Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **39** (2019) no. 1, 1–23. doi:10.1285/i15900932v39n1p1

Harmonic maps and biharmonic Riemannian submersions

Hajime Urakawa[i]
Institute for International Education, Tohoku University
Kawauchi 41, Sendai 980-8576, Japan.
The present address: Division of Mathematics, Graduate School of Information Sciences, Tohoku University
Aoba 6-3-09, Sendai 980-8579, Japan. urakawa@math.is.tohoku.ac.jp

Received: 21.9.2018; accepted: 24.1.2019.

Abstract. Characterizations for Riemannian submersions to be harmonic or biharmonic are shown. Examples of biharmonic but not harmonic Riemannian submersions are shown.

Keywords: Riemannian submersions, harmonic map, biharmonic map

MSC 2000 classification: primary 58E20, secondary 53C43

Introduction

Variational problems play central roles in geometry; Harmonic map is one of important variational problems which is a critical point of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2\, v_g$ for smooth maps φ of (M, g) into (N, h). The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [12] extended the notion of harmonic map to biharmonic map, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g. \tag{0.1}$$

After G.Y. Jiang [20] studied the first and second variation formulas of E_2, extensive studies in this area have been done (for instance, see [8], [24], [27], [37], [38], [15], [16], [19], etc.). Notice that harmonic maps are always biharmonic by definition. B.Y. Chen raised ([10]) so called B.Y. Chen's conjecture and later, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc raised ([8]) the generalized B.Y. Chen's conjecture.

B.Y. Chen's conjecture:

[i]This work is partially supported by the Grant-in-Aid for the Scientific Reserch (C) 18K03352, Japan Society for the Promotion of Science.

Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic (minimal).

The generalized B.Y. Chen's conjecture:

Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).

For the generalized Chen's conjecture, Ou and Tang gave ([36], [37]) a counter example in a Riemannian manifold of negative curvature. For the Chen's conjecture, affirmative answers were known for the case of surfaces in the three dimensional Euclidean space ([10]), and the case of hypersurfaces of the four dimensional Euclidean space ([14], [11]). K. Akutagawa and S. Maeta gave ([1]) showed a supporting evidence to the Chen's conjecture: *Any complete regular biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ is harmonic (minimal).* The affirmative answers to the generalized B.Y. Chen's conjecture were shown ([29], [30], [31]) under the L^2-condition and completeness of (M, g).

In [45], we treated with a principal G-bundle over a Riemannian manifold, and showed the following two theorems:

Theorem A. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold (M, h) with non-positive Ricci curvature. Assume P is compact so that M is also compact. If the projection π is biharmonic, then it is harmonic.*

Theorem B. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold with non-positive Ricci curvature. Assume that (P, g) is a non-compact complete Riemannian manifold, and the projection π has both finite energy $E(\pi) < \infty$ and finite bienergy $E_2(\pi) < \infty$. If π is biharmonic, then it is harmonic.*

We give two comments on the above theorems: For the generalized B.Y. Chen's conjecture, non-positivity of the sectional curvature of the ambient space of biharmonic submanifolds is necessary. However, it should be emphasized that for the principal G-bundles, we need not the assumption of non-positivity of the sectional curvature. We only assume *non-positivity of the Ricci curvature* of the domain manifolds in the proofs of Theorems A and B. Second, in Theorem B, finiteness of the energy and bienergy is necessary. Otherwise, one can see the following counter examples due to Loubeau and Ou ([25]):

Example C. (cf. [3], [25], p. 62) The inversion in the unit sphere $\phi : \mathbb{R}^n \backslash \{o\} \ni \mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2} \in \mathbb{R}^n$ is biharmonic if $n = 4$. It is not harmonic since $\tau(\phi) = -\frac{4\,\mathbf{x}}{|\mathbf{x}|^4}$.

Example D. (cf. [25], p. 70) Let (M^2, h) be a Riemannian surface, and let $\beta : M^2 \times \mathbb{R} \to \mathbb{R}^*$ and $\lambda : \mathbb{R} \to \mathbb{R}^*$ be two positive C^∞ functions. Consider the projection $\pi : (M^2 \times \mathbb{R}^*, g = \lambda^{-2}\, h + \beta^2\, dt^2) \ni (p,t) \mapsto p \in (M^2, h)$. Here, we take $\beta = c_2\, e^{\int f(x)\, dx}$, $f(x) = \frac{-c_1\,(1+e^{c_1 x})}{1-e^{c_1 x}}$ with $c_1,\, c_2 \in \mathbb{R}^*$, and $(M^2, h) = (\mathbb{R}^2, dx^2 + dy^2)$. Then,

$$\pi : (\mathbb{R}^2 \times \mathbb{R}^*, dx^2 + dy^2 + \beta^2(x)\, dt^2) \ni (x,y,t) \mapsto (x,y) \in (\mathbb{R}^2, dx^2 + dy^2)$$

gives a family of *proper biharmonic* (i.e., biharmonic but not harmonic) Riemannian submersions.

In this paper, we treat with a more general setting of Riemannian submersion $\pi : (P, g) \to (M, h)$ with a S^1 fiber over a compact Riemannian manifold (M, h). We first derive the tension field $\tau(\pi)$ and the bitension field $\tau_2(\pi)$ (Theorem 1). As a corollary of our main theorem, we show characterization theorems for a Riemannian submersion $\pi : (P, g) \to (M, h)$ over a compact Kähler-Einstein manifold (M, h), to be *biharmonic* (Theorems 2, 3, 4 and 5).

1 Preliminaries

1.1 Harmonic maps and biharmonic maps

We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi)\, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\left.\frac{d}{dt}\right|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{1.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M,g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X,Y) &= (\widetilde{\nabla} d\varphi)(X,Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \end{aligned} \tag{1.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^h, are Levi-Civita connections on TM, TN of (M,g), (N,h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \tag{1.3}$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{1.4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i} \overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i)) d\varphi(e_i)$, and R^N is the curvature tensor of (N,h) given by $R^h(U,V) = \nabla^h{}_U \nabla^h{}_V - \nabla^h{}_V \nabla^h{}_U - \nabla^h{}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [12] proposed polyharmonic (k-harmonic) maps and Jiang [20] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \tag{1.5}$$

where $|V|^2 = h(V,V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$\frac{d}{dt}\bigg|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g. \tag{1.6}$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \tag{1.7}$$

which is called the *bitension field* of φ, and J is given in (5).

A smooth map φ of (M,g) into (N,h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say, for an immersion $\varphi : (M,g) \to (N,h)$ to be *proper biharmonic* if it is biharmonic but not harmonic (minimal).

1.2 Riemannian submersions

We prepare with several notions on the Riemannian submersions. A C^∞ mapping π of a C^∞ Riemannian manifold (P, g) into another C^∞ Riemannian manifold (M, h) is called a *Riemannia submersion* if (0) π is surjective, (1) the differential $d\pi = \pi_* :\ T_uP \to T_{\pi(u)}M$ $(u \in P)$ of $\pi :\ P \to M$ is surjective for each $u \in P$, and (2) each tangent space T_uP at $u \in P$ has the direct decomposition:

$$T_uP = \mathcal{V}_u \oplus \mathcal{H}_u, \qquad (u \in P),$$

which is orthogonal decomposition with respect to g such that $\mathcal{V} = \mathrm{Ker}(\pi_{*\,u}) \subset T_uP$ and (3) the restriction of the differential $\pi_* = d\pi_u$ to $\mathcal{H}_u$ is a surjective isometry, $\pi_* :\ (\mathcal{H}_u, g_u) \to (T_{\pi(u)}M, h_{\pi(u)})$ for each $u \in P$ (cf. [4]). A manifold P is the total space of a Riemannian submersion over M with the projection $\pi : P \to M$ onto M, where $p = \dim P = k + m$, $m = \dim M$, and $k = \dim \pi^{-1}(x)$, $(x \in M)$. A Riemannian metric g on P, called *adapted metric* on P which satisfies

$$g = \pi^* h + k \tag{1.8}$$

where k is the Riemannian metric on each fiber $\pi^{-1}(x)$, $(x \in M)$. Then, T_uP has the orthogonal direct decomposition of the tangent space T_uP,

$$T_uP = \mathcal{V}_u \oplus \mathcal{H}_u, \qquad u \in P, \tag{1.9}$$

where the subspace $\mathcal{V}_u = \mathrm{Ker}(\pi_{*u})$ at $u \in P$, the *vertical subspace*, and the subspace $\mathcal{H}_u$ of P_u is called *horizontal subspace* at $u \in P$ which is the orthogonal complement of $\mathcal{V}_u$ in T_uP with respect to g.

