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Existence and multiplicity results for Dirichlet boundary value problems involving the $(p_1(x), p_2(x))$-Laplace operator

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Abstract. This paper is concerned with the existence and multiplicity of solutions for the following Dirichlet boundary value problems involving the $(p_1(x), p_2(x))$-Laplace operator of the form:

$$-\operatorname{div}(|\nabla u|^{p_1(x)-2}\nabla u) - \operatorname{div}(|\nabla u|^{p_2(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

By means of critical point theorems with Cerami condition and the theory of the variable exponent Sobolev spaces, we establish the existence and multiplicity of solutions.

Keywords: variational methods, generalized Lebesgue-Sobolev spaces

MSC 2000 classification: 35A15, 46E35

Introduction

The purpose of this article is to show the existence of solutions of the following Dirichlet problem involving the $(p_1(x), p_2(x))$-Laplace operator of the form

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p_1(x)-2}\nabla u) - \operatorname{div}(|\nabla u|^{p_2(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p_i \in C(\overline{\Omega})$ such that $1 < p_i^- := \inf_{x\in\overline{\Omega}} p_i(x) \leq p_i^+ := \sup_{x\in\overline{\Omega}} p_i(x) < +\infty$ for $i = 1, 2$, and $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

The operator $\triangle_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated properties than the p-Laplacian; for example, it is inhomogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [22], electrorheological fluids [1] or image restoration [7]. The differential operator $\operatorname{div}(|\nabla u|^{p_1(x)-2}\nabla u) + \operatorname{div}(|\nabla u|^{p_2(x)-2}\nabla u)$ is known as the $(p_1(x), p_2(x))$-Laplacian operator when $p_1 \neq p_2$. When $p_1(x)$ and $p_2(x)$ are constant, this operator arises in problems of mathematical physics (see [3]) and in plasma physics and biophysics (see [8]). If $p_1 = p_2 = p$, then we have a single operator the $p(x)$-Laplacian and problem (1) becomes the $p(x)$-Laplacian Dirichlet problem of the form

$$\begin{aligned} -\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

which have been studied sufficiently by several authors [14, 20] and the references therein.

Define the family of functions

$$\mathcal{F} = \left\{ G_\gamma : G_\gamma(x,t) = f(x,t)t - \gamma F(x,t), \gamma \in [2p_m^-, 2p_M^+] \right\},$$

where $p_m(x) = \min\{p_1(x), p_2(x)\}$, $p_M(x) = \max\{p_1(x), p_2(x)\}$ for all $x \in \overline{\Omega}$, $p_m^- = \inf_{x\in\overline{\Omega}} p_m(x)$, $p_M^+ = \sup_{x\in\overline{\Omega}} p_M(x)$ and $F(x,t) = \int_0^t f(x,s)ds$.

Noting that when $p_1(x) \equiv p_2(x) \equiv p$ is a constant, $\mathcal{F} = \{f(x,t)t - 2pF(x,t)\}$ consists of only one element.

Throughout this paper, we make the following assumptions on the function f:

(f₁) There exist $C > 0$ and $q \in C_+(\overline{\Omega})$ with $q(x) < p_M^*(x)$ for all $x \in \overline{\Omega}$ such that

$$|f(x,t)| \leq C(1 + |t|^{q(x)-1})$$

for each $(x,t) \in \Omega \times \mathbb{R}$, where

$$C_+(\overline{\Omega}) = \left\{ p(x) : p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}$$

and p_M^* is the critical exponent of p_M, i.e.,

$$p_M^*(x) = \begin{cases} \frac{Np_M(x)}{N - p_M(x)} & \text{if } p_M(x) < N, \\ \infty & \text{if } p_M(x) \geq N. \end{cases}$$

($\mathbf{f_2}$) $f(x,t) = o(|t|^{p_M^+-1})$ as $t \to 0$ uniformly for a.e. $x \in \Omega$.

($\mathbf{f_3}$) $f(x,-t) = -f(x,t)$ for $(x,t) \in \Omega \times \mathbb{R}$.

($\mathbf{f_4}$) $\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{p_M^+}} = +\infty$, uniformly for a.e. $x \in \Omega$.

