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Some congruences modulo 2, 8 and 12 for Andrews' singular overpartitions

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Abstract. Recently, G. E. Andrews defined combinatorial objects which he called (k,i)-singular overpartitions, overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Let the number of (k,i)-singular overpartitions of n be denoted by $\overline{C}_{k,i}(n)$. Andrews and Chen, Hirschhorn and Sellers noted numerous congruences modulo 2 for $\overline{C}_{3,1}(n)$. The object of this paper is to obtain new congruences modulo 2 for $\overline{C}_{20,5}(n)$ and modulo 8 and 12 for $\overline{C}_{3,1}(n)$.

Keywords: singular overpartition, congruence, generating function, sums of squares.

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. Let $p(n)$ be the number of partitions of n. For example $p(5) = 7$. The seven partitions of 5 are 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1},$$

where as customary, we define $f_k := (q^k;q^k)_\infty = \prod_{m=1}^{\infty}(1-q^{mk})$.
If l is a positive integer, then a partition of n is said to be l-regular if no part

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is divisible by l. If $b_l(n)$ denotes the number of l-regular partitions of n then

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{(q^l;q^l)_\infty}{(q;q)_\infty} = \frac{f_l}{f_1}.$$

Several interesting arithmetic properties of l-regular partitions are found by many mathematicians, see [2, 6, 10, 11, 15, 19, 21]. In [9], Corteel and Lovejoy developed a new aspect of the theory of partitions - overpartitions. A hint of such a subject can also been seen in Hardy and Ramanujan [13, p.304]. An overpartition of n is a non-increasing sequence of positive integers whose sum is n in which the first occurrence of a part may be overlined. If $\overline{p}(n)$ denotes the number of overpartitions of n then

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{f_2}{f_1^2}. \tag{1.1}$$

Lovejoy [17] investigated the function $\overline{A}_l(n)$ which counts the number of l-regular overpartitions of n. He also proved theorems for overpartitions analogous to Gordons celebrated generalization of the RogersRamanujan identities [12]. The generating function for $\overline{A}_l(n)$ is

$$\sum_{n=0}^{\infty} \overline{A}_l(n)q^n = \frac{(-q;q)_\infty(q^l;q^l)_\infty}{(q;q)_\infty(-q^l;q^l)_\infty} = \frac{f_2f_l^2}{f_1^2f_{2l}}.$$

Recently, G. E. Andrews [3] introduced (k,i)-singular overpartitions, overpatitions in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Let $\overline{C}_{k,i}(n)$ denote the number of such partitions of n. For example, $\overline{C}_{3,1}(4) = 10$. The ten $(3,1)$-singular overpartitions of 4 are 4, $\bar{4}$, $2+2$, $\bar{2}+2$, $2+1+1$, $\bar{2}+1+1$, $2+\bar{1}+1$, $\bar{2}+\bar{1}+1$, $1+1+1+1$ and $\bar{1}+1+1+1$. The generating function for $\overline{C}_{k,i}(n)$, where $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$ is

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{1.2}$$

In his paper [3], Andrews also proved that for $n \geq 0$,

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.3}$$

It is important to note that $\overline{C}_{3,1}(n) = \overline{A}_3(n)$. Later, Chen, Hirschhorn and Sellers [8] found infinite families of congruences modulo 3 for $\overline{C}_{3,1}(n)$, $\overline{C}_{6,1}(n)$, $\overline{C}_{6,2}(n)$ and parity results for $\overline{C}_{4,1}(n)$. For example, they proved the following congruences,

Theorem 1.1. Let $p \equiv 3 \pmod 4$ be prime. Then for all $k, m \geq 0$ with $p \nmid m$,

$$\overline{C}_{3,1}(p^{2k+1}m) \equiv 0 \pmod 3. \tag{1.4}$$

In Theorem 1.1 if we set $p = 3, k = 0$ and $m \equiv 1, 2 \pmod 3$, we can easily obtain (1.3). For recent works on singular overpartitions, see [1, 3, 7, 8, 16, 18, 20, 22]. The aim of this paper is to prove new congruences for $\overline{C}_{3,1}(n)$ and $\overline{C}_{20,5}(n)$. The following are our main results.