In the following, we fix a locally defined orthonormal frame field, called *adapted local orthonormal frame field* to the projection $\pi :\ P \to M$, $\{e_i\}_{i=1}^p$ corresponding to (10) in such a way that

- $\{e_i\}_{i=1}^m$ is a locally defined orthonormal basis of the horizontal subspace $\mathcal{H}_u$ $(u \in P)$, and
- $\{e_i\}_{i=1}^k$ is a locally defined orthonormal basis of the vertical subspace $\mathcal{V}_u$ $(u \in P)$.

Corresponding to the decomposition (10), the tangent vectors X_u, and Y_u in T_uP can be defined by

$$X_u = X_u^{\mathrm{V}} + X_u^{\mathrm{H}}, \quad Y_u = Y_u^{\mathrm{V}} + Y_u^{\mathrm{H}}, \tag{1.10}$$

$$X_u^{\mathrm{V}},\ Y_u^{\mathrm{V}} \in \mathcal{V}_u, \quad X_u^{\mathrm{H}},\ Y_u^{\mathrm{H}} \in \mathcal{H}_u \tag{1.11}$$

for $u \in P$.

Then, there exist a unique decomposition such that

$$g(X_u, Y_u) = h(\pi_* X_u, \pi_* Y_u) + k(X_u^{\rm V}, Y_u^{\rm V}), \quad X_u,\ Y_u \in T_u P,\ u \in P.$$

Then, let us recall the following definitions for our question:

Definition 1. (1) The projection $\pi : (P, g) \to (M, h)$ is to be *harmonic* if the tension field vanishes, $\tau(\pi) = 0$, and

(2) the projection $\pi : (P, g) \to (M, h)$ is to be *biharmonic* if, the bitension field vanishes, $\tau_2(\pi) = J(\tau(\pi)) = 0$.

We define the Jacobi operator J for the projection π by

$$J(V) := \overline{\Delta} V - \mathcal{R}(V), \qquad V \in \Gamma(\pi^{-1}TM). \tag{1.12}$$

Here,

$$\overline{\Delta} V := -\sum_{i=1}^{p} \left\{ \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\} = \overline{\Delta}_{\mathcal{H}} V + \overline{\Delta}_{\mathcal{V}} V. \tag{1.13}$$

where

$$\overline{\Delta}_{\mathcal{H}} V = -\sum_{i=1}^{m} \left\{ \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\}, \tag{1.14}$$

$$\overline{\Delta}_{\mathcal{V}} V = -\sum_{i=1}^{k} \{ \overline{\nabla}_{A^*_{m+i}}(\overline{\nabla}_{A^*_{m+i}} V) - \overline{\nabla}_{\nabla_{A^*_{m+i}} A^*_{m+i}} V \}, \tag{1.15}$$

for $V \in \Gamma(\pi^{-1}TM)$, respectively. Recall, $\{e_i\}_{i=1}^p$ is a local orthonormal frame field on (P, g), $\{e_i\}_{i=1}^m$ is a local orthonormal horizontal field on (M, h) and $\{e_{m+i,\,u}\}_{i=1}^k$ $(u \in P)$ is an orthonormal frame field on the vertical space $\mathcal{V}_u$ $(u \in P)$. We call $\overline{\Delta}_{\mathcal{H}}$, the *horizontal Laplacian*, and $\overline{\Delta}_{\mathcal{V}}$, the *vertical Laplacian*, respectively.

2 The reduction of the biharmonic equation

2.1 Horizontal vector fields

Hereafter, we treat with the above problem more precisely in the case that $\dim(\pi^{-1}(x)) = 1$, $(u \in P,\ \pi(u) = x)$. Let $\{e_1,\ e_1,\ \ldots,\ e_m\}$ be an adapted local orthonormal frame field being $e_{n+1} = e_m$, vertical. The frame fields $\{e_i : i = 1, 2, \ldots, n\}$ are the basic orthonormal frame field on (P, g) corresponds to an orthonormal frame field $\{\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_n\}$ on (M,g). Here, a vector field $Z\in\mathfrak{X}(P)$ is *basic* if Z is horizontal and π-related to a vector field $X\in\mathfrak{X}(M)$.

In this section, we determine the biharmonic equation precisely in the case that $p=m+1=\dim P$, $m=\dim M$, and $k=\dim\pi^{-1}(x)=1$ $(x\in M)$. Since $[V,Z]$ is a vertical field on P if Z is basic and V is vertical (cf. [33], p. 461). Therefore, for each $i=1,\ldots,n$, $[e_i,e_{n+1}]$ is vertical, so we can write as follows.

$$[e_i,e_{n+1}]=\kappa_i\,e_{n+1},\quad i=1,\ \ldots,\ n \tag{2.1}$$

where $\kappa_i\in C^\infty(P)$ $(i=1,\ldots,n)$. For two vector fields $X,\ Y$ on M, let $X^*,\ Y^*$, be the horizontal vector fields on P. Then, $[X^*,Y^*]$ is a vector field on P which is π-related to a vector field $[X,Y]$ on M (for instance, [46], p. 143). Thus, for $i,\ j=1,\ldots,n$, $[e_i,e_j]$ is π-related to $[\epsilon_i,\epsilon_j]$, and we may write as

$$[e_i,e_j]=\sum_{k=1}^{n+1}D_{ij}^k\,e_k, \tag{2.2}$$

where $D_{ij}^k\in C^\infty(P)$ $(1\leq i,\,j\leq n;\ 1\leq k\leq n+1)$.

2.2 The tension field

In this subsection, we calculate the tension field $\tau(\pi)$. We show that

$$\tau(\pi)=-d\pi\left(\nabla_{e_{n+1}}e_{n+1}\right)=-\sum_{i=1}^n\kappa_i\,\epsilon_i. \tag{2.3}$$

Indeed, we have

$$\begin{aligned}\tau(\pi)&=\sum_{i=1}^m\left\{\nabla^\pi_{e_i}d\pi(e_i)-d\pi\left(\nabla_{e_i}e_i\right)\right\}\\&=\sum_{i=1}^n\left\{\nabla^\pi_{e_i}d\pi(e_i)-d\pi\left(\nabla_{e_i}e_i\right)\right\}+\nabla^\pi_{e_{n+1}}d\pi(e_{n+1})-d\pi\left(\nabla_{e_{n+1}}e_{n+1}\right)\\&=-d\pi\left(\nabla_{e_{n+1}}e_{n+1}\right)\\&=-\sum_{i=1}^n\kappa_i\epsilon_i.\end{aligned}$$

Because, for $i,\ j=1,\ldots,n$, $d\pi(\nabla_{e_i}e_j)=\nabla^h_{\epsilon_i}\epsilon_j$, and $\nabla^\pi_{e_i}d\pi(e_iu)=\nabla^h_{d\pi(e_i)}d\pi(e_i)=\nabla^h_{\epsilon}\epsilon_i$. Thus, we have

$$\sum_{i=1}^n\left\{\nabla^\pi_{e_i}d\pi(e_i)-d\pi\left(\nabla_{e_i}e_i\right)\right\}=0. \tag{2.4}$$

Since $e_{n+1} = e_m$ is vertical, $d\pi(e_{n+1}) = 0$, so that $\nabla^{\pi}_{e_{n+1}} d\pi(e_{n+1}) = 0$.