(**g**) There exists a constant $\delta \geq 1$ such that for all $\gamma, \eta \in [2p_m^-, 2p_M^+]$ and $(s,t) \in [0,1] \times \mathbb{R}$, the inequality

$$\delta G_\gamma(x,t) \geq G_\eta(x,st) \qquad \text{holds for a.e. } x \in \overline{\Omega}.$$

In [18] the authors consider problem (1) in the particular case $f(x,u) = \lambda |u|^{q(x)-2}u$, where $\lambda > 0$. Under the assumptions $1 < p_2(x) < q(x) < p_1(x) < N$ and $\max_{y\in\overline{\Omega}} q(y) < \frac{Np_2(x)}{N-p_2(x)}$ for all $x \in \overline{\Omega}$, they established the existence of two positive constants λ_0, λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of the above problem.

In [16] the authors consider problem (1) i.e., which is the well-known anisotropic $\vec{p}(.)$-Laplacian problem (see, e.g., [4] and references therein) in the case $N = 1, 2$, that is,

$$-\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \right) = f(x,u).$$

Under proper growth condition and specially the well-known Ambrosetti-Rabinowitz type condition:

$$\exists \nu > p_M^+, \quad M > 0 \quad \text{such that}$$
$$x \in \Omega, |t| \geq M \quad \Rightarrow \quad 0 \leq \nu F(x,t) \leq f(x,t)t, \quad \text{(AR)}$$

they obtained some existence and multiplicity results.

The role of (AR) condition is to ensure the boundness of the Palais-Smale sequences of the Euler-Lagrange functional. This is very crucial in the applications of critical point theory. However, although (AR) is a quite natural condition, it is somewhat restrictive and eliminates many nonlinearities. Indeed, there are many superlinear functions which do not satisfy (AR) condition. For instance when $p_1(x) = p_2(x) \equiv 2$ and $\delta = 2$, the function below does not satisfy (AR), while it satisfies the aforementioned conditions.

$$f(x,t) = 2t \log(1 + |t|). \tag{3}$$

But it is easy to see the above function (3) satisfies ($\mathbf{f_1}$)–($\mathbf{f_4}$).

As far as we are aware, elliptic problems like (1) involving the $(p_1(x), p_2(x))$-Laplace operator without the (AR) type condition, have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction. The purpose of this work is to improve the results of the above mentioned papers. Without assuming the Ambrosetti-Rabinowitz type conditions (AR), we prove the existence of solutions.

Remark 1. If $f(x,t)$ is increasing in t, then (AR) implies (**g**) when t is large enough, in fact, we can take $\delta = \frac{1}{1-\frac{p_M^+}{\nu}} > 1$, then

$$\delta G_\gamma(x,t) - G_\eta(x,st) \geq f(x,t)t - f(x,st)st \geq 0.$$

But, in general, (AR) does not imply (g), see [20, Remark 3.4] when $p_1(x) \equiv p_2(x) \equiv p$.

Now we are ready to state our results.

Theorem 1. *Suppose that the conditions* ($\mathbf{f_1}$), ($\mathbf{f_2}$),($\mathbf{f_4}$) *and* (**g**) *are satisfied. If $q^- > p_M^+$, then the problem (1) has at least one nontrivial solution.*

Theorem 2. *Assume that* ($\mathbf{f_1}$), ($\mathbf{f_3}$),($\mathbf{f_4}$) *and* (**g**) *hold. If $q^- > p_M^+$, then problem* (1) *has a sequence of weak solutions with unbounded energy.*

The present paper is divided into three sections, organized as follows: In section 2, we introduce some basic properties of the Lebesgue and Sobolev spaces with variable exponents and some min-max theorems like mountain pass theorem and fountain theorem with the Cerami condition that will be used later. In section 3, we prove our main results.

Throughout the sequel, the letters $c, c_i, i = 1, 2, ...,$ denotes positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

Preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [9, 10, 11, 12] for details. Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$p^+ = \max\{p(x) : x \in \overline{\Omega}\}, \quad p^- = \min\{p(x) : x \in \overline{\Omega}\},$$

$$L^{p(x)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1 \right\},$$

becomes a Banach space [15]. We also define the space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

Proposition 1 ([9, 12]). *(i) The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv|dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}.$$

(ii) If $p_1, p_2 \in C_+(\overline{\Omega})$ and $p_1(x) \leq p_2(x)$ for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2 ([12]). *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)}dx$, then for $u \in X$ and $(u_k) \subset X$, we have*

(1) $\|u\| < 1$ (respectively $= 1; > 1$) if and only if $\rho(u) < 1$ (respectively $= 1; > 1$);

(2) for $u \neq 0$, $\|u\| = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;

(3) if $\|u\| > 1$, then $\|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

(4) if $\|u\| < 1$, then $\|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

(5) $\|u_k\| \to 0$ (respectively $\to \infty$) if and only if $\rho(u_k) \to 0$ (respectively $\to \infty$).

