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# Semi-equivelar maps on the surface of Euler characteristic $-1$

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**Abstract.** Semi-Equivelar maps are generalizations of Archimedean solids to surfaces other than the 2-sphere. In an earlier work a complete classification of semi-equivelar map of type $(3^5, 4)$ on the surface of Euler characteristic $-1$ was given. Vertex transitive semi-equivelar maps on the double torus have been classified. In this article we study these types of maps on the surface of Euler characteristic $-1$. We classify them and show that none of them is vertex transitive.

**Keywords:** vertex transitive maps, Archimedean solids, semi-Equivelar Maps

**MSC 2000 classification:** primary 52B70, secondary 52C20

## Introduction

A triangulation of a surface is called $d$-covered if each edge of the triangulation is incident with a vertex of degree $d$. We got interested in studying the content presented in this article while attempting to answer a question of Negami and Nakamoto [17] about existence of $d$-covered triangulations for closed surfaces. We had answered their question in affirmative [19] for the surfaces of Euler characteristic $-127 \leq \chi \leq -2$ and further became interested in looking at what happens for surfaces with $\chi = -1$. It was here that due to curvature considerations of this surface we had to construct a map on this surface which we named as Semi-equivelar map [24]. Such maps have also been studied in various forms (see [1], [7, 8, 12, 11]). In the meantime we came to learn that Nedela and Karabas [13], [14] have worked along similar lines and classified all the vertex transitive Archimedean maps on orientable surfaces of Euler characteristics $-2$,

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$-4$ and $-6$ (see also [15]). In particular, they have shown that there are seventeen isomorphism classes of Archimedean maps on the orientable surface of Euler characteristic $-2$, out of which exactly fourteen are semi-equivelar maps with eleven distinct face sequences of types: $(3^5, 4)$, $(3^4, 4^2)$, $(3^4, 8)$, $(3^2, 4, 3, 6)$, $(3, 4^4)$, $(3, 4, 8, 4)$, $(3, 6, 4, 6)$, $(4^3, 6)$, $(4, 6, 16)$, $(4, 8, 12)$, $(6^2, 8)$. An orientable closed surface of Euler characteristic $-2$ is the double cover of non orientable closed surface of Euler characteristic $-1$. This motivated us to explore the existence of above eleven types of semi-equivelar maps on non orientable surface of Euler characteristic $-1$. In the article [24] we have classified the semi-equivelar map of type $(3^5, 4)$ on this surface. Here, we investigate the remaining types of semi-equivelar maps on this surface. In the next few paragraphs we describe the definitions and terminologies used in this article. These definitions and terminologies are given in [10] and we are giving them here for the sake of ready reference. A standard reference on the subject of polyhedral maps is the article [3] of Brehm and Schulte. For graph theory related terminologies one may also refer to[21] and for topological preliminaries and terminologies one may refer to [20].

Throughout this article the term graph will mean a finite simple graph. A cycle of length $m$ or a *m-Cycle*, usually denoted by $C_m$, is by definition a connected 2-regular graph with $m$ vertices. A 2-dimensional *Polyhedral Complex* $K$ is a finite collection of $m_i$-cycles, where $\{m_i \colon 1 \le i \le n \text{ and } m_i \ge 3\} \subseteq \mathbb{N}$, together with vertices and edges of the cycles such that the non-empty intersection of any two cycles is either a vertex or is an edge. The cycles are called faces of $K$. The notations $V(K)$ and $EG(K)$ are used to denote the set of vertices and edges of $K$ respectively. A geometric object, called *geometric career* of $K$, denoted by $|K|$ can be associated to a polyhedral complex $K$ in the following manner: corresponding to each $m$-cycle $C_m$ in $K$, consider a $m$-gon $D_m$ whose boundary cycle is $C_m$. Then $|K|$ is the union of all such $m$-gons. The complex $K$ is said to be connected (resp. compact or orientable) if $|K|$ is a connected (resp. compact or orientable) topological space. A polyhedral complex $K$ is called a *Polyhedral 2-manifold* if for each vertex $v$ the faces containing $v$ are of the form $C_{m_1}, \dots, C_{m_p}$ where $C_{m_1} \cap C_{m_2}, \dots C_{m_{p-1}} \cap C_{m_p}$, and $C_{m_p} \cap C_{m_1}$ are edges for some $p \ge 3$. A connected polyhedral 2-manifold is called a *Polyhedral Map*. We will also use the term *map* for a polyhedral map. Among any two complexes $K_1$ and $K_2$ we define an isomorphism to be a bijective map $f \colon V(K_1) \longrightarrow V(K_2)$ for which $f(\sigma)$ is a face in $K_2$ if and only if $\sigma$ is a face in $K_1$. If $K_1 = K_2$ then $f$ is said to be an automorphism of $K_1$. The set of all automorphisms of a polyhedral complex $K$ forms a group under the operation of composition of maps. This group is called the automorphism group of $K$. If this group acts transitively on the set $V(K)$ then the complex is called a *vertex transitive* complex. Some vertex transitive maps of Euler characteristic 0 have been studied in [4] and many others in [2], [5], [6], [16] and [18].