Furthermore, we have, by definition of the Levi-Civita connection, we have, for $i = 1, \ldots, n$,

$$2g(\nabla_{e_{n+1}e_{n+1}}, e_i) = 2g(e_{n+1}, [e_i, e_{n+1}]) = 2\kappa_i,$$

and $2g(\nabla_{e_{n+1}} e_{n+1}, e_{n+1}) = 0$. Therefore, we have

$$\nabla_{e_{n+1}} e_{n+1} = \sum_{i=1}^{n} \kappa_i e_i,$$

and then,

$$d\pi\left(\nabla_{e_{n+1}} e_{n+1}\right) = \sum_{i=1}^{n} \kappa_i \epsilon_i. \tag{2.5}$$

Thus, we obtain (19). QED

2.3 The bitension field

Let us recall first the bitension field $\tau_2(\pi)$ is given by

$$\begin{aligned} \tau_2(\pi) = &- \sum_{i=1}^{m} \left\{ \nabla^{\pi}_{e_i} \left(\nabla^{\pi}_{e_i} \tau(\pi) \right) - \nabla^{\pi}_{\nabla_{e_i} e_i} \tau(\pi) \right\} \\ &- \sum_{i=1}^{m} R^h(\tau(\pi), d\pi(e_i)) d\pi(e_i). \end{aligned} \tag{2.6}$$

First, since $d\pi(e_i) = \epsilon_i,\ i = 1, \ldots, n$, we have

$$\begin{aligned} \sum_{i=1}^{n} R^h(\tau(\pi), d\pi(e_i)) d\pi(e_i) &= \sum_{i=1}^{n} R^h(\tau(\pi), \epsilon_i)\epsilon_i \\ &= \mathrm{Ric}^h(\tau(\pi)). \end{aligned} \tag{2.7}$$

On the other hand, we calculate the first term of (22) for $\tau_2(\pi)$.

(The first step) To calculate $\nabla^{\pi}_{e_i} \tau(\pi)$ $(i = 1, \ldots, m = n + 1)$, we want to show

$$\nabla^{\pi}_{e_i} \tau(\pi) = \begin{cases} -\displaystyle\sum_{j=1}^{n} \left\{ (e_i \kappa_j)\epsilon_j + \kappa_j \nabla^h_{\epsilon_i} \epsilon_j \right\} & (i = 1, \ldots, n), \\ 0 & (i = n + 1). \end{cases} \tag{2.8}$$

Because, if $i=1,\dots,n$, by noticing $\kappa_j\in C^\infty(P)$, $(j=1,\dots,n))$, we have by (19),

$$\begin{aligned}\nabla^{\pi}_{e_i}\tau(\pi)&=\nabla^{\pi}_{e_i}\left(-\sum_{j=1}^{n}\kappa_j\,\epsilon_j\right)\\&=-\sum_{j=1}^{n}\{(e_i\,\kappa_j)\,\epsilon_j+\kappa_j\,\nabla^{\pi}_{e_i}\epsilon_j\}\\&=-\sum_{j=1}^{n}\left\{(e_i\,\kappa_j)\,\epsilon_j+\kappa_j\,\nabla^{h}_{\epsilon_i}\epsilon_j\right\},\end{aligned}\tag{2.9}$$

since $\nabla^{\pi}_{e_i}\epsilon_j=\nabla^{h}_{d\pi(e_i)}\epsilon_j=\nabla^{h}_{\epsilon_i}\epsilon_j$. Furthermore, for $i=n+1$, we have

$$\nabla^{\pi}_{e_{n+1}}\tau(\pi)=\nabla^{h}_{d\pi(e_{n+1})}\tau(\pi)=0.\tag{2.10}$$

To show (26), recalling the definition of the parallel displacement of the connection, let $P_{\pi\circ\sigma(t)}:\ T_{\pi(\sigma(0))}M\to T_{\pi(\sigma(t))}M$ be the parallel transport with respect to (M,h) along a smooth curve in P. Then, since $\sigma(t)\in P$, $\epsilon<t<\epsilon$ with $\sigma(0)=x\in P$ and $\dot\sigma(0)=e_{n+1\,x}\in T_xP$, for every $V\in\Gamma(\pi^{-1}TM)$, and then,

$$\nabla^{\pi}_{e_{n+1}}V(x)=\frac{d}{dt}\bigg|_{t=0}P^{-1}_{\pi\circ\sigma(t)}V(\sigma(t))=\frac{d}{dt}\bigg|_{t=0}P^{-1}_{\pi(x)}V(\sigma(t))=0,\tag{2.11}$$

since $\pi(\sigma(t))=\pi(\sigma(0))=\pi(x)\in P$ because e_{n+1} is a vertical vector field of the Riemannian submersion $\pi:\ (P,g)\to(M,h)$.

(The second step) To calculate $\nabla^{\pi}_{\nabla_{e_i}e_i}\tau(\pi)$ $(i=1,\dots,m=n+1)$, we have

$$\nabla^{\pi}_{\nabla_{e_i}e_i}\tau(\pi)=\begin{cases}-\displaystyle\sum_{j=1}^{n}\left\{(\nabla_{e_i}e_i\,\kappa_j)\epsilon_j+\kappa_j\,\nabla^{h}_{\nabla^{h}_{\epsilon_i}\epsilon_i}\epsilon_j\right\}\ (i=1,\dots,n),\\-\displaystyle\sum_{\ell,j=1}^{n}\left\{\kappa_\ell\,(e_\ell\kappa_j)\,\epsilon_j+\kappa_\ell\kappa_j\,\nabla^{h}_{\epsilon_\ell}\epsilon_j\right\}\ (i=n+1).\end{cases}\tag{2.12}$$

Indeed, for a vector field $\nabla_{e_i}e_i$ on P $(i=1,\dots,n)$, we only have to see that

$$d\pi(\nabla_{e_i}e_i)=\nabla^{h}_{\epsilon_i}\epsilon_i,\tag{2.13}$$

which yields the first equation of (28). To see (29), we have to see the following equations:

$$\begin{aligned}\nabla_{e_i} e_i &= \mathcal{V}(\nabla_{e_i} e_i) + \mathcal{H}(\nabla_{e_i} e_i)\\ &= A_{e_i} e_i + \mathcal{H}(\nabla_{e_i} e_i) \quad \text{(cf. the fourth of Lemma 3 in [33], p. 461)}\\ &= \frac{1}{2}\mathcal{V}[e_i, e_i] + \mathcal{H}(\nabla_{e_i} e_i) \qquad \text{(cf. Lemma 2 in [33], p. 461)}\\ &= \mathcal{H}(\nabla_{e_i} e_i). \end{aligned} \tag{2.14}$$

Here, since $\mathcal{H}(\nabla_{e_i} e_i)$ is a basic vector field corresponding to $\nabla^h_{\epsilon_i}\epsilon_i$ (cf. the third of Lemma 1 in [33], p. 460), we have $d\pi(\nabla_{e_i} e_i) = d\pi(\mathcal{H}(\nabla_{e_i} e_i)) = \nabla^h_{\epsilon_i}\epsilon_i$, i.e., (29). Then, we have

$$\begin{aligned}\nabla^\pi_{\nabla_{e_i} e_i}\tau(\pi) &= \sum_{\nabla_{e_i} e_i}(-\sum_{j=1}^n \kappa_j\epsilon_j)\\ &= -\sum_{j=1}^n\left\{(\nabla_{e_i} e_i\,\kappa_j)\,\epsilon_j + \kappa_j\nabla^\pi_{\nabla_{e_i} e_i}\epsilon_j\right\}\\ &= -\sum_{j=1}^n\left\{(\nabla_{e_i} e_i\,\kappa_j)\,\epsilon_j + \kappa_j\,\nabla^h_{\nabla^h_{\epsilon_i}\epsilon_i}\epsilon_j\right\},\end{aligned} \tag{2.15}$$

which is the first equation of (28). To see the second equation of (28), recall (21) $d\pi(\nabla_{e_{n+1}} e_{n+1}) = \sum_{i=1}^n \kappa_i\epsilon_i$ and also the first equation of (24). Then, we have

$$\begin{aligned}\nabla^\pi_{\nabla_{e_{n+1}} e_{n+1}}\tau(\pi) &= -\nabla^h_{\left(\sum_{i=1}^n \kappa_i\epsilon_i\right)}\sum_{j=1}^n \kappa_j\epsilon_j\\ &= -\sum_{i,j=1}^n\left\{\kappa_i\,\epsilon_i(\kappa_j)\,\epsilon_j + \kappa_i\kappa_j\,\nabla^h_{\epsilon_i}\epsilon_j\right\},\end{aligned} \tag{2.16}$$

which implies the second equation of (28).