Theorem 1.2. For all $k, n \geq 0$,

$$\overline{C}_{3,1}(4^k(72n + 21)) \equiv 0 \pmod{12}, \tag{1.5}$$

$$\overline{C}_{3,1}(4^k(72n + 39)) \equiv 0 \pmod{12}, \tag{1.6}$$

$$\overline{C}_{3,1}(4^k(72n + 57)) \equiv 0 \pmod{12}. \tag{1.7}$$

Theorem 1.3. Let $p \geq 5$ be prime and $1 \leq s \leq p-1$ with $6s+1$ a quadratic nonresidue modulo p. Then, for all $m \geq 0$,

$$\overline{C}_{3,1}\left(18(pm + s) + 3\right) \equiv 0 \pmod{12}. \tag{1.8}$$

Theorem 1.4. For all $k, n \geq 0$,

$$\overline{C}_{3,1}(4^k(12n + 5)) \equiv 0 \pmod 8, \tag{1.9}$$

$$\overline{C}_{3,1}(4^k(12n + 11)) \equiv 0 \pmod 8. \tag{1.10}$$

Theorem 1.5. Let $p \geq 5$ be prime. Then for all $\alpha \geq 1$ and $n \geq 0$,

$$\overline{C}_{3,1}\left(48p^{2\alpha}n + (48j + 2p)p^{2\alpha-1}\right) \equiv 0 \pmod 8,\ j = 1, 2, \ldots, p-1. \tag{1.11}$$

Theorem 1.6. For all $\alpha, n \geq 0$,

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+1}n + \frac{31 \cdot 5^{2\alpha} - 7}{12}\right) \equiv 0 \pmod 2, \tag{1.12}$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+1}n + \frac{79 \cdot 5^{2\alpha} - 7}{12}\right) \equiv 0 \pmod 2, \tag{1.13}$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+2}n + \frac{83 \cdot 5^{2\alpha+1} - 7}{12}\right) \equiv 0 \pmod 2, \tag{1.14}$$

$$\overline{C}_{20,5}\left(2 \cdot 5^{2\alpha+2}n + \frac{107 \cdot 5^{2\alpha+1} - 7}{12}\right) \equiv 0 \pmod 2. \tag{1.15}$$

For an odd prime , the Legendre symbol is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is quadratic residue modulo } p \text{ and } \ a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } a \text{ is quadratic non-residue modulo } p \ , \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

We also prove the following infinite family of congruences for $\overline{C}_{20,5}(n)$.

Theorem 1.7. Let $p \geq 5$ be prime, $\left(\frac{-10}{p}\right) = -1$. Then for all $\alpha, n \geq 0$,

$$\overline{C}_{20,5}\left(2p^{2\alpha+1}(pn+j) + 7 \times \frac{p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{2},\ j = 1, 2, \ldots, p-1. \tag{1.16}$$

In order to prove our main results, we collect a few definitions and Lemmas in section 2. In section 3-5 we prove our main results.

2 Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections.
For $\mid ab \mid < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

Using Jacobi's triple product identity [5, Entry 19, p. 35], (2.1) becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The most important special cases of $f(a, b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{2.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \tag{2.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{2.5}$$

By the binomial theorem, we see that for any positive integer k,

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{2.6}$$

Lemma 2.1. (Hirschhorn, Garvan and Borwein [14]) The following 2-dissection holds

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.7}$$

Lemma 2.2. (Hirschhorn and Sellers [15, Theorem 2.1, 2.3, 2.4]) We have,

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv \frac{f_5 f_{20}}{f_1 f_{10}} \pmod{2}, \tag{2.9}$$

$$b_5(20n+5) \equiv 0 \pmod{2}, \tag{2.10}$$

$$b_5(20n+13) \equiv 0 \pmod{2}. \tag{2.11}$$

Lemma 2.3. (Cui and Gu [10, Theorem 2.2]) If $p \geq 5$ is a prime and

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}, \end{cases}$$

then

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2}; q^{p^2})_\infty. \tag{2.12}$$

Furthermore, if $-\frac{p-1}{2} \leq k \leq \frac{p-1}{2}, k \neq \frac{\pm p-1}{6}$ then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}$.