The *face sequence* (see [24]) of a vertex $v$ in a map is a finite cyclically ordered sequence $(a^p, b^q, ...., m^r)$ of powers of positive integers $a, b, ..., m \geq 3$ and $p, q, ..., r \geq 1$, such that through the vertex $v$, $p$ numbers of $C_a$, $q$ numbers of $C_b$, $\ldots$, $r$ numbers of $C_m$ are incident in the given cyclic order. A map $K$ is said to be *Semi-Equivelar* if face sequence of each vertex of $K$ is same. A SEM with face sequence $(a^p, b^q, ...., m^r)$, is also called SEM of type $(a^p, b^q, ...., m^r)$. In [22], maps with face sequence $(3^3, 4^2)$ and $(3^2, 4, 3, 4)$ have been considered.

Let $EG(K)$ be the edge graph of a map $K$ and $V(K) = \{v_1, v_2, \ldots, v_n\}$. Let $L_K(v_i) = \{v_j \in V(K) \colon v_iv_j \in EG(K)\}$. For $0 \leq t \leq n$ define a graph $G_t(K)$ with $V(G_t(K)) = V(K)$ and $v_iv_j \in EG(G_t(K))$ if $|L_K(v_i) \bigcap L_K(v_j)| = t$. In other words the number of elements in the set $L_K(v_i) \bigcap L_K(v_j)$ is $t$. This graph was introduced in [6] by B. Datta. Moreover if $K$ and $K'$ are two isomorphic maps then $G_i(K) \cong G_i(K')$ for each $i$. We have used these graphs in this article to determine whether two SEMs are isomorphic.

In the article [24] it has been shown that:

**Proposition 1.** *There are exactly three non isomorphic semi-equivelar maps of type* $(3^5, 4)$ *on the surface of Euler characteristic* $-1$.

□

In the present article we show:

**Lemma 1.** *If $K$ is a semi-equivelar map of type (3, 4, 8, 4) on the surface of Euler characteristic* $-1$, *then $K$ is isomorphic to* $K_1(3, 4, 8, 4)$ *or* $K_2(3, 4, 8, 4)$, *see the examples described in Section 1.*

**Lemma 2.** *If $M$ is a semi-equivelar map of type* $(4, 6, 16)$ *on the surface of Euler characteristic* $-1$, *then $M$ is isomorphic to* $M_1(4, 6, 16)$ *or* $M_2(4, 6, 16)$, *see the examples described in Section 1.*

**Lemma 3.** *If $N$ is a semi-equivelar map of type* $(6^2, 8)$ *on the surface of Euler characteristic* $-1$, *then $N$ is isomorphic to* $N_1(6^2, 8)$ *or* $N_2(6^2, 8)$, *see the examples described in Section 1.*

It has been shown further that the maps described in the above Lemmas are non-isomorphic, see Claim 1.1, 1.2 in page 5. Considering this fact together with the above Proposition 1 it follows that:

**Theorem 1.** *There are at least nine non-isomorphic semi-equivelar maps on the surface of Euler characteristic* $-1$.

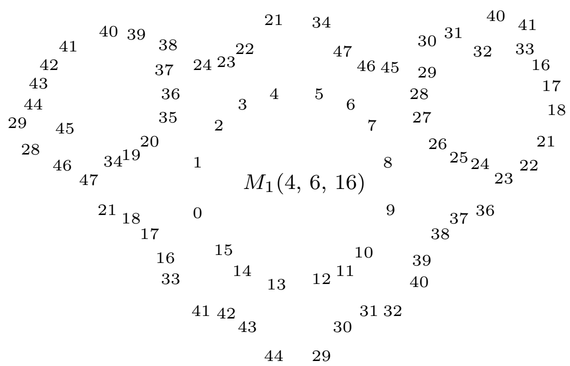
□

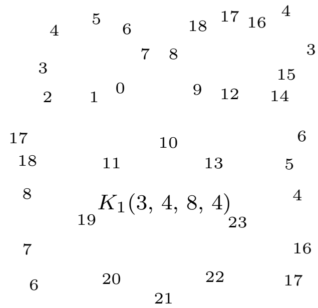
In the article we also show that:

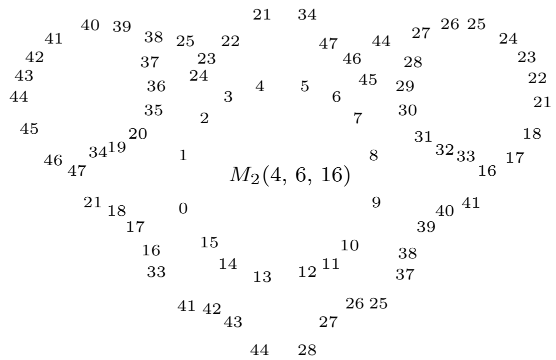
**Theorem 2.** *There exist no semi-equivelar maps of types* $(3^4, 8)$, $(3^2, 4, 3, 6)$, *(3, 6, 4, 6), $(4^3, 6)$ and (4, 8, 12) on the surface of Euler characteristic -1.*