(The third step) We calculate $\nabla^\pi_{e_i}(\nabla^\pi_{e_i}\tau(\pi))$. Indeed, we have

$$\begin{aligned}\nabla^\pi_{e_i}\left(\nabla^\pi_{e_i}\tau(\pi)\right) &= \nabla^\pi_{e_i}\left(-\sum_{j=1}^n\left\{(e_i\kappa_j)\,\epsilon_j + \kappa_j\,\nabla^h_{\epsilon_i}\epsilon_j\right\}\right)\\ &= -\sum_{j=1}^n\left\{e_i(e_i\kappa_j)\,\epsilon_j + (e_i\kappa_j)\nabla^\pi_{e_i}\epsilon_j + (e_i\kappa_j)\,\nabla^h_{\epsilon_i}\epsilon_j + \kappa_j\nabla^\pi_{e_i}\left(\nabla^h_{\epsilon_i}\epsilon_j\right)\right\},\end{aligned} \tag{2.17}$$

where

$$\begin{cases}\nabla^\pi_{e_i}\epsilon_j = \nabla^h_{d\pi(e_i)}\epsilon_j = \nabla^h_{\epsilon_i}\epsilon_j,\\ \nabla^\pi_{e_i}(\nabla^h_{\epsilon_i}\epsilon_j) = \nabla^h_{d\pi(e_i)}(\nabla^h_{\epsilon_i}\epsilon_j) = \nabla^h_{\epsilon_i}(\nabla^h_{\epsilon_i}\epsilon_j).\end{cases} \tag{2.18}$$

Then we have, for $i=1,\ldots,n$,

$$\begin{cases}\nabla^{\pi}_{e_i}\left(\nabla^{\pi}_{e_i}\tau(\pi)\right)=-\sum_{j=1}^{n}\Big\{e_i(e_i\kappa_j)\,\epsilon_j+2(e_i\kappa_j)\,\nabla^h_{\epsilon_i}\epsilon_j+\kappa_j\,\nabla^h_{\epsilon_i}(\nabla^h_{\epsilon_i}\epsilon_j)\Big\},\\ \nabla^{\pi}_{e_{n+1}}(\nabla^{\pi}_{e_{n+1}}\tau(\pi))=0,\\ \nabla^{\pi}_{\nabla_{e_i}e_i}\tau(\pi)=-\sum_{j=1}^{n}\Big\{(\nabla_{e_i}e_i\,\kappa_j)\,\epsilon_j+\kappa_j\,\nabla^h_{\nabla^h_{\epsilon_i}\epsilon_i}\epsilon_j\Big\},\\ \nabla^{\pi}_{\nabla_{e_{n+1}}e_{n+1}}\tau(\pi)=-\sum_{i,j=1}^{n}\Big\{\kappa_i(e_i\kappa_j)\,\epsilon_j+\kappa_i\kappa_j\,\nabla^h_{\epsilon_i}\epsilon_j\Big\}.\end{cases}\tag{2.19}$$

(The fourth step) Therefore, we have

$$\begin{aligned}\tau_2(\pi)&=\overline{\Delta}^h\,\tau(\pi)-\mathrm{Ric}^h(\tau(\pi))\\&=-\sum_{i=1}^{m}\Big\{\nabla^{\pi}_{e_i}\left(\nabla^{\pi}_{e_i}\tau(\pi)\right)-\nabla^{\pi}_{\nabla_{e_i}e_i}\tau(\pi)\Big\}-\mathrm{Ric}^h(\tau(\pi))\\&=\sum_{i,j=1}^{n}\Big\{e_i(e_i\kappa_j)\,\epsilon_j+2(e_i\kappa_j)\nabla^h_{\epsilon_i}\epsilon_j+\kappa_j\,\nabla^h_{\epsilon_i}(\nabla^h_{\epsilon_i}\epsilon_j)\\&\qquad-(\nabla_{e_i}e_i\,\kappa_j)\epsilon_j-\kappa_j\,\nabla^h_{\nabla^h_{\epsilon_i}\epsilon_i}\epsilon_j-\kappa_i(e_i\kappa_j)\,\epsilon_j-\kappa_i\kappa_j\,\nabla^h_{\epsilon_i}\epsilon_j\Big\}\\&\quad+\mathrm{Ric}^h\left(\sum_{j=1}^{n}\kappa_j\epsilon_j\right)\\&=\sum_{j=1}^{n}\sum_{i=1}^{n}\Big\{e_i(e_i\,\kappa_j)-\nabla_{e_i}e_i\,\kappa_j\Big\}\epsilon_j+2\sum_{j=1}^{n}\nabla^h_{(\sum_{i=1}^{n}(e_i\kappa_j)\epsilon_i)}\epsilon_j\\&\quad+\sum_{j=1}^{n}\kappa_j\sum_{i=1}^{n}\Big\{\nabla^h_{\epsilon_i}\nabla^h_{\epsilon_i}\epsilon_j-\nabla^h_{\nabla^h_{\epsilon_i\epsilon_i}}\epsilon_j\Big\}-\nabla_{(\sum_{i=1}^{n}\kappa_i\epsilon_i)}\sum_{j=1}^{n}\kappa_j\epsilon_j\\&\quad+\mathrm{Ric}^h(\sum_{j=1}^{n}\kappa_j\epsilon_j)\end{aligned}$$

$$\begin{aligned}&=\sum_{j=1}^{n}\Big\{-\Delta\kappa_j-e_{n+1}(e_{n+1}\kappa_j)+\nabla_{e_{n+1}}e_{n+1}\,\kappa_j\Big\}\epsilon_j\\&\quad+2\sum_{j=1}^{n}\nabla^h_{(\sum_{i=1}^{n}(e_i\kappa_j)\epsilon_i)}\epsilon_j-\sum_{j=1}^{n}\kappa_j(\overline{\Delta}^h\epsilon_j)-\nabla_{(\sum_{i=1}^{n}\kappa_i\epsilon_i)}\sum_{j=1}^{n}\kappa_j\epsilon_j\end{aligned}$$

$$
\begin{aligned}
&\quad + \mathrm{Ric}^h(\sum_{j=1}^n \kappa_j\epsilon_j) \\
&= \sum_{j=1}^n (-\Delta^h \kappa_j)\,\epsilon_j \\
&\quad + 2\sum_{j=1}^n \nabla^h_{(\sum_{i=1}^n (e_i\kappa_j)\epsilon_i)}\epsilon_j - \sum_{j=1}^n \kappa_j(\overline{\Delta}^h\epsilon_j) - \nabla_{(\sum_{i=1}^n \kappa_i\epsilon_i)}\sum_{j=1}^n \kappa_j\epsilon_j \\
&\quad + \mathrm{Ric}^h(\sum_{j=1}^n \kappa_j\epsilon_j).
\end{aligned} \tag{2.20}
$$

Since

$$
\overline{\Delta}^h(\kappa_j\epsilon_j) = (\overline{\Delta}^h\kappa_j)\epsilon_j - 2\sum_{i=1}^n (e_i\kappa_j)\,\nabla^h_{\epsilon_i}\epsilon_j + \kappa_j(\overline{\Delta}^h\epsilon_j), \tag{2.21}
$$

we obtain

$$
\tau_2(\pi) = -\Delta^h\big(\sum_{j=1}^n \kappa_j\epsilon_j\big) - \nabla^h_{(\sum_{i=1}^n \kappa_i\epsilon_i)}\sum_{j=1}^n \kappa_j\epsilon_j + \mathrm{Ric}^h\big(\sum_{j=1}^n \kappa_j\epsilon_j\big). \tag{2.22}
$$

Thus, we obtain the following theorem:

Theorem 1. *Let $\pi : (P,g) \to (M,h)$ be a Riemannian submersion over (M,h). Then,*

(1) *The tension field $\tau(\pi)$ of π is given by*

$$
\tau(\pi) = -\sum_{i=1}^n \kappa_i\epsilon_i, \tag{2.23}
$$

where $\kappa_i \in C^\infty(P)$, $(i = 1, \dots, n)$.

(2) *The bitension field $\tau_2(\pi)$ of π is given by*

$$
\tau_2(\pi) = -\overline{\Delta}^h\big(\sum_{j=1}^n \kappa_j\epsilon_j\big) + \nabla^h_{(\sum_{i=1}^n \kappa_i\epsilon_i)}\sum_{j=1}^n \kappa_j\epsilon_j + \mathrm{Ric}^h\big(\sum_{j=1}^n \kappa_j\epsilon_j\big). \tag{2.24}
$$

Remark 1. The bitension field $\tau_2(\pi)$ for π has been obtained in a different way by Akyol and Ou [2] in which has referenced our paper.

