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On the reciprocity theorem of Ramanujan and its applications

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Abstract. In this paper, we give two new proofs of the reciprocity theorem of Ramanujan found in his lost notebook. We derive the well-known quintuple product identity using the reciprocity theorem. Further we obtain two interesting partition theoretic identities from the reciprocity theorem.

Keywords: Basic hypergeometric series, reciprocity theorem, Jacobi's triple product identity, quintuple product identity, partition function.

MSC 2000 classification: primary 33D15, secondary 11A15.

1 Introduction

On page 40 of his lost notebook [7] [3, entry(6.3.3)], Ramanujan has given the following beautiful reciprocity theorem of two variables.

Theorem 1. *If* $a,\ b$ *are complex numbers other than* 0 *and* $-q^{-n}$*, then*

$$\rho(a,\ b)-\rho(b,\ a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,q)_\infty}{(-aq,-bq)_\infty}, \tag{1.1}$$

where

$$\rho(a,b):=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n}$$

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$$:= 1 + \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n-1}}{(-aq)_{n+1}}. \tag{1.2}$$

Throughout this paper, we assume $|q| < 1$ and employ the customary notations

$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \quad n, \textit{ an integer.}$$

We use the notation

$$(a_1, a_2, a_3, ..., a_m)_n = (a_1)_n (a_2)_n (a_3)_n ... (a_m)_n, \quad n, \text{ an integer or } \infty,$$

and the $q-$shifted factorial identity [6, equation (I.2), p.351]

$$(a)_{-n} = \frac{1}{(aq^{-n})_n} = \frac{(-q/a)^n}{(q/a)_n} q^{n(n-1)/2}. \tag{3}$$

Jacobi's triple product identity [6, equation (II.28), p.357] is given by

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -q/z; q^2)_\infty, \quad z \neq 0 \tag{4}$$

and Heine's transformation [6, equation (III.2), p.359] is given by

$$\sum_{n=0}^{\infty} \frac{(A,B)_n}{(q,C)_n} Z^n = \frac{(C/B, BZ)_\infty}{(C,Z)_\infty} \sum_{n=0}^{\infty} \frac{(ABZ/C, B)_n}{(q, BZ)_n} \left(\frac{C}{B}\right)^n. \tag{5}$$

The q-binomial theorem [6, equation (II.3), p.354]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{6}$$

The Roger's Fine identity [8, equation (12), p.576] is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a)_n (azq/b)_n b^n z^n q^{n^2-n} (1 - azq^{2n})}{(b)_n (z)_{n+1}}. \tag{7}$$

The transformation lemma [1]: Subject to suitable convergence conditions, if $\sum_{m=0}^{\infty} a_{m+n} b_m = c_n$, then

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} c_n. \tag{8}$$

For,

$$\sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_n = \sum_{m=0}^{\infty} b_m \sum_{n=-\infty}^{\infty} a_{n+m} = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} b_m a_{n+m} = \sum_{n=-\infty}^{\infty} c_n.$$

Tannery's theorem [4]: If $f_k(n) \to L_k$ for each k, as $n \to \infty$ and if $|f_k(n)| \le M_k$ with $\sum_{k=0}^{\infty} M_k$ convergent, then $\lim_{n\to\infty} \sum_{k=1}^{p} f_k(n) = \sum_{k=1}^{\infty} L_k$, provided that $p \to \infty$ as $n \to \infty$.

Andrews [2] was the first to establish (1.1) by employing his four-free variable identity and the well-known Jacobi's triple product identity which is a special case of (1.1). Somashekara and Fathima [9] used Ramanujan's ${}_1\psi_1$ summation formula and Heine's transformation formula to establish an equivalent version of (1.1). Since then many authors have contributed to the proof and applications of (1.1). For more details one may refer the book by Andrews and Berndt [3] or a recent paper [10] by Somashekara, Narasimha Murthy and Shalini.

The main objective of this paper is to give two proofs of (1.1) and derive therefrom the well-known quintuple product identity and to prove two interesting partition theoretic identities. One may refer the paper [5] by S. Cooper for more details about the quintuple product identity. In fact, we give two proofs of (1.1) in Section 2. In Section 3, we derive the quintuple product identity and prove two partition theoretic identities.

2 Proofs of the reciprocity theorem of Ramanujan

In this Section we give two proofs of (1.1).