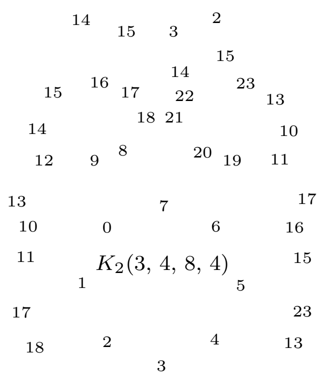
The article is organized in the following manner. In the next section, we present examples of semi-equivelar maps on the surface of Euler characteristic −1. In the section following it, we describe the results and their proofs. The technique of the proofs involve exhaustive search for the desired objects. This leads to a case by case considerations and enumeration of these objects. Since the verifications are routine in almost all cases we have given this completely in [23] and only referred to this here.

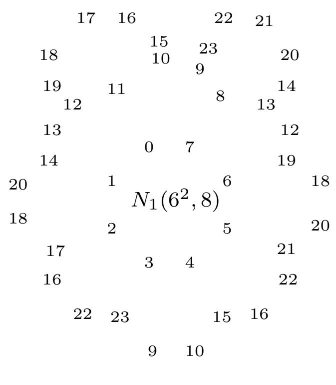
# 1 Examples: Semi-equivelar maps on the surface of Euler characteristic −1

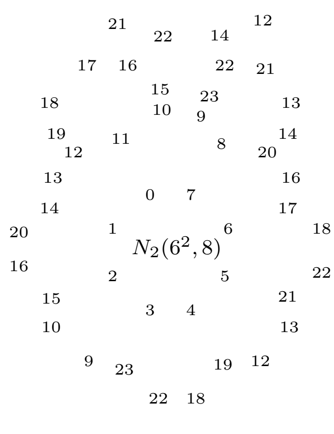


**Claim 1.1.** *$N_1(6^2, 8) \not\cong N_2(6^2, 8)$ and $K_1(3, 4, 8, 4) \not\cong K_2(3, 4, 8, 4)$, also $N_1$ $(6^2, 8)$, $N_2(6^2, 8)$, $K_1(3, 4, 8, 4)$ and $K_2(3, 4, 8, 4)$ are not vertex transitive.*

**Proof.** Consider the graphs $EG(G_{12}(N_1(6^2, 8))) = \{[0, 7], [3, 4], [8, 13], [11, 12], [15, 16], [22, 23]\}$, $EG(G_{12}(N_2(6^2, 8))) = \{[4, 5], [18, 19], [21, 22]\}$, $EG(G_2(K_1(3, 4, 8, 4))) = C_{12}(1, 10, 12, 6, 19, 18, 2, 21, 14, 5, 23, 17) \cup C_6(4, 13, 9, 7, 20, 15)$ and $EG(G_2(K_2(3, 4, 8, 4))) = C_{21}(0, 8, 21, 3, 12, 10, 1, 18, 20, 6, 15, 22, 2, 17, 19, 7, 9, 13, 5, 16, 11)$. From these graphs and discussions in page 3 it is evident that $N_1(6^2, 8) \not\cong N_2(6^2, 8)$ and $K_1(3, 4, 8, 4) \not\cong K_2(3, 4, 8, 4)$. Also from these graphs one can deduce that the above four maps are not vertex transitive. This proves the claim. □

**Claim 1.2.** *$M_1(4, 6, 16) \not\cong M_2(4, 6, 16)$ and $M_1(4, 6, 16)$, $M_2(4, 6, 16)$ are not vertex transitive.*

**Proof.** Let $A(EG(M_1))$ and $A(EG(M_2))$ denote the adjacency matrices associated to the edge graphs of $M_1(4, 6, 16)$ and $M_2(4, 6, 16)$, respectively. Let $P_1(x)$ and $P_2(x)$ denote the characteristic polynomials of $A(EG(M_1))$ and $A(EG(M_2))$ respectively. If the map $M_1(4, 6, 16)$ and $M_2(4, 6, 16)$ are isomorphic, then $P_1(x) = P_2(x)$, (see [16]). We have (using Maple):

$P_1(x) = x^{48} - 73x^{46} + 2454x^{44} - 50419x^{42} + 708648x^{40} - 63x^{39} + 3326x^{37} + 55370675x^{36} - 78998x^{35} - 325536254x^{34} + 1117272x^{33} + 1488079446x^{32} - 10498532x^{31} - 5328759647x^{30} + 69274014x^{29} + 15001009001x^{28} - 330979906x^{27} - 33214008513x^{26} + 1164748518x^{25} + 57733175145x^{24} - 3045404365x^{23} - 78484320585x^{22} + 5935770108x^{21} + 82965261974x^{20} - 8621690840x^{19} - 67636071362x^{18} + 9302657658x^{17} + 42014823892x^{16} - 7407374240x^{15} - 19530592234x^{14} + 4302417304x^{13} + 6604154516x^{12} - 1787400560x^{11} - 1549106652x^{10} + 513857976x^9 + 230136488x^8 - 96466160x^7 - 17066976x^6 + 10545344x^5 - 49440x^4 - 495936x^3 + 67264x^2 - 1920x$;