Proposition 1. *Let $\pi : (P, g) \to (M, h)$ be a Riemannian submersion whose base manifold (M, h) has non-positive Ricci curvature. Assume that $\pi : (P, g) \to (M, h)$ is biharmonic. Then the tension field $X := \tau(\pi)$ is parallel, i.e., $\nabla^h X = 0$ if we assume div$(X) = 0$.*

Proof Assume that $\pi : (P, g) \to (M, h)$ is biharmonic, i.e.,

$$0 = \tau_2(\pi) = -\overline{\Delta}^h X - \nabla^h_X X + \mathrm{Ric}^h(X).$$

Then, we have

$$\begin{aligned}
0 &\leq \int_M \overline{\nabla}^h X, \overline{X}^h X)\, v_h \\
&= \int_M h(\overline{\Delta}^h X, X)\, g_h \\
&= -\int_M h(\nabla^h_X X, X)\, v_h + \int_M h(\mathrm{Ric}^h(X), X)\, v_h \\
&= -\frac{1}{2}\int_M X \cdot h(X, X)\, v_h + \int_M h(\mathrm{Ric}^h(X), X)\, v_h \\
&= \int_M (\mathrm{Ric}^h(X), X)\, v_h \leq 0. \qquad (2.25)
\end{aligned}$$

The second equality from below holds, due to Gaffney's theorem (cf. Theorem 2.2 in [35]), $\int_M X f\, v_h = 0$ ($f \in C^1(M)$) if $\mathrm{div}(X) = 0$. The last inequality holds for non-positive Ricci curvature of (M, h). Therefore, we have

$$0 = h(\mathrm{Ric}(X), X)\, v_h = \int_M h(\overline{X}^h, \overline{X}^h)\, v_h.$$

Thus, we have $\overline{\nabla}^h X = 0$. QED

3 Einstein manifolds

3.1 Riemannian submersions over Einstein manifolds

Regarding the orthogonal direct decomposition:

$$\mathfrak{X}(M) = \{X \in \mathfrak{X}(M) |\ \mathrm{div}(X) = 0\} \oplus \{\nabla f \in \mathfrak{X}(M) |\ f \in C^\infty(M)\}, \qquad (3.1)$$

we obtain the following theorems:

Theorem 2. *Let $\pi : (P, g) \to (M, h)$ be a compact Riemannian submersion over a weakly stable Einstein manifold (M, g) whose Ricci tensor ρ^h satisfies $\rho^h = c\,\mathrm{Id}$ for some constant c. Assume that π is biharmonic, i.e.,*

$$\tau_2(\pi) = -\overline{\Delta}^h X + \nabla^h_X X + \mathrm{Ric}^h(X) = 0, \qquad (3.2)$$

where $X=\sum_{i=1}^n \kappa_i\epsilon_i$. *Assume that* $\mathrm{div}X=0$. *Then,*

$$\begin{cases}\overline{\Delta}^h X=cX,\\ \nabla^h_X X=0.\end{cases}\tag{3.3}$$

Proof Let $X=\sum_{i=1}^{\infty} X_i$ where $\Delta^H X_i=\lambda_i X_i$ satisfying that $\int_M h(X_i,X_j)\,v_h=\delta_{ij}$. Δ^H corresponds to the Laplacian Δ^1 acting on the space $A^1(M)$ of 1-forms on (M,h). By (42),

$$\begin{aligned}-\nabla^h_X X&=\overline{\Delta}^h X-cX\\ &=\sum_{i=1}^{\infty}\lambda_i X_i-2c\sum_{i=1}^{\infty}X_i \qquad (3.4)\\ &=\sum_{i=1}^{\infty}(\lambda_i-2c)X_i \qquad (3.5)\end{aligned}$$

since $\Delta^H=\overline{\Delta}^h+\rho^h=\overline{\Delta}+c\,\mathrm{Id}$. Since $\mathrm{div}(X)=0$,

$$\begin{aligned}0&=-\frac{1}{2}\int_M X\,\cdot\,h(X,X)\,v_h\\ &=-\int_M h(\nabla^h_X X,X)\,v_h\\ &=\int_M h(\sum_{i=1}^{\infty}(\lambda_i-2c)X_i,\sum_{j=1}^{\infty}X_j)\,v_h\\ &=\sum_{i=1}^{\infty}(\lambda_i-2c). \qquad (3.6)\end{aligned}$$

If (M,h) is weakly stable, i.e., $2c\leq\lambda_1^1(h)\leq\lambda_i$ $(i=1,2,\ldots)$, then we have

$$\lambda_i=2c\qquad(i=1,2,\ldots).$$

Therefore, we have

$$\overline{\Delta}^h X+cX=\Delta^H X=\sum_{i=1}^{\infty}\lambda_i X_i=2c\sum_{i=1}^{\infty}X_i=2cX.$$

Therefore,

$$\begin{cases}\overline{\Delta}^h X=cX,\\ \nabla^h_X X=0.\end{cases}$$

We have Theorem 2. *QED*

We have immediately the following theorem and corollary:

Theorem 3. *Let $\pi : (P,g) \to (M,h)$ be a compact Riemannian submersion over an irreducible compact Hermitian symmetric space $(M,h) = (K/H,h)$ where K is a compact semi-simple Lie group, and H, a closed subgroup of K, h, an invariant Riemannan metric on $M = K/H$, respectively. Let $X \in \mathfrak{k}$ be an invariant vector field on M. Then,* $\mathrm{div} X = 0$*, and that*

$$\begin{cases} \overline{\Delta}^h X = cX, \\ \nabla^h_X X = 0. \end{cases} \tag{3.7}$$

Corollary 1. *Let $\pi : (P,g) \to (M,h)$ be a principal S^1- bundle over an n-dimensional compact Hermitian symmetric space (M,h). Then,*

$$\tau(\pi) = -\sum_{j=1}^{n} \kappa_j \widetilde{\epsilon}_j \in \Gamma(\pi^{-1}TM). \tag{3.8}$$

If $X = \sum_{i=1}^{n} \kappa_j \epsilon_j$ is a non-vanishing Killing vector field on (M,h), $\pi : (P,g) \to (M,h)$ is biharmonic, but not harmonic.

3.2 Analytic vector fields and the first eigenvalue

Regarding (42), we now consider the case $\{\nabla f \in \mathfrak{X}(M) | \ f \in C^\infty(M)\}$. Recall a theorem of M. Obata on a compact Kähler-Einstein Riemannian manifold (M,h) ([46], p. 181), the first non-zero positive eigenvalue $\lambda_1(h)$ of (M,h) satisfies that

$$\lambda_1(h) \geq 2c, \tag{3.9}$$

and if the equality $\lambda_1(h) = 2c$ holds, the corresponding eigenfunction f with the eigenvalue $2c$ satisfies that ∇f is an analytic vector field on M ([46], p. 174) and

$$J_{\mathrm{id}}(\nabla f) = 0, \tag{3.10}$$

where J_{id} is the Jacobi operator given by $J_{\mathrm{id}} := \overline{\Delta}^h - 2\,\mathrm{Ric}$.

We apply the above to the our situation that $\pi : (P,g) \to (M,h)$ is a compact Riemannian submersion over a compact Kähler-Einstein manifold (M,h) with $\mathrm{Ric}^h = c\,\mathrm{Id}$, and assume that $\pi : (P,g) \to (M,h)$ is biharmonic, i.e.,

$$\overline{\Delta}^h X + \nabla^h{}_X X - \mathrm{Ric}^h(X) = 0, \tag{3.11}$$

where $X = \tau(\pi) \in \Gamma(\pi^{-1}TM)$.

Thus, we can summarize the above as follows:

Theorem 4. *Assume that our $X = \tau(\pi)$ is of the form, $X = \nabla f$, where f is the eigenfunction of the Laplacian Δ_h acting on $C^\infty(M)$ with the first eigenvalue $\lambda_1(h) = 2c$.*

Then X is an analytic vector field on M ([46], p. 174) and

$$J_{\rm id}(X) = 0, \tag{3.12}$$

where $J_{\rm id}$ is the Jacobi operator given by $J_{\rm id} := \overline{\Delta}^h - 2\,\mathrm{Ric}$.

