a quadric cone, Journal of Interdisciplinary Mathematics **20** (2017), 1711-1714.

[3] N. Durante, V. Napolitano and D. Olanda, On quadrics of $\mathrm{PG}(3, q)$, Quaderni di Matematica (ARACNE – Rome) **19** (2010), 67–75.

[4] S. Ferri, On some (l, m, n)–type sets of $\mathrm{PG}(3, q)$, Journal of Discrete Mathematical Sciences and Cryptography, **5** (2002), 227–230.

[5] S. Innamorati, M. Zannetti, In $\mathrm{PG}(3, p^{2^h})$ any $(p^{2^{h+1}}+1)$–set of class $[0, m, n]_1$ is an ovoid and any $(p^{2^{h+1}} + p^{2^h} + 1)$–set of class $[1, m, n]_2$ containing at least two lines is either a plane or a cone projecting an oval from a point, Journal of Combinatorial Mathematics and Combinatorial Computing **110** (2019), pp. 103–108.

[6] V. Napolitano, On $(q^2 + q + 1)$–sets of class $[1, m, n]_2$ in $\mathrm{PG}(3, q)$, J. Geom. **105** (2014), 449–455.

[7] V. Napolitano, A characterization of the Hermitian variety in finite 3-dimensional projective spaces, The Electronic Journal of Combinatorics **22** (1) (2015), 1–22.