$P_2(x) = x^{48} - 72x^{46} + 2388x^{44} - 48424x^{42} + 672018x^{40} - 28x^{39} - 6770448x^{38} + 1464x^{37} + 51267848x^{36} - 34548x^{35} - 298108536x^{34} + 486936x^{33} + 1348802145x^{32} - 4573164x^{31} - 4785171566x^{30} + 30247956x^{29} + 13360329054x^{28} - 145305100x^{27} - 29376425928x^{26} + 515828328x^{25} + 50783351168x^{24} - 1365657624x^{23} - 68773076142x^{22} + 2706801464x^{21} + 72559583454x^{20} - 4017232620x^{19} - 59173427088x^{18} + 4451481228x^{17} + 36880710516x^{16} - 3658879076x^{15} - 17277557628x^{14} + 2204369472x^{13} + 5931587385x^{12} - 953952300x^{11} - 1432856946x^{10} + 286671228x^9 + 226687857x^8 - 56423208x^7 - 20151768x^6 + 6499968x^5 + 573840x^4 - 330368x^3 + 26880x^2$.

Therefore $M_1(4, 6, 16) \not\cong M_2(4, 6, 16)$. Also, we have $EG(G_{15}(M_1(4, 6, 16))) = EG\ (G_{15}(M_2(4, 6, 16))) = C_8(0, 2, 4, 6, 8, 10, 12, 14) \cup C_8(1, 3, 5, 7, 9, 11, 13, 15) \cup C_8(16, 18, 22, 24, 26, 28, 30, 32) \cup C_8(17, 21, 23, 25, 27, 29, 31, 33) \cup C_8(19, 35, 37, 39, 41, 43, 45, 47) \cup C_8(20, 36, 38, 40, 42, 44, 46, 34)$. Let

$\alpha \in Aut(M_1(4,6,16))$ such that $\alpha(1) = 2$ then $\alpha$ induces an automorphism on $EG(G_{15}(M_1(4,6,16)))$. So $\alpha(\{3,15\}) = \{0,4\}$. This implies $\alpha(3) = 0$ or 4. But from the links of 1 and 2 it is easy to see that $\alpha(3) \neq 0$. So we have $\alpha(3) = 4$, this implies $\alpha(13) = 6$ and $\alpha(35) = 20$. From $\alpha(35) \mapsto 20$ we get $\alpha(42) = 43$. Now considering lk(42) and lk(43) and the map $\alpha(42) \mapsto 43$, we see that $\alpha(13) = 14$, a contradiction. Thus there is no automorphism which maps 1 to 2. Hence $M_1(4,6,16)$ is not vertex transitive. Similarly for $M_2(4,6,16)$ we get no automorphism such that $\alpha(1) = 2$. This proves the Claim 1.2. □

## 2 Enumeration of SEMs on the surface of Euler characteristic −1

Considering the Euler characteristic equation, it is easy to see that semi-equivelar maps of types $(3^4,4^2)$ and $(3,4^4)$ do not exist on the surface of Euler characteristic −1. As, in these cases the number of vertices required to complete a link of a vertex is bigger than the number of vertices of the SEMs. From the study of remaining eight types: $(3^4,8)$, $(3^2,4,3,6)$, $(3,4,8,4)$, $(3,6,4,6)$, $(4^3,6)$, $(4,6,16)$, $(4,8,12)$ and $(6^2,8)$, we show the following :

**Lemma 4.** *There exists no SEM of type* $(3^4,8)$ *on the surface of Euler characteristic* −1*.*

**Proof.** Let $M$ be a SEM of type $(3^4,8)$ on the surface of Euler characteristic −1. The notation $\text{lk}(i) = C_{10}([i_1,i_2,i_3,i_4,i_5,i_6,i_7],i_8,i_9,i_{10})$ for the link of a vertex $i$ will mean that $[i,i_1,i_{10}]$, $[i,i_9,i_{10}]$, $[i,i_8,i_9]$, $[i,i_7,i_8]$ form triangular faces and $[i,i_1,i_2,i_3,i_4,i_5,i_6,i_7]$ forms an octagonal face. If $|V|$ denotes the number of vertices in $V(M)$, $E(M)$ denotes the number of edges, $T(M)$ denotes the number of triangular faces and $O(M)$ denotes the number of octagonal faces in map $M$, respectively, then it is easy to see that $E(M) = \frac{5|V|}{2}$, $T(M) = \frac{4|V|}{3}$ and $O(M) = \frac{|V|}{8}$. By Euler's equation we get, $-1 = |V| - \frac{5|V|}{2} + (\frac{4|V|}{3} + \frac{|V|}{8})$, *i.e.* $-1 = |V|(\frac{-1}{24})$. From the equation we see, if the map exists then $|V| = 24$. Let $V = V(M) = \{0, 1, \ldots, 23\}$. Now, we prove the lemma by exhaustive search for all $M$. Assume without loss of generality that $\text{lk}(0) = C_{10}([1,2,3,4,5,6,7],8,9,10)$. This implies $\text{lk}(7) = C_{10}([6,5,4,3,2,1,0],8,a,b)$ for $a,b \in V$. Observe that $(a,b) \in \{(10,9),(11,12)\}$. In case $(a,b) = (10,9)$, considering lk(10) we see that 1 lies in two octagonal faces, which is not allowed. In case $(a,b) = (11,12)$ we get $\text{lk}(7) = C_{10}([0,1,2,3,4,5,6],12,11,8)$, $\text{lk}(6) = C_{10}([7,0,1,2,3,4,5],14,13,12)$, $\text{lk}(5) = C_{10}([6,7,0,1,2,3,4],16,15,14)$, $\text{lk}(4) = C_{10}([5,6,7,0,1,2,3],18,17,16)$, $\text{lk}(3) = C_{10}([4,5,6,7,0,1,2],20,19,18)$, $\text{lk}(2) = C_{10}([3,4,5,6,7,0,1],22,21,20)$ and $\text{lk}(1) = C_{10}([2,3,4,5,6,7,0],10,23,22)$. This implies $\text{lk}(8) = C_{10}([9,c,d,e,f,g,h],11,7,0)$ for $c,d,e,f,g,h \in V$. In this case we have $(h,g) \in \{(17,18),$