For $x \in \Omega$, let us define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 3 ([12, 13]). *(i) $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

(ii) If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ ($q(x) < p^*(x)$) for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $X \hookrightarrow L^{q(x)}(\Omega)$.

(iii) There is constant $C > 0$ such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \quad \forall u \in W_0^{k,p(x)}(\Omega).$$

We recall the definition of the Cerami condition (C) introduced by G. Cerami (see [5]).

Definition 1 ([5])**.** Let $(X, \|.\|)$ be a Banach space and $J \in C^1(X, \mathbb{R})$, given $c \in \mathbb{R}$, we say that J satisfies the Cerami c condition (we denote condition (C_c)), if

- (i) any bounded sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence;
- (ii) there exist constants $\delta, R, \beta > 0$ such that

$$\|J'(u)\| \|u\| \geq \beta \quad \forall u \in J^{-1}([c-\delta, c+\delta]) \quad \text{with} \quad \|u\| \geq R.$$

If $J \in C^1(X, \mathbb{R})$ satisfies condition (C_c) for every $c \in \mathbb{R}$, we say that J satisfies condition (C).

Note that condition (C) is weaker than the (PS) condition. However, it was shown in [2] that from condition (C) it is possible to obtain a deformation lemma, which is fundamental in order to get some critical point theorems. More precisely, let us recall the version of the mountain pass lemma with Cerami condition which is used in the sequel.

Proposition 4 ([2])**.** *Let $(X, \|.\|)$ a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$, be such that $||e|| > r$ and*

$$b := \inf_{\|u\|=r} J(u) > J(0) \geq J(e).$$

If J satisfies the condition (C_c) with

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

$$\Gamma := \{\gamma \in C([0,1], X) | \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of J.

We also introduce the fountain theorem with the Condition (C) which is a variant of [19, 23]. Let X be a reflexive and separable Banach space. Then, from [21] there are $\{e_i\} \subset X$ and $\{e_i^*\} \subset X^*$ such that

$$X = \overline{\langle e_i, i \in \mathbb{N}^* \rangle}, \quad X^* = \overline{\langle e_i^*, i \in \mathbb{N}^* \rangle}, \quad \langle e_i, e_j^* \rangle = \delta_{i,j},$$

where $\delta_{i,j}$ denotes the Kroneker symbol. For $k \in \mathbb{N}^*$, put

$$X_k = \mathbb{R} e_k, \quad Y_k = \overset{k}{\underset{i=1}{\oplus}} X_i, \quad Z_k = \overline{\overset{\infty}{\underset{i=k}{\oplus}} X_i}.$$

We have the following lemma.

Lemma 1. *([16]). For $q \in C_+(\overline{\Omega})$ and $q(x) < p^*_M(x)$ for all $x \in \overline{\Omega}$, define*

$$\beta_k = \sup\{|u|_{q(x)} : \ \|u\| = 1, \quad u \in Z_k\}.$$

Then $\lim_{k\to+\infty} \beta_k = 0$.

Proposition 5 ([17])**.** *Assume that $(X, \|.\|)$ is a separable Banach space, $J \in C^1(X, \mathbb{R})$ is an even functional satisfying the Cerami condition. Moreover, for each $k = 1, 2, \dots,$ there exist $\rho_k > r_k > 0$ such that*

(A1) $\inf_{\{u\in Z_k:\ \|u\|=r_k\}} J(u) \to +\infty$ *as* $k \to \infty$*;*

(A2) $\max_{\{u\in Y_k:\ \|u\|=\rho_k\}} J(u) \leq 0$.

Then J has a sequence of critical values which tends to $+\infty$.

Consider the following functional

$$\Phi(u) = \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx, \quad \text{for all} \quad u \in X,$$

where $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega)$ with the norm $\|u\| = \|u\|_{p_1(x)} + \|u\|_{p_2(x)}$, $\forall x \in \overline{\Omega}$. It is obvious that $(X, \|.\|)$ is also a separable and reflexive Banach space.