Proof 1. Employing (1.2), the left hand side of (1.1) can be written as

$$\rho(a,b) - \rho(b,a) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n-1}}{(-aq)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} b^n a^{-n-1}}{(-bq)_{n+1}}. \tag{9}$$

Changing $n \to -n-1$ in the first term on the right side of (9) and then using (3) we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n-1}}{(-aq)_{n+1}} = \frac{-1}{a} \sum_{n=-1}^{-\infty} (-1/a)_n (-bq)^n. \tag{10}$$

Next setting $A = -1/a$, $B = q$, $Z = -bq$ in (5) then letting $c \to 0$ and multiplying the resulting identity throughout by $1/a$, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} b^n a^{-n-1}}{(-bq)_{n+1}} = \frac{1}{a} \sum_{n=0}^{\infty} (-1/a)_n (-bq)^n. \tag{11}$$

Using (10) and (11) in (9) we obtain

$$\rho(a,b) - \rho(b,a) = \frac{-1}{a} \sum_{n=-\infty}^{\infty} (-1/a)_n (-bq)^n. \tag{12}$$

Now define

$$F(a,b) := \sum_{n=-\infty}^{\infty} (-1/a)_n (-bq)^n.$$

Then we have,

$$(1+aq)(1+bq)F(a,b) = F(aq,bq)$$

or equivalently

$$F(a,b) = \frac{F(aq,bq)}{(1+aq)(1+bq)}.$$

Iterate this relation $m \geq 0$ times to obtain

$$F(a,b) = \frac{F(aq^m, bq^m)}{(-aq,-bq)_m}. \tag{13}$$

We have

$$\lim_{m\to\infty} F(aq^m, bq^m) = \sum_{n=-\infty}^{\infty} (-1)^n b^n a^{-n} q^{n(n+1)/2} = (bq/a, a/b, q; q)_\infty, \tag{14}$$

on using Tannery's theorem [4] and (4) with q changed to $q^{1/2}$ and then z changed to $-bq^{1/2}/a$. Letting $m \to \infty$ in (13) and using (14) we obtain

$$F(a,b) = \frac{(bq/a, a/b, q; q)_\infty}{(-aq,-bq;q)_\infty}$$

or equivalently

$$\sum_{n=-\infty}^{\infty} (-1/a)_n (-bq)^n = \frac{(bq/a, a/b, q; q)_\infty}{(-aq,-bq;q)_\infty}. \tag{15}$$

Substituting this in (12) we obtain (1.1) on some simplifications.

Proof 2. In the transformation lemma (8), take

$$a_n = (-1/a)_n (-bq)^n, \quad b_n = \frac{(a/b)^n}{(q;q)_n}, \quad c_n = \frac{(q;q)_\infty}{(-aq;q)_\infty} \frac{(-1/a)_n (-bq)^n}{(q;q)_n}.$$

Then $c_n = \sum_{m=0}^{\infty} a_{m+n} b_m$ is a restatement of the q-binomial theorem(6). Hence by the transformation lemma we obtain (15). Substituting (15) in (12) we obtain (1.1), on some simplifications.

3 Applications of (1.1)

Theorem 2 (Quintuple product identity)**.**

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}(a^{3n+1}+a^{-3n})=\frac{(a^2,q/a^2,q)_\infty}{(a,q/a)_\infty}. \quad (16)$$

Proof . Using (1.2) on the right side of (1.1) and after some simplifications, we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n-1}}{(-aq)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}b^n a^{-n-1}}{(-bq)_{n+1}}$$
$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,q)_\infty}{(-aq,-bq)_\infty}. \quad (17)$$

Changing a to $-a/q$ and b to $-1/a$, we obtain, after some simplifications

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}a^{2n}}{(a)_{n+1}}-\frac{q}{a^2}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}a^{-2n}}{(q/a)_{n+1}}=\frac{(a^2,q/a^2,q)_\infty}{(a,q/a)_\infty}. \quad (18)$$

Changing B to q and then $C\to 0$ in (5), we obtain

$$\sum_{n=0}^{\infty}(A;q)_n Z^n=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}(AZ)^n}{(Z;q)_n}. \quad (19)$$

Changing A to a and Z to a in (19), we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}a^{2n}}{(a)_{n+1}}=\sum_{n=0}^{\infty}(a;q)_n a^n. \quad (20)$$

Changing A to q/a and Z to q/a in (19), we obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+3)/2}a^{-2n}}{(q/a)_{n+1}}=\sum_{n=0}^{\infty}(q/a;q)_n(q/a)^n. \quad (21)$$

Letting $b\to 0$ in (7), we obtain

$$\sum_{n=0}^{\infty}(a;q)_n z^n=\sum_{n=0}^{\infty}\frac{(a)_n(-1)^n a^n z^{2n}q^{n(3n-1)/2}(1-azq^{2n})}{(z)_{n+1}}. \quad (22)$$

Changing z to a in (22), we obtain

$$\sum_{n=0}^{\infty}(a;q)_n a^n=\sum_{n=0}^{\infty}\frac{(a)_n(-1)^n a^{3n}q^{n(3n-1)/2}(1-a^2q^{2n})}{(a)_{n+1}}. \quad (23)$$