(21, 22)}. But for both the cases of $(h, g)$ we see easily that no map exists, for detailed calculation we refer the readers to see [23]. This proves the lemma. □

**Lemma 5.** *There exists no SEM of type* $(3^2, 4, 3, 6)$ *on the surface of Euler characteristic* $-1$.

**Proof.** Let $G$ be a SEM of type $(3^2, 4, 3, 6)$ on the surface of Euler characteristic $-1$. The notation $\text{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], i_6, i_7, i_8, i_9)$ for the link of $i$ will mean that $[i, i_1, i_9]$, $[i, i_5, i_6]$, $[i, i_6, i_7]$ form triangular faces, $[i, i_7, i_8, i_9]$ forms a quadrangular face and $[i, i_1, i_2, i_3, i_4\ , i_5]$ forms a hexagonal face. Let $|V|$ denote the number of vertices in $V(G)$. If $E(G)$, $T(G)$, $Q(G)$ and $H(G)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of hexagonal faces in the map $G$, respectively, then it is easy to see that $E(G) = \frac{5|V|}{2}$, $T(G) = \frac{3|V|}{3}$, $Q(G) = \frac{|V|}{4}$ and $H(G) = \frac{|V|}{6}$. By Euler's equation we see, if the map exists then $|V| = 12$. Let $V = V(G) = \{0, 1, \ldots, 11\}$. Now, we prove the lemma by exhaustive search for all $G$. Assume that $\text{lk}(0) = C_{11}([1, 2, 3, 4, 5], 6, 7, 8, 9)$ then $\text{lk}(7) = C_{11}([a, b, c, d, e], 6, 0, 9, 8)$ or $\text{lk}(7) = C_{11}([6, a, b, c, d], e, 8, 9, 0)$ for some $a, b, c, d, e \in V$. But, for both the cases we need more than twelve vertices to complete lk(7). This is not allowed. So the map does not exist. □

**Lemma 6.** *There exists no SEM of type* $(3, 6, 4, 6)$ *on the surface of Euler characteristic* $-1$.

**Proof.** Let $E$ be a SEM of type $(3, 6, 4, 6)$ on the surface of Euler characteristic $-1$. The notation $\text{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], [i_6, i_7, i_8, i_9, i_{10}], i_{11})$ for the link of $i$ will mean that $[i, i_5, i_6]$ forms a triangular face, $[i, i_1, i_{11}, i_{10}]$ forms a quadrangular face and $[i, i_1, i_2, i_3, i_4, i_5]$, $[i, i_6, i_7, i_8, i_9, i_{10}]$ form hexagonal faces. Let $|V|$ denote the number of vertices in $V(E)$. If $E(E)$, $T(E)$, $Q(E)$ and $H(E)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of hexagonal faces, respectively, then we see that $E(E) = \frac{4|V|}{2}$, $T(E) = \frac{|V|}{3}$, $Q(E) = \frac{|V|}{4}$ and $H(E) = \frac{2|V|}{6}$. By Euler's equation we see, if the map exists then $|V| = 12$. For this, let $V = V(E) = \{0, 1, \ldots, 11\}$. Now, we prove the lemma by exhaustive search for all $E$. For this assume that $\text{lk}(0) = C_{11}([1, 2, 3, 4, 5], [6, 7, 8, 9, 10], 11)$, then $\text{lk}(1) = C_{11}([0, 5, 4, 3, 2], [a, b, c, d, 11], 10)$ for some $a, b, c, d \in V$. Now, it is easy to see that lk(1) can not be completed, as $a$, $b$, $c$, $d$ have no suitable values in $V(E)$. Therefore the required map does not exist. Thus the lemma is proved. □