By using standard arguments, it can be proved that $\Phi \in C^1(X, \mathbb{R})$ (see [6]), and the $(p_1(x), p_2(x))$-Laplace operator is the derivative operator of Φ in the weak sense. Denote $L = \Phi' : X \to X^*$, then

$$\langle L(u), v\rangle = \int_\Omega |\nabla u|^{p_1(x)-2} \nabla u \nabla v \, dx + \int_\Omega |\nabla u|^{p_2(x)-2} \nabla u \nabla v \, dx,$$

for all $u,\ v \in X$, and $\langle ., .\rangle$ is the dual pair between X and its dual X^*.

Proposition 6 ([16])**.** *(1) L is a continuous, bounded homeomorphism and strictly monotone operator.*

(2) L is a mapping of type (S_+), namely: $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} L(u_n)(u_n - u) \leq 0$, imply $u_n \to u$.

Let us denote

$$p_M(x) = \max\{p_1(x), p_2(x)\}, \quad p_m(x) = \min\{p_1(x), p_2(x)\}, \quad \forall x \in \overline{\Omega}.$$

It is easy to see that $p_M(.), p_m(.) \in C_+(\overline{\Omega})$. For $q(.) \in C_+(\overline{\Omega})$ such that $q(x) < p_M(x)$ for any $x \in \overline{\Omega}$ we have $X := W_0^{1,p_1(x)}(\Omega) \cap W_0^{1,p_2(x)}(\Omega) = W_0^{1,p_M(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and the imbedding is continuous and compact.

Definition 2. A function $u \in X$ is said to be a weak solution of (1) if

$$\int_\Omega |\nabla u|^{p_1(x)-2}\nabla u \nabla v\, dx + \int_\Omega |\nabla u|^{p_2(x)-2}\nabla u \nabla v\, dx - \int_\Omega f(x,u)v\, dx = 0,$$

for all $v \in X$.

Define

$$\Phi(u) = \int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}dx, \qquad \Psi(u) = \int_\Omega F(x,u)dx.$$

The Euler-Lagrange functional associated to problem (1) is

$$J(u) = \Phi(u) - \Psi(u).$$

Under the hypothesis $(\mathbf{f_1})$, the functional J is well defined, of class C^1, and the Fréchet derivative is given by

$$\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p_1(x)-2}\nabla u \nabla v\, dx + \int_\Omega |\nabla u|^{p_2(x)-2}\nabla u \nabla v\, dx - \int_\Omega f(x,u)v\, dx$$

for all $u, v \in X$. Moreover a weak solution of problem (1) corresponds to a critical point of the functional J.

Proof of main results

First of all, we start with the following compactness result which plays the most important role.

Lemma 2. *Under assumptions* $(\mathbf{f_1})$, $(\mathbf{f_4})$ *and* $(\mathbf{g})$, *J satisfies the Cerami condition.*

Proof. For all $c \in \mathbb{R}$, we show that J satisfies (i) of Cerami condition. Let $\{u_n\} \subset X$ be bounded, $J(u_n) \to c$ and $J'(u_n) \to 0$. Without loss of generality, we assume that $u_n \rightharpoonup u$, then $J'(u_n)(u_n - u) \to 0$. Thus we have

$$\begin{aligned} J'(u_n)(u_n - u) &= \int_\Omega |\nabla u_n|^{p_1(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\ &+ \int_\Omega |\nabla u_n|^{p_2(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \\ &- \int_\Omega f(x,u_n)(u_n - u)dx \\ &\to 0. \end{aligned}$$

From $(\mathbf{f_1})$, Propositions 1 and 3, we can easily get that $\int_\Omega f(x,u_n)(u_n-u)dx \to 0$. Therefore, we have

$$\sum_{i=1}^{2}\int_\Omega |\nabla u_n|^{p_i(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx \to 0. \tag{4}$$

Furthermore, since $u_n \rightharpoonup u$ in X, we have

$$\sum_{i=1}^{2}\int_\Omega |\nabla u|^{p_i(x)-2}\nabla u(\nabla u_n-\nabla u)dx \to 0. \tag{5}$$