Changing a to q/a and z to q/a in (22), we obtain

$$\sum_{n=0}^{\infty}(q/a;q)_n(q/a)^n = \sum_{n=0}^{\infty}\frac{(q/a)_n(-1)^n(q/a)^{3n}q^{n(3n-1)/2}(1-q^{2n+2}/a^2)}{(q/a)_{n+1}}. \quad (24)$$

Using (20), (21), (23) and (24), the left side of (18) can be written as

$$\sum_{n=0}^{\infty}(-1)^n a^{3n}q^{n(3n-1)/2}(1+aq^n) - \sum_{n=0}^{\infty}(-1)^n a^{-3n-2}q^{(n+1)(3n+2)/2}(1+q^{n+1}/a)$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{3n}q^{n(3n-1)/2}(1+aq^n) + \sum_{n=0}^{\infty}(-1)^n a^{-3n+1}q^{n(3n-1)/2}(1+q^n/a)$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}(a^{3n+1}+a^{-3n}) \quad (25)$$

Using (25) in (18), we obtain (16).

Definition 1. Given a partition π, let $e(\pi)$ denote the number of parts in π. Define $P_m(n)$ to be the set of partitions of n in which all parts are less than or equal to m. Let $q_m(n)$ be the number of partitions of n in which all parts are less than or equal to m. Define

$$p_m(n) = \sum_{\pi\in P_m(n)}(-1)^{e(\pi)} \quad (26)$$

so that

$$\frac{1}{(-q;q)_m} = \sum_{n=0}^{\infty}p_m(n)q^n \quad (27)$$

and

$$\frac{1}{(q;q)_m} = \sum_{n=0}^{\infty}q_m(n)q^n. \quad (28)$$

Definition 2. Given a partition π, let $e(\pi)$ denote the number of parts in π. Define $P_{o,m}(n)$ to be the set of partitions of n into odd parts and all parts are less than or equal to $2m$. Let $q_{e,m}(n)$ be the number of partitions of n into even parts in which all parts are less than or equal to $2m$. Define

$$p_{o,m}(n) = \sum_{\pi\in P_{o,m}(n)}(-1)^{e(\pi)} \quad (29)$$

so that

$$\frac{1}{(-q;q^2)_m} = \sum_{n=0}^{\infty}p_{o,m}(n)q^n \quad (30)$$

and

$$\frac{1}{(q^2;q^2)_m} = \sum_{n=0}^{\infty} q_{e,m}(n) q^n. \tag{31}$$

Theorem 3. *If $p_d(n)$ denotes the number of partitions of n into distinct parts and if $p_m(n)$ and $q_m(n)$ are as defined above, then*

$$p_d(n) = \frac{1}{2} \sum_{1 \leq m \leq \frac{1+\sqrt{8n+1}}{2}} \left\{ p_m\left[n - m(m-1)/2\right] + q_m\left[n - m(m-1)/2\right] \right\}. \tag{32}$$

Proof. Changing a to 1 and b to 1 and n to $m-1$ in (17), we obtain after some simplifications

$$\frac{1}{2} \left\{ \sum_{m=1}^{\infty} \frac{q^{m(m-1)/2}}{(-q;q)_m} + \sum_{m=1}^{\infty} \frac{q^{m(m-1)/2}}{(q;q)_m} \right\} = (-q;q)_\infty. \tag{33}$$

Using the definition (1) in (33), we obtain

$$\frac{1}{2} \sum_{n=0}^{\infty} \left\{ \sum_{1 \leq m \leq \frac{1+\sqrt{8n+1}}{2}} \left[p_m(n - m(m-1)/2) + q_m(n - m(m-1)/2) \right] q^n \right\} = \sum_{n=0}^{\infty} p_d(n) q^n. \tag{34}$$

By comparing the coefficients of q^n , we obtain (32).

Theorem 4. *If $p_{o,d}(n)$ denotes the number of partitions of n into distinct odd parts and if $p_{o,m}(n)$ and $q_{e,m}(n)$ are as defined above, then*

$$p_{o,d}(n) = \sum_{m=1}^{\infty} \left\{ p_{o,m}[n-(m-1)^2] + q_{e,m}(n-m^2) \right\}. \tag{35}$$

Proof. Changing q to q^2 in (17) and then changing a to $1/q$ and b to -1 and n to $m-1$ in the resulting identity, we obtain after some simplifications

$$\frac{1}{2} \left\{ \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q^2;q^2)_m} + \sum_{m=1}^{\infty} \frac{q^{(m-1)^2}}{(-q;q^2)_m} \right\} = (-q;q^2)_\infty. \tag{36}$$

Using the definition (2) in (36), we obtain

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ q_{e,m}(n-m^2) + p_{o,m}[n-(m-1)^2] \right\} q^n = \sum_{n=0}^{\infty} p_{o,d}(n) q^n. \tag{37}$$

By comparing the coefficients of q^n, we obtain (35).

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