**Lemma 7.** *There exists no SEM of type* $(4^3, 6)$ *on the surface of Euler characteristic* $-1$.

**Proof:** Let $F$ be a SEM of type $(4^3, 6)$ on the surface of Euler characteristic $-1$. The notation $\text{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5], i_6, i_7, i_8, i_9, i_{10})$ for the link of $i$ will mean that $[i, i_1, i_{10}, i_9]$, $[i, i_5, i_6, i_7]$, $[i, i_7, i_8, i_9]$ form quadrangular faces and $[i, i_1, i_2, i_3, i_4, i_5]$ forms a hexagonal face. Let $|V|$ denote the number of vertices in $V(F)$. If $E(F)$, $Q(F)$ and $H(F)$ denote the number of edges, number of quadrangular faces and number of hexagonal faces, respectively, then $E(F) = \frac{4|V|}{2}$, $Q(F) = \frac{3|V|}{4}$ and $H(F) = \frac{|V|}{6}$. By Euler's equation we see if the map exists then $|V| = 12$. For this, let $V = V(F) = \{0, 1, \ldots, 11\}$. Now we prove the lemma by exhaustive search for all $F$. Assume that $\text{lk}(0) = C_{11}([1,2,3,4,5],6,7,8,9,10)$. This implies, $\text{lk}(7) = C_{11}([b,c,d,e,6],5,0,9,8,a)$ or $\text{lk}(7) = C_{11}([b,c,d,e,8],9,0,5,6,a)$ for some $a,b,c,d,e \in V$. Then for both the cases of lk(7) we need more than twelve vertices to complete. But this is not allowed. So we do not get the required map. Thus the lemma is proved. □

**Lemma 8.** *There exists no SEM of type* $(4,8,12)$ *on the surface of Euler characteristic* $-1$.

**Proof.** Let $M$ be a SEM of type $(4,8,12)$ on the surface of Euler characteristic $-1$. The notation $\text{lk}(i) = C_{18}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, \boldsymbol{i_{11}}], i_{12}, [\boldsymbol{i_{13}}, i_{14}, i_{15}, i_{16}, i_{17}, i_{18}])$ for the link of $i$ will mean that $[i, i_{11}, i_{12}, i_{13}]$, $[i, i_1, i_{18}, i_{17}, i_{16}, i_{15}, i_{14}, i_{13}]$ and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, i_{11}]$ form a 4-gonal, a 8-gonal and a 12-gonal faces. If $|V|$, $E(M)$, $Q(M)$, $O(M)$ and $R(M)$ denote the number of vertices, number of edges, number of 4-gonal faces, number of 8-gonal faces and number of 12-gonal faces in $M$, respectively, then we see that $E(M) = \frac{3|V|}{2}$, $Q(M) = \frac{|V|}{4}$, $O(M) = \frac{|V|}{8}$ and $R(M) = \frac{|V|}{12}$. By Euler's equation we see, if the map exists then $|V| = 24$. For this, let $V = V(M) = \{0, 1, \ldots, 23\}$. Assume that $\text{lk}(0) = C_{18}([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], 12, [13, 14, 15, 16, 17, 18])$. This implies $\text{lk}(1) = C_{18}([0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2], 19, [18, 17, 16, 15, 14, 13])$ and $\text{lk}(2) = C_{18}([3, 4, 5, 6, 7, 8, 9, 10, 11, 0, 1], 18, [19, \text{a}, \text{b}, \text{c}, \text{d}, \text{e}])$ for some $a,b,c,d,e \in V$. Observe that $a \in \{12, 20\}$. If $a = 12$ then $b = 13$, for otherwise $\deg(12) > 3$. But, then 13 appears in two octagonal faces, which is not allowed. So we have $a = 20$, this implies $b \in \{12, 21\}$. If $b = 12$ then $c = 13$ and we get 13 in two octagonal faces. So $b = 21$. This implies $c \in \{12, 22\}$. In case $c = 12$, $d = 13$. This implies 13 appears in two octagonal faces. If $c = 22$ then $d = 23$, now we see that $e$ has no suitable value in $V$ so that lk(2) can be completed. So, the required map does not exist. □

**Proof of Theorem 2.** The proof of Theorem 2 follows from Lemmas 4, 5, 6, 7 and 8. □

**Proof of Lemma 1.** Let $K$ be a SEM of type $(3,4,8,4)$ on the surface of Euler characteristic $-1$. The notation $\text{lk}(i) = C_{11}([i_1, i_2, i_3, i_4, i_5, i_6, i_7], i_8, [i_9, i_{10}], i_{11})$ for the link of $i$ will mean that $[i, i_9, i_{10}]$ forms a triangular face, $[i, i_7, i_8, i_9]$, $[i, i_1, i_{11}, i_{10}]$ form quadrangular faces and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7]$ forms a octagonal face. Let $|V|$ denote the number of vertices in $V(K)$. If $E(K)$, $T(K)$, $Q(K)$ and $O(K)$ denote the number of edges, number of triangular faces, number of quadrangular faces and number of octagonal faces in the map $K$, respectively, then we see that $E(K) = \frac{4|V|}{2}$, $T(K) = \frac{|V|}{3}$, $Q(K) = \frac{2|V|}{4}$ and $H(K) = \frac{|V|}{8}$. By Euler's equation we see, if the map exists then $|V| = 24$. Let $V = V(K) = \{0, 1, \dots, 23\}$. Now, we prove the result by exhaustive search for all $K$.