Furthermore, we have

$$\Delta_H X = 2c\,X, \text{ i.e., } \quad \overline{\Delta}^h X = c\,X, \tag{3.13}$$

and also

$$\nabla^h{}_X X = 0. \tag{3.14}$$

Here, Δ_H is the operator acting on $\mathfrak{X}(M)$ corresponding to the standard Laplacian $\Delta := d\delta + \delta d$ on the space $A^1(M)$ of 1-forms on (M, h).

3.3 The divergence of an analytic vector field

In this part, we show

Proposition 2. *Under the above situation, we have, at each point $p \in P$,*

$$\mathrm{div}(X)(p) = \sum_{i=1}^n e_i\,\kappa_i\,(p), \tag{3.15}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal frame field on a neighborhood of each point $p \in P$ satisfying that $(\nabla_Y e_i)(p) = 0$, $\forall\ Y \in T_pP$ $(i = 1, \ldots, n)$.

Proof. Let us recall $X := \tau(\pi) = -\sum_{i=1}^n \kappa_i\,\widetilde{\epsilon}_i \in \Gamma(\pi^{-1}TM)$, where $\kappa_i \in C^\infty(P)$, $\widetilde{\epsilon}_i = \pi^{-1}\epsilon_i \in \Gamma(\pi^{-1}TM)$ defined by

$$\widetilde{\epsilon}_i(p) := (\pi^{-1}\epsilon_i)(p) = \epsilon_{i\,\pi(p)}, \quad (p \in P),$$

and $\{\epsilon_i\}_{i=1}^n$ is a locally defined orthonormal frame field on (M, h). Here, note that, for $p \in P$, $\pi(p) = x \in M$,

$$X(p) = -\sum_{i=1}^n \kappa_i(p)\widetilde{\epsilon}_i(p) = -\sum_{i=1}^n \kappa_i(p)\epsilon_i(x) \in T_xM.$$

Let $\widetilde{\nabla}$ be the induced connection on $\Gamma(\pi^{-1}TM)$ from the Levi-Civita connection ∇^h of (M,h), and define $\mathrm{div}(X)\in C^{\infty}(P)$ by

$$\begin{aligned}\mathrm{div}(X)(p) :&= \sum_{i=1}^{m} g_p(e_{i\,p},\,(\widetilde{\nabla}_{e_i}X)(p)) = \sum_{i=1}^{m} g_p(e_{i\,p},\nabla^h_{\pi_* e_i}X)\\ &= \sum_{i=1}^{n} g_p(e_{i\,p},(\widetilde{\nabla}_{e_i}X)(p)),\end{aligned} \tag{3.16}$$

where $m=n+1=\dim(P)$. Because $\widetilde{\nabla}_{e_{n+1}}X(p)=0$ since, for a C^1 curve σ in P with $\sigma(0)=p,\ \sigma'(0)=(e_{n+1})_p\in T_pP$, we have $\pi\,\circ\,\sigma_t(s)=x,\ \forall\ 0\leq s\leq t$. Therefore, we have

$$(\widetilde{\nabla}_{e_{n+1}}X)(x)=\nabla^h_{\pi_* e_{n+1}}X=\frac{d}{dt}\bigg|_{t=0} {P^h_{\pi\,\circ\,\sigma_t}}^{-1}X(\sigma(t))=0,$$

where $P^h_{\pi\,\circ\,\sigma_t}\,:\,T_{\pi(p)}M\,\to\,T_{\pi(\sigma(t))}M$ is the parallel displacement along a C^1 curve $\pi\,\circ\,\sigma_t$ with respect to ∇^h on (M,h). Then, for the RHS of (57), we have

$$\begin{aligned}\mathrm{div}(X)(p) &= \sum_{i=1}^{n} g_p(e_{i\,p},(\widetilde{\nabla}_{e_i}X)(p))\\ &= \sum_{i=1}^{n} g_p(e_{i\,p},\widetilde{\nabla}_{e_i}(\sum_{j=1}^{n}\kappa_j\widetilde{\epsilon}_j))\\ &= \sum_{i=1}^{n} g_p\big(e_{i\,p},\sum_{j=1}^{n}\big\{e_i\kappa_j(p)\,\widetilde{\epsilon}_j(p)+\kappa_j(p)(\widetilde{\nabla}_{e_i}\widetilde{\epsilon}_j)(p)\big\}\big)\\ &= \sum_{i,j=1}^{n}(e_i\kappa_j)(p)\,g_p(e_{i\,p},\widetilde{\epsilon}_j(p))+\sum_{i,j=1}^{n}\kappa_j(p)\,g_p(e_{i\,p},(\widetilde{\nabla}_{e_i}\widetilde{\epsilon}_j)(p))\\ &= \sum_{i=1}^{n}(e_i\kappa_i)(p)-g_p(\sum_{i=1}^{n}\nabla^g_{e_i}e_i,\sum_{j=1}^{n}\kappa_j(p)\widetilde{\epsilon}_j)\\ &= \sum_{i=1}^{n}e_i\kappa_i+g(\sum_{i=1}^{n}\nabla^g_{e_i}e_i,X),\end{aligned} \tag{3.17}$$

since

$$g_p(e_{i\,p},(\widetilde{\nabla}_{e_i}\widetilde{\epsilon}_j)(p)=e_{i\,p}g(e_i,\widetilde{\epsilon}_j)-g_p(\nabla^g_{e_i}e_i,\widetilde{\epsilon}_j)=-g_p(\nabla^g_{e_i}e_i,\widetilde{\epsilon}_j)$$

by means of $e_{i\,p}\,g(e_i,\widetilde{\epsilon}_j)=0$. By noticing that $g(\sum_{i=1}^n\nabla^g_{e_i}e_i,X)=0$ at the point $p\in P$ because of a choice of $\{e_i\}$, we obtain (56). $\boxed{QED}$

4 Kähler-Einstein flag manifolds

Let $(M, h) = (K/T, h)$ be a Kähler-Einstein flag manifold with $\mathrm{Ric}^h = c\,\mathrm{Id}$ for some $c > 0$, where T be a maximal torus in K, and let E_λ, the line bundle over K/T associated to non-trivial homomorphism $\lambda:\ T \to \mathbb{C}^*$. Then, E_λ is the totality of all equivalence classes $[k, v]$ including (k, v) with $k \in K$ and $v \in \mathbb{C}^*$ under the equivalence relation $(k', v') \sim (k, v)$, i.e., $k' = ka,\ v' = \lambda(a^{-1})v$ for some $a \in T$. Let $\mathcal{S}_\lambda := \{[k, u]|\, k \in K,\ u \in S^1\} = \{(k, u)|\, k \in K,\ u \in S^1\}/\sim$. Then, $\mathcal{S}_\lambda$ is the circle bundle over a flag manifold K/T associted to $\lambda:\ T \to S^1$, where $S^1 = \{u \in \mathbb{C}|\ |u| = 1\}$. Note that $m := \dim \mathcal{S} = n+1$, with $n = \dim M = \dim K/T$.

Example 1. For $r = 1, 2, \ldots$, let

$$K = SU(r+1) \supset T = \left\{ \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta_1} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & e^{2\pi\sqrt{-1}\,\theta_{r+1}} \end{bmatrix} \Bigg| \right.$$

$$\left. \theta_1,\ \ldots,\ \theta_{r+1} \in \mathbb{R},\ \theta_1 + \cdots + \theta_{r+1} = 0 \right\},$$

and for $\mathcal{I} = (a_1,\ \ldots, a_{r+1}) \in \mathbb{Z}^{r+1}$, let

$$\lambda_{\mathcal{I}}:\ T \ni \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta_1} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & e^{2\pi\sqrt{-1}\,\theta_{r+1}} \end{bmatrix} \mapsto e^{2\pi\sqrt{-1}\,(a_1\theta_1+\cdots+a_{s+1}\theta_{r+1})} \in S^1,$$

where $S^1 = \{z \in \mathbb{C}|\,|z| = 1\}$, and $a_1,\ \ldots, a_{r+1} \in \mathbb{Z}$. The action of T on $K \times S^1 = SU(r+1) \times S^1$ by

$$(x, e^{2\pi\sqrt{-1}\,\theta}) \cdot a = (xa,\ \lambda_{\mathcal{I}}(a^{-1})\, e^{2\pi\sqrt{-1}\,\theta}), \qquad a \in T.$$