3 Congruences modulo 12 for $\overline{C}_{3,1}(n)$

In this section we prove some infinite families of congruences modulo 12 for $\overline{C}_{3,1}(n)$.
From [19, Theorem 2.7, Eq. 2.22], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n = 6\frac{f_2^8 f_3^{15}}{f_1^{17} f_6^6} + 96q\frac{f_2^5 f_3^6 f_6^3}{f_1^{14}}. \tag{3.1}$$

Using (2.6), we have

$$\frac{f_2^8 f_3^{15}}{f_1^{17} f_6^6} \equiv \frac{f_3^3}{f_1} \pmod{2}. \tag{3.2}$$

Using (3.2) in (3.1), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \equiv 6\frac{f_3^3}{f_1} \pmod{12}. \tag{3.3}$$

Substituting the identity (2.7) in (3.3) and then simplifying, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \equiv 6f_8 + 6q\frac{f_{12}^3}{f_4} \pmod{12}. \tag{3.4}$$

Equating the coefficients of q^{4n+1} from both sides of (3.4), dividing both sides by q and then replacing q^4 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(36n+12)q^n \equiv 6\frac{f_3^3}{f_1} \equiv \sum_{n=0}^{\infty} \overline{C}_{3,1}(9n+3)q^n \pmod{12}, \tag{3.5}$$

which yields,

$$\overline{C}_{3,1}(36n+12) \equiv \overline{C}_{3,1}(9n+3) \pmod{12}. \tag{3.6}$$

By (3.6) and mathematical induction , we find that for $n, k \geq 0$,

$$\overline{C}_{3,1}(4^k(9n+3)) \equiv \overline{C}_{3,1}(9n+3) \pmod{12}. \tag{3.7}$$

Equating the coefficients of q^{2n} from both sides of (3.4), and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^n \equiv 6f_4 \pmod{12}. \tag{3.8}$$

Equating the coefficients of q^{4n+1}, q^{4n+2}, q^{4n+3} from the both sides of (3.8), we obtain

$$\overline{C}_{3,1}(72n+21) \equiv 0 \pmod{12}, \tag{3.9}$$

$$\overline{C}_{3,1}(72n+39) \equiv 0 \pmod{12}, \tag{3.10}$$

$$\overline{C}_{3,1}(72n+57) \equiv 0 \pmod{12}. \tag{3.11}$$

Proof of Theorem 1.2. Replacing n by $8n+2$ in (3.7) and using (3.9), we obtain (1.5). Replacing n by $8n+4$ in (3.7) and using (3.10), we have (1.6). Replacing n by $8n+6$ in (3.7) and then employing (3.11), we obtain (1.7). QED

Theorem 3.1. For all $n \geq 0$,

$$\overline{C}_{3,1}(18n+3) \equiv \begin{cases} 6 \pmod{12} & \text{if } n = 2k(3k-1)\ , \\ 0 \pmod{12} & \text{otherwise.} \end{cases} \tag{3.12}$$

Proof. Using Euler's Pentagonal Number Theorem [4, p. 12] in (3.8), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^n \equiv 6 \sum_{k=-\infty}^{\infty} q^{2k(3k-1)} \pmod{12}. \tag{3.13}$$

QED

Proof of Theorem 1.3. Replacing q by q^6 in both sides of (3.13) and then multiplying both sides by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^{6n+1} \equiv 6 \sum_{k=-\infty}^{\infty} q^{12k(3k-1)+1} \pmod{12}, \tag{3.14}$$

which yields

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(18n+3)q^{6n+1} \equiv 6 \sum_{k=-\infty}^{\infty} q^{(6k-1)^2} \pmod{12}. \tag{3.15}$$

Let $n = pm+s$, then $6n+1 = 6pm+6s+1 \equiv 6s+1 \pmod{p}$ is not a quadratic residue modulo p. Thus, $6n+1$ is not a square and $\overline{C}_{3,1}(18n+3) \equiv 0 \pmod{12}$.