Let $\mathrm{lk}(0) = C_{11}([1,2,3,4,5,6,7],8,[9,10],11)$, this implies $\mathrm{lk}(9) = C_{11}([b,c,d,e,f,g,8],7,[0,10],a)$ and $\mathrm{lk}(10) = C_{11}([11,l,k,j,i,h,a],12,[9,0],1)$ for some $a,b,c,d,e,f,g,h,i,j,k,l \in V$. Observe that $b = 12$ and $a = 13$, then octagonal faces of the map $K$ are $O_1 = [0,1,2,3,4,5,6,7]$, $O_2 = [8,9,12,c,d,e,f,g]$ and $O_3 = [13,10,11,l,k,j,i,h]$. As these faces share no vertex with each other, successively we get $c = 14$, $d = 15$, $e = 16$, $f = 17$, $g = 18$, $l = 19$, $k = 20$, $j = 21$, $i = 22$ and $h = 23$. This implies $\mathrm{lk}(9) = C_{11}([12,14,15,16,17,18,8],7,[0,10],13)$, $\mathrm{lk}(10) = C_{11}([11,19,20,21,22,23,13],12,[9,0],1)$ and $\mathrm{lk}(8) = C_{11}([18,17,16,15,14,12,9],0,[7,x],y)$ for some $x,y \in V$. In this case, $(x,y) \in \{(19, 11), (19, 20), (20, 19), (20, 21), (21, 20), (21, 22), (22, 21), (22, 23), (23, 13), (23, 22)\}$. If $(x,y) = (23,13)$ then considering lk(8) and lk(13) successively we see 12 18 as an edge and a non-edge both. Also, $(19,20) \cong (23,13)$; $(20,19) \cong (22,21)$ and $(20,21) \cong (22,23)$ by the map (0, 9)(1, 12)(2, 14)(3, 15)(4, 16)(5, 17) (6, 18)(7, 8)(11, 13)(19, 23)(20, 22); $(20,19) \cong (21,22)$ by the map (0, 8)(1, 18)(2, 17)(3, 16)(4, 15)(5, 14)(6, 12)(7, 9)(10, 21) (11, 22)(13, 20)(19, 23); $(19,11) \cong (21,20)$ by the map (0, 8)(1, 18)(2, 17)(3, 16)(4, 15)(5, 14)(6, 12)(7, 9)(10, 21)(11, 20)(13, 22). So, it is enough to search the map for $(x,y) \in \{(19, 11), (20, 19), (20, 21), (23, 22)\}$.

**Case 1.** If $(x,y) = (19,11)$ then constructing the map successively we get $\mathrm{lk}(8) = C_{11}([18,17,16,15,14,12,9],0,[7,19],11)$, $\mathrm{lk}(7) = C_{11}([0,1,2,3,4,5,6],20,[19,8],9)$, $\mathrm{lk}(19) = C_{11}([20,21,22,23,13,10,11],18,[8,7],6)$, $\mathrm{lk}(11) = C_{11}([19,20,21,22,23,13,10],0,[1,18],8)$, $\mathrm{lk}(1) = C_{11}([0,7,6,5,4,3,2],17,[18,11],10)$, $\mathrm{lk}(18) = C_{11}([17,16,15,14,12,9,8],19,[11,1],2)$ and $\mathrm{lk}(6) = C_{11}([5,4,3,2,1,0,7],19,[20,m],n)$ for some $m,n \in V$. In this case, $(m,n) \in \{(14, 12), (14, 15), (15, 14), (15, 16), (16, 15), (16, 17)\}$. If $(m,n) = (16,17)$ then considering lk(17) we see 25 as an edge and a non-edge both and, if $(m,n) = (14,15)$ then considering lk(6) and lk(20) we see that lk(21) can not be completed. For the remaining values of $(m,n)$, we have following subcases.

When $(m,n) = (15,14)$ then $\mathrm{lk}(6) = C_{11}([5,4,3,2,1,0,7],19,[20,15],14)$, $\mathrm{lk}(15) = C_{11}([16,17,18,8,9,12,14],5,[6,20],21)$ and $\mathrm{lk}(20) = C_{11}([21,22,23,13,10,11,19],7,[6,15],16)$. This implies $\mathrm{lk}(21) = C_{11}([22,23,13,10,11,19,20],15,[16,o],p)$ for some $o,p \in V$. Observe that $(o,p) \in \{(3, 2), (3, 4), (4, 3), (4, 5)\}$. In case $(o,p) \in \{(3, 2), (3, 4)\}$ considering lk(21), lk(16) and lk(3) successively we see that lk(4) or lk(17) can not be completed. If $(o,p) = (4,3)$ then considering lk(21), lk(4), lk(16) successively we see that lk(22) can not be completed. If $(o,p) = (4,5)$ then successively considering lk(21), lk(5) and lk(22), we get $C_9(9,10,11,19,20,21,22,23,12) \subseteq \mathrm{lk}(13)$. A contradiction. So,