The orbit space

$$\begin{aligned} P = \mathcal{S}_\lambda &= SU(s+1) \times S^1/\!\sim \\ &= \{(x, e^{2\pi\sqrt{-1}\,\theta})|x \in SU(r+1),\ \theta \in \mathbb{R}\}/\!\sim \end{aligned}$$

whose equivalence relation is given by $(x', e^{2\pi\sqrt{-1}\,\theta'}) \sim (x, e^{2\pi\sqrt{-1}\,\theta})$ is equivalent to that: $x' = xt$ and $e^{2\pi\sqrt{-1}\,\theta'} = e^{2\pi\sqrt{-1}\,\theta}\lambda_{\mathcal{I}}(t^{-1})$. We denote the equivalence class including $(x, e^{2\pi\sqrt{-1}\,\theta})$ by $[x, e^{2\pi\sqrt{-1}\,\theta}]$. Then, we have the principal S^1-bundle $P = \mathcal{S}_\lambda$ over K/T associated to $\lambda_{\mathcal{I}}$, which is the space of all T-orbits through $(x, e^{2\pi\sqrt{-1}\,\theta})$, $x \in SU(r+1),\ \theta \in \mathbb{R}$, namely,

$$P = \mathcal{S}_\lambda = \{[x, e^{2\pi\sqrt{-1}\,\theta}]|\, x \in SU(r+1),\ \theta \in \mathbb{R}\}.$$

Example 2. In particular, let us consider the case $r = 1$. Let

$$K = SU(2) \supset T = \left\{ \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\,\theta} \end{bmatrix} \mid \theta \in \mathbb{R} \right\},$$

$\dim(K/T) = 2$ and $\dim P = 3$. For $a_1,\ a_2 \in \mathbb{Z}$, and $\ell = a_1 - a_2$, let

$$\lambda_{\mathcal{I}} :\ T \ni \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\,\theta} \end{bmatrix} \mapsto e^{2\pi\sqrt{-1}\,(a_1-a_2)\theta} = e^{2\pi\sqrt{-1}\,\ell\,\theta} \in S^1$$

and T acts on $SU(2) \times S^1$ by

$$(x, e^{2\pi\sqrt{-1}\,\xi}) \cdot a := (xa, e^{2\pi\sqrt{-1}\,\ell\,\theta}\, e^{2\pi\sqrt{-1}\,\xi}),$$

for $a = \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\,\theta} \end{bmatrix} \in T$, $x \in SU(2)$, $\xi \in \mathbb{R}$. Then, P is diffeomorphic with S^3, and $M = K/T$ is diffeomorphic with $P^1(\mathbb{C})$, and we have $\pi :\ P = \mathcal{S}_{\lambda_{\mathcal{I}}} \to M = K/T = SU(2)/S^1 = P^1(\mathbb{C})$. Let

$$\begin{aligned}
\mathfrak{k} &= \mathfrak{s}u(2) = \{X \in \mathfrak{g}l(2,\mathbb{C}) |\, {}^{\mathrm{t}}\overline{X} + X = 0,\ \mathrm{Tr}(X) = 0\}, \\
\mathfrak{t} &= \mathfrak{g}(\mathfrak{u}(1) \times \mathfrak{u}(1)) = \left\{ \begin{pmatrix} \sqrt{-1}\,\theta & 0 \\ 0 & -\sqrt{-1}\,\theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}, \\
\mathfrak{m} &= \left\{ \begin{pmatrix} 0 & -\overline{z} \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\},
\end{aligned}$$

respectively. Let $\langle\, \cdot\,, \cdot\,\rangle$ be the inner product on $\mathfrak{k}$ defined by

$$\langle X, Y \rangle := -\frac{1}{2}\mathrm{Tr}(X\,Y), \qquad X,\ Y \in \mathfrak{k}.$$

Then, for $X = \begin{pmatrix} 0 & -\overline{z} \\ z & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -\overline{w} \\ w & 0 \end{pmatrix} \in \mathfrak{m}$,

$$\langle X, Y \rangle = x\xi + y\eta, \qquad z = x + \sqrt{-1}\,y,\ w = \xi + \sqrt{-1}\,\eta,\ \ x,\ y,\ \xi,\ \eta \in \mathbb{R},$$

and h, the G-invariant Riemannian metric on $M = K/T = P^1(\mathbb{C})$ in such a way that

$$h_o(X_o, Y_o) = \langle X, Y \rangle, \qquad X,\ Y \in \mathfrak{m},$$

where $o = \{T\} \in M = K/T$. Let $\{H_1,\ X_1,\ X_2\}$ be an orthonormal basis of $\mathfrak{k}$ with respect to $\langle\cdot\,,\cdot\rangle$ where

$$H_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

satisfying that

$$[H_1, X_1] = 2X_2, \quad [X_2, H_1] = 2X_1, \quad [X_1, X_2] = 2H_1.$$

In our case, taking

$$SU(2) \ni k\exp(sX_1 + tX_2)\exp(uH_1) \mapsto (s,t,u) \in \mathbb{R}^3,$$

as a local coordinate around $k \in SU(2)$, and let us write a locally defined orthonormal frame field $\{e_i\}_{i=1}^3$ on $SU(2)$ around the identity e in $SU(2)$ by

$$e_1 = a\,\frac{\partial}{\partial s} + b\,\frac{\partial}{\partial t},\ e_2 = c\,\frac{\partial}{\partial s} + d\,\frac{\partial}{\partial t},\ e_3 = e^{C\ell(\ell-1)u\,(As+Bt)}\,\frac{\partial}{\partial u},$$

where $a,\ b,\ c,\ d,\ A,\ B,\ C$ are real constants.

For $X = \tau(\pi) = -(\kappa_1\widetilde{\epsilon}_1 + \kappa_2\widetilde{\epsilon}_2)$, and $\{e_i\}_{i=1}^3$ an orthonormal frame field on P such that the vertical subspace $\mathcal{V}_p = \mathbb{R}e_{3\,p}$ and the horizontal subspace $\mathcal{H}_p = \mathbb{R}\,e_{1\,p} \oplus \mathbb{R}\,e_{2\,p}$ of T_pP $(p \in P)$ satisfies

$$[e_i,\, e_3] = \kappa_i\, e_3 \quad (i = 1,2)$$

with $\kappa_i \in C^\infty(P)$ $(i = 1,2)$, where $\kappa_1 = C\ell(\ell-1)u(aA + bB)$, $\kappa_2 = C\ell(\ell-1)u(cA + dB)$. It holds that

$$\mathrm{d}iv(X) = e_1\kappa_1 + e_2\kappa_2 \equiv 0. \tag{4.1}$$

Furthermore, we obtain

$$\begin{aligned} X = \tau(\pi) &= -(\kappa_1\widetilde{\epsilon}_1 + \kappa_2\widetilde{\epsilon}_2) \\ &= -C\ell(\ell-1)u\{(aA+bB)\widetilde{\epsilon}_1 + (cA+dB)\widetilde{\epsilon}_2\}. \end{aligned} \tag{4.2}$$

Therefore, if $\ell = 0$ or $\ell = 1$,

$$X = \tau(\pi) = 0,$$

namely, $\pi:\ P = S_{\lambda_{\mathcal{I}}} \to M = K/T = P^1(\mathbb{C})$ is the direct product if $\ell = 0$, and it is the standard Hopf fiberring is harmonic if $\ell = 1$.

If $\ell = 2, 3, \cdots$, our $X = \tau(\pi) \not\equiv 0$ satisfies that $\overline{\Delta}^h X = cX$ with $\nabla^h_X X = 0$ which is equivalent to

$$\overline{\Delta}^h X + \nabla^h_X X - \mathrm{R}ic^h(X) = 0,$$

which is equivalent to that

$$\overline{\Delta}^h X = cX, \quad \nabla^h_X X = 0, \tag{4.3}$$

and $\pi : P = \mathcal{S}_{\lambda_{\mathcal{I}}} \to M = K/T = \mathbb{C}^1P$ is biharmonic, however it is not harmonic. Notice that $(M, h) = (\mathbb{C}^1P, h)$ satisfies that $\mathrm{R}ic^h = \frac{1}{2}\,\mathrm{I}d$ with $c = \frac{1}{2}$ and $\lambda_1(M, h) = 1$ ([42], p. 213, and [43], p. 67, Type A III in Table A2 and also p. 70).