QED

4 Congruences modulo 8 for $\overline{C}_{3,1}(n)$

In this section, we prove some arithmetic properties modulo 8 satisfied by $\overline{C}_{3,1}(n)$.
From [19, Theorem 2.6, Eq. 2.16], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(3n+2)q^n = 4\frac{f_2 f_6^3}{f_1^4}. \tag{4.1}$$

Using (2.6), it follows that

$$\frac{f_2 f_6^3}{f_1^4} \equiv \frac{f_3^6}{f_1^2} \pmod{2}. \tag{4.2}$$

Using (4.2) in (4.1), we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+2)q^n \equiv 4\frac{f_3^6}{f_1^2} \pmod 8. \tag{4.3}$$

Substituting the identity (2.7) in (4.3) and then using (2.6) , we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+2)q^n &\equiv 4\left(\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}\right)^2\\ &\equiv 4\left(\frac{f_4^6f_6^4}{f_2^4f_{12}^2}+q^2\frac{f_{12}^6}{f_4^2}\right)\\ &\equiv 4\left(f_4^4+q^2\frac{f_{12}^6}{f_4^2}\right) \pmod 8. \end{aligned}\tag{4.4}$$

Extracting the terms containing q^{4n+2} from both sides of (4.4), dividing both sides by q^2 and then replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+8)q^n \equiv 4\frac{f_3^6}{f_1^2} \equiv \sum_{n=0}^{\infty}\overline{C}_{3,1}(3n+2)q^n \pmod 8, \tag{4.5}$$

which yields,

$$\overline{C}_{3,1}(12n+8) \equiv \overline{C}_{3,1}(3n+2) \pmod 8. \tag{4.6}$$

By (4.6) and mathematical induction we have , for $n, k \geq 0$,

$$\overline{C}_{3,1}(4^k(3n+2)) \equiv \overline{C}_{3,1}(3n+2) \pmod 8. \tag{4.7}$$

Extracting the terms containing q^{4n} from both sides of (4.4) and then replacing q^4 by q we have,

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+2)q^n \equiv 4f_1^4 \equiv 4f_4 \pmod 8. \tag{4.8}$$

Equating the coefficients of q^{4n+1} and q^{4n+3} from both sides of (4.4), we have

$$\overline{C}_{3,1}(12n+5) \equiv 0 \pmod 8, \tag{4.9}$$

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod 8. \tag{4.10}$$

Proof of Theorem 1.4. Replacing n by $4n+1$ in (4.7) and using (4.9), we obtain (1.9). Again by replacing n by $4n+3$ in (4.7) and using (4.10), we have (1.10).

QED

Theorem 4.1. Let $p \geq 5$ be prime. Then for all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(48p^{2\alpha}n + 2p^{2\alpha}\right) q^n \equiv 4(q;q)_\infty \pmod 8. \tag{4.11}$$

Proof. Extracting the terms involving q^{4n}, from the both sides of (4.8) and then replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(48n+2)q^n \equiv 4f_1 \pmod 8, \tag{4.12}$$

which is the case $\alpha = 0$ of (4.11). Suppose that (4.11) is true for some $\alpha \geq 0$. Substituting (2.12) into (4.11), extracting the terms containing $q^{pn+\frac{p^2-1}{24}}$ from both sides of the identity, dividing both sides by $q^{\frac{p^2-1}{24}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(48p^{2\alpha}\left(pn + \frac{p^2-1}{24}\right) + 2p^{2\alpha}\right) q^n \equiv 4(q^p;q^p)_\infty \pmod 8. \tag{4.13}$$

Extracting the terms containing of q^{pn} from both sides of (4.13) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(48p^{2(\alpha+1)}n + 2p^{2(\alpha+1)}\right) q^n \equiv 4(q;q)_\infty \pmod 8, \tag{4.14}$$

which is (4.11) with $\alpha+1$ for α. This completes the proof of (4.11) by induction. QED

Proof of Theorem 1.5. Comparing the coefficients of q^{pn+j}, for $1 \leq j \leq p-1$, from both sides of (4.13), we arrive at (1.11). QED

5 Congruences modulo 2 for $\overline{C}_{20,5}(n)$

In this section, we prove a number of arithmetic properties modulo 2 satisfied by $\overline{C}_{20,5}(n)$.