$(m,n) \neq (15,14)$.
**Subcase 1.1.** If $(m,n) = (14,12)$ then successively we get $\mathbf{K_1(3,4,8,4)}$ as given in Section 1.
**Subcase 1.2.** When $(m,n) = (15,16)$ then we get an object which is isomorphic to $\mathbf{K_2(3,4,8,4)}$, as given in Section 1, by the map (0, 23, 7, 22)(1, 13, 6, 21)(2, 10, 5, 20)(3, 11, 4, 19)(8, 14, 17, 9, 15, 18, 12, 16). Consideration of other cases is a similar enumeration the details of which are given in [23]. We do not get any new object in this process. To save space we are hence referring the reader to [23] for further details. □

**Proof of Lemma 2.** Let $M$ be a SEM of type $(4,6,16)$ on the surface of Euler characteristic $-1$. The notation $\mathrm{lk}(i) = C_{20}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, i_{11}, i_{12}, i_{13}, i_{14}, \boldsymbol{i_{15}}, i_{16}, \boldsymbol{i_{17}}, i_{18}, i_{19}, i_{20})$ for the link of $i$ will mean that $[i, i_{15}, i_{16}, i_{17}]$, $[i, i_1, i_{20}, i_{19}, i_{18}, i_{17}]$ and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}, i_{11}, i_{12}, i_{13}, i_{14}, i_{15}]$ form a 4-gonal face, a 6-gonal face and a 16-gonal face, respectively. Let $|V|$ denote the number of vertices in $V(M)$. If $E(M)$, $Q(M)$, $H(M)$ and $P(M)$ denote the number of edges, number of 4-gonal faces, number of 6-gonal faces and number of 16-gonal faces, respectively, then $E(M) = \frac{3|V|}{2}$, $Q(M) = \frac{|V|}{4}$, $H(M) = \frac{|V|}{6}$ and $P(M) = \frac{|V|}{16}$. By Euler's equation we see if the map exists then $|V| = 48$. For this, let $V = V(M) = \{0, 1, \ldots, 47\}$. Now we can prove the result by exhaustive search for all $M$.

From a case by case consideration for different values of $(c,d)$ similar to that of Lemma 1 we get $M_1$ and $M_2$ as given in Section 1 and we do not get any other object different than these. The detailed enumeration is given in [23]. To save space we are hence referring the reader to [23] for further details. □

**Proof of Lemma 3.** Let $N$ be a SEM of type $(6^2,8)$ on the surface of Euler characteristic $-1$. The notation $\mathrm{lk}(i) = C_{14}([\boldsymbol{i_1}, i_2, i_3, i_4, i_5, i_6, \boldsymbol{i_7}], i_8, i_9, i_{10}, \boldsymbol{i_{11}}, i_{12}, i_{13}, i_{14})$ for the link of $i$ will mean that $[i, i_7, i_8, i_9, i_{10}, i_{11}]$, $[i, i_1, i_{14}, i_{13}, i_{12}, i_{11}]$ form hexagonal faces and $[i, i_1, i_2, i_3, i_4, i_5, i_6, i_7]$ forms a octagonal face. If $|V|$, $E(N)$, $H(N)$ and $O(N)$ denote the number vertices, number of edges, the number of hexagonal faces and the number of octagonal faces in the map $N$, respectively, then $E(N) = \frac{3|V|}{2}$, $H(N) = \frac{2|V|}{6}$ and $O(N) = \frac{|V|}{8}$. Using Euler's equation we see that if the map exists then $|V| = 24$. A case by case consideration similar to that of Lemma 1 gives us $N_1$ and $N_2$ as given in Section 1. We do not get any other object different than these. The detailed enumeration of this is given in [23]. To save space we are hence referring the reader to [23] for further details. These proofs are also available with authors and may be supplied if required. □

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# Some companions of Ostrowski type inequalities for twice differentiable functions

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**Abstract.** The main aim of this paper is to establish some companions of Ostrowski type integral inequalities for functions whose second derivatives are bounded. Moreover, some Ostrowski type inequalities are given for mappings whose first derivatives are of bounded variation. Some applications for special means and quadrature formulae are also given.

**Keywords:** Function of bounded variation, Ostrowski type inequalities, Riemann-Stieltjes integral.

**MSC 2000 classification:** primary 26D15, 26A45 secondary 26D10, 41A55

## 1 Introduction

In 1938, Ostrowski [27] established a following useful inequality:

**Theorem 1.** *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on $(a,b)$ whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on $(a,b)$, i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[ \frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty , \tag{1.1}$$

*for all $x\in[a,b]$.*

The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) is referred to, in the literature, as the Ostrowski inequality. Numerous studies were devoted to extensions and generalizations of this inequality in both the integral and discrete case. For some examples, please refer to ([10], [11], [17]-[26], [28]-[35])