From (4) and (5), we deduce that

$$\sum_{i=1}^{2}\int_\Omega \left(|\nabla u_n|^{p_i(x)-2}\nabla u_n-|\nabla u|^{p_i(x)-2}\nabla u\right).(\nabla u_n-\nabla u)\,dx \to 0. \tag{6}$$

Next, we apply the following well-known inequality

$$\left(|\xi|^{r-2}\xi-|\psi|^{r-2}\psi\right).(\xi-\psi)\ge 2^{-r}|\xi-\psi|^r, \quad \xi,\psi\in\mathbb{R}^N, \tag{7}$$

valid for all $r\ge 2$. From the relations (6) and (7), we infer that

$$\sum_{i=1}^{2}\int_\Omega |\nabla u_n-\nabla u|^{p_i(x)}dx \to 0, \tag{8}$$

and, consequently, $u_n \to u$ in X.

Now, we check that J satisfies the assertion (ii) of Cerami condition. Arguing by contradiction, there exist $c\in\mathbb{R}$ and $\{u_n\}\subset X$ satisfying:

$$J(u_n)\to c, \qquad \|u_n\|\to\infty, \qquad \|J'(u_n)\|\|u_n\|\to 0. \tag{9}$$

Let

$$\overline{p}_n=\frac{\int_\Omega\left(|\nabla u_n|^{p_1(x)}+|\nabla u_n|^{p_2(x)}\right)dx}{\int_\Omega \frac{1}{p_1(x)}|\nabla u_n|^{p_1(x)}dx+\int_\Omega\frac{1}{p_2(x)}|\nabla u_n|^{p_2(x)}dx}.$$

Choosing $\|u_n\|>1$, for $n\in\mathbb{N}$, thus

$$\begin{aligned}c&=\lim_{n\to+\infty}\left(J(u_n)-\frac{1}{\overline{p}_n}J'(u_n)(u_n)\right)\\ &=\lim_{n\to+\infty}\left(\frac{1}{\overline{p}_n}\int_\Omega f(x,u_n)u_n dx-\int_\Omega F(x,u_n)dx\right).\end{aligned} \tag{10}$$

Denote $w_n = \frac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$, so $\{w_n\}$ is bounded. Up to a subsequence, for some $w \in X$, we get

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{in } X, \\ &w_n \to w \quad \text{in } L^{q(x)}(\Omega), \\ &w_n(x) \to w(x) \quad \text{a.e. in } \Omega. \end{aligned}$$

If $w \equiv 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$J(t_n u_n) = \max_{t\in[0,1]} J(tu_n).$$

For any $B > \frac{1}{2p_M^+}$, let $b_n = \left(2Bp_M^+\right)^{\frac{1}{p_M^-}} w_n$, since $b_n \to 0$ in $L^{q(x)}(\Omega)$ and $|F(x,t)| \leq C(1 + |t|^{q(x)})$, by the continuity of the Nemitskii operator, we see that $F(.,b_n) \to 0$ in $L^1(\Omega)$ as $n \to +\infty$, therefore

$$\lim_{n\to\infty} \int_\Omega F(x,b_n)dx = 0. \tag{11}$$

Then for n large enough, $\left(2Bp_M^+\right)^{\frac{1}{p_M^-}} / \|u_n\| \in (0,1)$ and

$$\begin{aligned} J(t_n u_n) &\geq J(b_n) \\ &= \int_\Omega \frac{|\nabla b_n|^{p_1(x)}}{p_1(x)}dx + \int_\Omega \frac{|\nabla b_n|^{p_2(x)}}{p_2(x)}dx - \Psi(b_n) \\ &\geq \frac{1}{p_1^+}\int_\Omega \left(2Bp_M^+\right)|\nabla w_n|^{p_1(x)}dx + \frac{1}{p_2^+}\int_\Omega \left(2Bp_M^+\right)|\nabla w_n|^{p_2(x)}dx \\ &\quad - \int_\Omega F(x,b_n)dx \\ &\geq 2B\int_\Omega |\nabla w_n|^{p_1(x)}dx + 2B\int_\Omega |\nabla w_n|^{p_2(x)}dx - \int_\Omega F(x,b_n)dx \\ &\geq 2c_1B\|w_n\|^{p_1^-} + 2c_2B\|w_n\|^{p_2^-} - \int_\Omega F(x,b_n)dx \\ &\geq 2c_1B + 2c_2B - \int_\Omega F(x,b_n)dx. \end{aligned}$$