Theorem 5.1. For all $\alpha, n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2 \cdot 5^{2\alpha}n + 7 \times \frac{5^{2\alpha}-1}{12}\right) q^n \equiv (q^2;q^2)_\infty (q^5;q^5)_\infty \pmod 2. \tag{5.1}$$

Proof. From (1.2), we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}(n)q^n = \frac{(q^{20};q^{20})_\infty(-q^5;q^{20})_\infty(-q^{15};q^{20})_\infty}{(q;q)_\infty} \equiv \frac{f_{10}f_5}{f_1} \pmod 2. \tag{5.2}$$

Using (2.6) in (2.8), we obtain

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \equiv f_4 + q\frac{f_{10}f_{20}}{f_2} \pmod 2. \tag{5.3}$$

Using (5.3) in (5.2), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}(n)q^n \equiv f_4 f_{10} + q\frac{f_{10}^2 f_{20}}{f_2} \pmod 2. \tag{5.4}$$

Extracting the terms containing q^{2n} from both sides of (5.4) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}(2n)q^n \equiv f_2 f_5 \pmod 2, \tag{5.5}$$

which is the case $\alpha = 0$ of (5.1). Now suppose (5.1) holds for some $\alpha \geq 0$. Recall Ramanujan's beautiful identity [4, p. 161]:

$$\frac{f_1}{f_{25}} = R(q^5)^{-1} - q - q^2 R(q^5), \tag{5.6}$$

where

$$R(q) = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

Replacing q by q^2 in (5.6), we get

$$\frac{f_2}{f_{50}} = R(q^{10})^{-1} - q^2 - q^4 R(q^{10}). \tag{5.7}$$

Using (5.7) in (5.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2\cdot 5^{2\alpha}n + 7\times\frac{5^{2\alpha}-1}{12}\right)q^n \equiv (q^2;q^2)_\infty(q^5;q^5)_\infty \pmod 2$$
$$= f_5 f_{50}\left(R(q^{10})^{-1} - q^2 - q^4R(q^{10})\right). \tag{5.8}$$

Extracting the terms containing q^{5n+2} from both sides of (5.8), then dividing both sides by q^2 and finally replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2\cdot 5^{2\alpha}(5n+2) + 7\times\frac{5^{2\alpha}-1}{12}\right)q^n \equiv f_1 f_{10} \pmod 2. \tag{5.9}$$

Using (5.6) in (5.9), we get

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2\cdot 5^{2\alpha+1}n + \frac{11\cdot 5^{2\alpha+1}-7}{12}\right)q^n \equiv f_1 f_{10} \pmod 2$$
$$= f_{10} f_{25}\left(R(q^5)^{-1} - q - q^2R(q^5)\right). \tag{5.10}$$

Extracting the terms containing q^{5n+1} from both sides (5.10), then dividing both sides by q and finally replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2\cdot 5^{2\alpha+1}(5n+1) + \frac{11\cdot 5^{2\alpha+1}-7}{12}\right)q^n$$
$$= \sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2\cdot 5^{2(\alpha+1)}n + 7\times\frac{5^{2(\alpha+1)}-1}{12}\right)q^n \equiv f_2 f_5 \pmod 2, \tag{5.11}$$

which is (5.1) with $\alpha+1$ for α. This completes the proof of (5.1) by induction. QED

Proof of Theorem 1.6. Comparing the coefficients of q^{5n+1} and q^{5n+3} from both sides of (5.8), we obtain the first two congruences of Theorem 1.6. Comparing the coefficients of q^{5n+3} and q^{5n+4} from both sides of (5.10), we obtain the remaining two congruences of Theorem 1.6. QED

Theorem 5.2. Let $p\ge 5$ be prime, $\left(\frac{-10}{p}\right) = -1$. Then for all $\alpha, n\ge 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2p^{2\alpha}n + 7\times\frac{p^{2\alpha}-1}{12}\right)q^n \equiv (q^2;q^2)_\infty(q^5;q^5)_\infty \pmod 2. \tag{5.12}$$