Therefore, we can summarize:

Theorem 5. *For $\ell = 1, 2, \ldots$, let*

$$\lambda_{\mathcal{I}} :\ T \ni \begin{bmatrix} e^{2\pi\sqrt{-1}\,\theta} & 0 \\ 0 & e^{-2\pi\sqrt{-1}\,\theta} \end{bmatrix} \mapsto e^{2\pi\sqrt{-1}\,\ell\,\theta} \in S^1$$

be a homomorphism of T into S^1, and let $\pi : P = \mathcal{S}_{\lambda_{\mathcal{I}}} \to M = K/T = SU(2)/S^1 = P^1(\mathbb{C})$ be the principal S^1-bundle over K/T associated to $\lambda_{\mathcal{I}}$. Then, for every $\ell = 2, 3, \ldots$, the projection $\pi : (P, g) \to (M, h)$ is biharmonic but not harmonic.

References

[1] K. Akutagawa and S. Maeta: *Complete biharmonic submanifolds in the Euclidean spaces*, Geometriae Dedicata, **164** (2013), 351–355.

[2] M.A. Akyol and Y.L. Ou: *Biharmonic Riemannian submersions*, 2018, arXiv: 1805.04754v1.

[3] P. Paird and D. Kamissoko: *On constructing biharmonic maps and metrics*, Ann. Global Anal. Geom. **23** (2003), 65–75.

[4] P. Baird and J.C. Wood: *Harmonic morphisms between Riemannian manifolds*, London Math. Soc. Monographs, Oxford, 2003.

[5] A. Balmus, S. Montaldo and C. Oniciuc: *Classification results for biharmonic submanifolds in spheres*, Israel J. Math., **168** (2008), 201–220.

[6] A. Balmus, S. Montaldo and C. Oniciuc: *Biharmonic hypersurfaces in 4-dimensional space forms*, Math. Nachr., **283** (2010), 1696–1705.

[7] C. Boyer and K. Galicki: *Sasakian Geometry*, Oxford Sci. Publ., 2008.

[8] R. Caddeo, S. Montaldo, P. Piu: *On biharmonic maps*, Contemp. Math., **288** (2001), 286–290.

[9] I. Castro, H.Z. Li and F. Urbano: *Hamiltonian-minimal Lagrangian submanifolds in complex space forms*, Pacific J. Math., **227** (2006), 43–63.

[10] B.Y. Chen: *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math., **17** (1991), 169–188.

[11] F. Defever: *Hypersurfaces in $\mathbb{E}^5$ with harmonic mean curvature vector*, Math. Nachr., **196** (1998), 61–69.

[12] J. Eells and L. Lemaire: *Selected Topics in Harmonic Maps*, CBMS, Regional Conference Series in Math., Amer. Math. Soc., **50**, 1983.

[13] D. Fetcu and C. Oniciuc: *Biharmonic integral $\mathcal{C}$-parallel submanifolds in 7-dimensional Sasakian space forms*, Tohoku Math. J., **64** (2012), 195–222.

[14] T. Hasanis and T. Vlachos: *Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field*, Math. Nachr., **172** (1995), 145–169.

[15] T. Ichiyama, J. Inoguchi, H. Urakawa: *Biharmonic maps and bi-Yang-Mills fields*, Note di Mat., **28**, (2009), 233–275.

[16] T. Ichiyama, J. Inoguchi, H. Urakawa: *Classifications and isolation phenomena of biharmonic maps and bi-Yang-Mills fields*, Note di Mat., **30**, (2010), 15–48.

[17] J. Inoguchi: *Submanifolds with harmonic mean curvature vector filed in contact 3-manifolds*, Colloq. Math., **100** (2004), 163–179.

[18] H. Iriyeh: *Hamiltonian minimal Lagrangian cones in $\mathbb{C}^m$*, Tokyo J. Math., **28** (2005), 91–107.

[19] S. Ishihara and S. Ishikawa: *Notes on relatively harmonic immersions*, Hokkaido Math. J., **4** (1975), 234–246.

[20] G.Y. Jiang: *2-harmonic maps and their first and second variational formula*, Chinese Ann. Math., **7A** (1986), 388–402; Note di Mat., **28** (2009), 209–232.

[21] T. Kajigaya: *Second variation formula and the stability of Legendrian minimal submanifolds in Sasakian manifolds*, Tohoku Math. J., **65** (2013), 523–543.

[22] S. Kobayashi: *Transformation Groups in Differential Geometry*, Springer, 1972.

[23] E. Loubeau, C. Oniciuc: *The index of biharmonic maps in spheres*, Compositio Math., **141** (2005), 729–745.

[24] E. Loubeau and C. Oniciuc: *On the biharmonic and harmonic indices of the Hopf map*, Trans. Amer. Math. Soc., **359** (2007), 5239–5256.

[25] E. Loubeau and Y-L. Ou: *Biharmonic maps and morphisms from conformal mappings*, Tohoku Math. J., **62** (2010), 55–73.

[26] S. Maeta and U. Urakawa: *Biharmonic Lagrangian submanifolds in Kähler manifolds*, Glasgow Math. J. , **55** (2013), 465–480.

[27] S. Montaldo, C. Oniciuc: *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina **47** (2006), 1–22.

[28] Y. Nagatomo: *Harmonic maps into Grassmannians and a generalization of do Carmo-Wallach theorem*, Proc. the 16th OCU Intern. Academic Symp. 2008, OCAMI Studies, **3** (2008), 41–52.

[29] N. Nakauchi and H. Urakawa : *Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature*, Ann. Global Anal. Geom., **40** (2011), 125–131.

[30] N. Nakauchi and H. Urakawa: *Biharmonic submanifolds in a Riemannian manifold with non-positive curvature*, Results in Math.,**63** (2013), 467–474.

[31] N. Nakauchi, H. Urakawa and S. Gudmundsson: *Biharmonic maps into a Riemannian manifold of non-positive curvature*, Geom. Dedicata, **169**. (2014), 263–272.

[32] C. Oniciuc: *Biharmonic maps between Riemannian manifolds*, Ann. Stiint Univ. Aℓ. I. Cuza Iasi, Mat. (N.S.), **48** No. 2, (2002), 237–248.

[33] B. O'Neill: *The fundamental equation of a submersion*, Michigan Math. J., **13** (1966), 459–469.

[34] S. Ohno, T. Sakai and H. Urakawa: *Biharmonic homogeneous hypersurfaces in compact symmetric spaces*, Differ. Geom. Appl., **43** (2015), 155–179.

[35] S. Ohno, T. Sakai and H. Urakawa: *Rigidity of transversally biharmonic maps between foliated Riemannian manifolds*, to appear in Hokkaido Math. J.

[36] Ye-Lin Ou and Liang Tang: *The generalized Chen's conjecture on biharmonic submanifolds is false*, arXiv: 1006.1838v1.

[37] Ye-Lin Ou and Liang Tang: *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J., **61** (2012), 531–542.

[38] T. Sasahara: *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors*, Publ. Math. Debrecen, **67** (2005), 285–303.

[39] T. Sasahara: *Stability of biharmonic Legendrian submanifolds in Sasakian space forms*, Canad. Math. Bull. **51** (2008), 448–459.

[40] T. Sasahara: *A class of biminimal Legendrian submanifolds in Sasaki space forms*, a preprint, 2013, to appear in Math. Nach.

[41] T. Takahashi: *Minimal immersions of Riemannian manifoplds*, J. Math. Soc. Japan, **18** (1966), 380–385.

[42] M. Takeuchi and S. Kobayashi: *Minimal imbeddings of* $\mathbb{R}$*-space*, J. Differ. Geom., **2** (1968), 203–213.

[43] H. Urakawa: *The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold*, Compositio Math., **59** (1986), 57–71.

[44] H. Urakawa: *CR rigidity of pseudo harmonic maps and pseudo biharmonic maps*, Hokkaido Math. J., **46** (2017), 141–187.

[45] H. Urakawa: *Harmonic maps and biharmonic maps on principal bundles and warped products*, J. Korean Math. Soc. **55** (2018), **55**, 553–574.

[46] H. Urakawa: *Calculus of Variations and Harmonic Maps*, Vol. 132, Amer. Math. Soc., 1990.

[47] Z-P Wang and Y-L Ou: *Biharmonic Riemannian submersions from 3-manifolds*, Math. Z., **269** (2011), 917–925.