That is,

$$J(t_n u_n) \to +\infty. \tag{12}$$

From $J(0) = 0$ and $J(u_n) \to c$, we know that $t_n \in (0,1)$ and

$$\langle J'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} J(tu_n) = 0.$$

Thus, from (12), we obtain that

$$\frac{1}{\overline{p}_{t_n}}\langle \Psi'(t_nu_n), t_nu_n\rangle - \Psi(t_nu_n) = \frac{1}{\overline{p}_{t_n}}\langle \Phi'(t_nu_n), t_nu_n\rangle - \Psi(t_nu_n) = J(t_nu_n) \to \infty, \quad (13)$$

as $n \to \infty$, where $\overline{p}_{t_n} = \frac{\langle \Phi'(t_nu_n), t_nu_n\rangle}{\Phi(t_nu_n)}$.
Let $\gamma_{t_nu_n} = \overline{p}_{t_n}$ and $\gamma_{u_n} = \overline{p}_n$, then $\gamma_{t_nu_n}, \gamma_{u_n} \in [2p_m^-, 2p_M^+]$. Hence, $G_{\gamma_{t_nu_n}}, G_{\gamma_{u_n}} \in \mathcal{F}$. Using (**g**), (13) and the fact that $\inf_n \frac{\overline{p}_{t_n}}{\overline{p}_n\delta} > 0$, we get

$$\begin{aligned}\frac{1}{\overline{p}_n}\int_\Omega f(x,u_n)u_n dx - \int_\Omega F(x,u_n)dx &= \frac{1}{\overline{p}_n}\int_\Omega G_{\gamma_{u_n}}(x,u_n)dx \\ &\geq \frac{1}{\overline{p}_n\delta}\int_\Omega G_{\gamma_{t_nu_n}}(x,t_nu_n)dx \\ &\geq \frac{\overline{p}_{t_n}}{\overline{p}_n\delta}\left(\frac{1}{\overline{p}_{t_n}}\langle \Psi'(t_nu_n), t_nu_n\rangle - \Psi(t_nu_n)\right) \to +\infty,\end{aligned}$$

which contradicts (10).

If $w \neq 0$, from (9), we write

$$\int_\Omega |\nabla u_n|^{p_1(x)}dx + \int_\Omega |\nabla u_n|^{p_2(x)}dx - \int_\Omega f(x,u_n)u_n dx = \langle J'(u_n), u_n\rangle = o(1)\|u_n\|, \quad (14)$$

that is,

$$\begin{aligned}1 - o(1) &= \int_\Omega \frac{f(x,u_n)u_n}{\int_\Omega |\nabla u_n|^{p_1(x)}dx + \int_\Omega |\nabla u_n|^{p_2(x)}dx}dx \\ &\geq \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^{p_M^+}}dx \\ &= \int_\Omega \frac{f(x,u_n)u_n}{|u_n|^{p_M^+}}|w_n|^{p_M^+}dx.\end{aligned} \quad (15)$$

Define the set $\Lambda_0 = \{x \in \Omega : w(x) = 0\}$. Then, for $x \in \Lambda\backslash\Lambda_0 = \{x \in \Omega : w(x) \neq 0\}$ we have $|u_n(x)| \to +\infty$ as $n \to +\infty$. Hence by (**f$_4$**) we deduce

$$\frac{f(x,u_n(x))u_n(x)}{|u_n(x)|^{p_M^+}}|w_n(x)|^{p_M^+} \to +\infty \qquad \text{as} \quad n \to +\infty.$$

In view of $|\Lambda\backslash\Lambda_0| > 0$, we deduce via the Fatou Lemma that

$$\int_{\Lambda\backslash\Lambda_0} \frac{f(x,u_n)u_n}{|u_n|^{p_M^+}}|w_n|^{p_M^+}dx \to +\infty \qquad \text{as} \quad n \to +\infty. \tag{16}$$

On the other hand, from $(\mathbf{f_1})$ and $(\mathbf{f_4})$, there exists $d > -\infty$ such that $\frac{f(x,t)t}{|t|^{p_M^+}} \geq d$ for $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Moreover, we have $\int_{\Lambda_0} |w_n(x)|^{p_M^+}dx \to 0$. Thus, there exists $m > -\infty$ such that

$$\int_{\Lambda_0} \frac{f(x,u_n)u_n}{|u_n|^{p_M^+}}|w_n|^{p_M^+}dx \geq d\int_{\Lambda_0} |w_n|^{p_M^+}dx \geq m > -\infty. \tag{17}$$

Combining (15), (16) and (17), there is a contradiction. This completes the proof of Lemma 2.