Proof. Now (5.5) is the $\alpha = 0$ case of (5.12). Suppose (5.12) is true for some $\alpha \geq 0$. Using (2.12) on the right hand of (5.12), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{20,5}\left(2p^{2\alpha}n + 7 \times \frac{p^{2\alpha}-1}{12}\right) q^n$$
$$\equiv \left[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{2\cdot\frac{3k^2+k}{2}} f\left(-q^{2\cdot\frac{3p^2+(6k+1)p}{2}}, -q^{2\cdot\frac{3p^2-(6k+1)p}{2}}\right) \right.$$
$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{2\cdot\frac{p^2-1}{24}} (q^{2p^2}; q^{2p^2})_\infty \right]$$
$$\times \left[\sum_{\substack{m=-\frac{p-1}{2} \\ m\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^m q^{5\cdot\frac{3m^2+m}{2}} f\left(-q^{5\cdot\frac{3p^2+(6m+1)p}{2}}, -q^{5\cdot\frac{3p^2-(6m+1)p}{2}}\right) \right.$$
$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{5\cdot\frac{p^2-1}{24}} (q^{5p^2}; q^{5p^2})_\infty \right] \pmod{2}. \tag{5.13}$$

For a prime p with $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, let us consider

$$2 \cdot \frac{3k^2+k}{2} + 5 \cdot \frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p},$$

which equivalent to

$$(12k+2)^2 + 10(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-10}{p}\right) = -1$, the only solution of the above condition is $k, m = \dfrac{\pm p - 1}{6}$. Therefore extracting the terms containing $q^{pn+7\cdot\frac{p^2-1}{24}}$ from both sides of (5.13), then dividing both sides by $q^{7\cdot\frac{p^2-1}{24}}$ and replacing q^{pn} by q we obtain,

$$\sum_{n=0}^{\infty}\overline{C}_{20,5}\left(2p^{2\alpha}\left(pn+7\cdot\frac{p^2-1}{24}\right)+7\times\frac{p^{2\alpha}-1}{12}\right)q^n$$
$$=\sum_{n=0}^{\infty}\overline{C}_{20,5}\left(2p^{2\alpha+1}n+7\times\frac{p^{2\alpha+2}-1}{12}\right)q^n\equiv(q^{2p};q^{2p})_\infty(q^{5p};q^{5p})_\infty\pmod 2. \tag{5.14}$$

Extracting the terms containing q^{pn} from both sides of (5.14) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{20,5}\left(2p^{2(\alpha+1)}n+7\times\frac{p^{2(\alpha+1)}-1}{12}\right)q^n\equiv(q^2;q^2)_\infty(q^5;q^5)_\infty\pmod 2, \tag{5.15}$$

which is (5.12) with $\alpha+1$ for α. This completes the proof of (5.12) by induction. QED

Proof of Theorem 1.7. Comparing the coefficients of q^{pn+j}, for $1\le j\le p-1$ from both sides of (5.14), we arrive at (1.16). QED

We close this section by briefly noting the following corollary.

Corollary 1. For all $n\ge 0$,

$$\overline{C}_{20,5}(n)\equiv b_5(2n+1)\pmod 2, \tag{5.16}$$

$$\overline{C}_{20,5}(10n+2)\equiv 0\pmod 2, \tag{5.17}$$

$$\overline{C}_{20,5}(10n+6)\equiv 0\pmod 2. \tag{5.18}$$

Proof. From (2.9) we have,

$$\sum_{n=0}^{\infty}b_5(2n+1)q^n\equiv\frac{f_5f_{20}}{f_1f_{10}}\equiv\frac{f_5f_{10}}{f_1}\pmod 2, \tag{5.19}$$

(5.16) follows from (5.2) and (5.19). Replacing n by $10n+2$ in (5.16), using (2.10), we have (5.17). Again replacing n by $10n+6$ in (5.16) and using (2.11), we obtain (5.18). Observe that (5.17) is the $\alpha=0$ case of (1.12) and (5.18) is the $\alpha=0$ case of (1.13). QED

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