Proof of Theorem 1. By Lemma 2, J satisfies conditions (C) in X. To apply Proposition 4, we will show that J possesses the mountain pass geometry.

First, we claim that there exist $\mu, \nu > 0$ such that

$$J(u) \geq \mu > 0, \quad \text{for all} \quad u \in X \quad \text{with} \quad \|u\| = \nu.$$

Let $\|u\| \leq 1$. Then by Proposition 2, we have

$$\begin{aligned} J(u) &\geq \frac{c_1}{p_1^+}\|u\|^{p_1^+} + \frac{c_2}{p_2^+}\|u\|^{p_2^+} - \int_\Omega F(x,u)dx \\ &\geq \frac{c^*}{p_M^+}\|u\|^{p_M^+} - \int_\Omega F(x,u)dx, \end{aligned} \tag{18}$$

where $c^* = \min\{c_1, c_2\}$. Since $p_M^+ < q^- \leq q(x) < p_M^*(x)$ for all $x \in \Omega$, we have the continuous embedding $X \hookrightarrow L^{p_M^+}(\Omega)$ and $X \hookrightarrow L^{q^-}(\Omega)$ and also there are two positive constants, c_3 and c_4 such that

$$|u|_{p_M^+} \leq c_3\|u\| \;\; \text{and} \;\; |u|_{q^-} \leq c_4\|u\| \;\; \text{for all} \;\; u \in X. \tag{19}$$

Let $\epsilon > 0$ be small enough such that $\epsilon c_3^{p_M^+} < \frac{c^*}{2p_M^+}$. By the assumptions $(\mathbf{f_1})$ and $(\mathbf{f_2})$, we have $F(x,t) \leq \epsilon|t|^{p_M^+} + c_\epsilon|t|^{q(x)}$, for all $(x,t) \in \Omega \times \mathbb{R}$. Then, for $\|u\| \leq 1$ it follows that

$$\begin{aligned} J(u) &\geq \frac{c^*}{p_M^+}\|u\|^{p_M^+} - \epsilon\int_\Omega |u|^{p_M^+}dx - c_\epsilon\int_\Omega |u|^{q(x)}dx \\ &\geq \frac{c^*}{p_M^+}\|u\|^{p_M^+} - \epsilon c_3^{p_M^+}\|u\|^{p_M^+} - c_\epsilon c_5\|u\|^{q^-}. \end{aligned} \tag{20}$$

Therefore, there exist two positive real numbers μ and ν such that $J(u) \geq \mu > 0$, for all $u \in X$ with $\|u\| = \nu$.

Next, we affirm that there exists $e \in X\backslash\overline{B_\nu(0)}$ such that $J(e) < 0$.

Let $v_0 \in X \setminus \{0\}$, by ($\mathbf{f_4}$), we can choose a constant

$$A > \frac{\frac{1}{p_1^-}\int_\Omega |\nabla v_0|^{p_1(x)}dx + \frac{1}{p_2^-}\int_\Omega |\nabla v_0|^{p_2(x)}dx}{\int_\Omega |v_0|^{p_M^+}dx},$$

such that

$$F(x,t) \geq A|t|^{p_M^+} \qquad \text{uniformly in} \quad x \in \Omega,\ |t| > C_A,$$

where $C_A > 0$ is a constant depending on A. Let $s > 1$ large enough, we have

$$\begin{aligned}
J(sv_0) &= \int_\Omega \frac{1}{p_1(x)}|\nabla sv_0|^{p_1(x)}dx + \int_\Omega \frac{1}{p_2(x)}|\nabla sv_0|^{p_2(x)}dx - \int_\Omega F(x,sv_0)dx \\
&\leq \frac{s^{p_1^+}}{p_1^-}\int_\Omega |\nabla v_0|^{p_1(x)}dx + \frac{s^{p_2^+}}{p_2^-}\int_\Omega |\nabla v_0|^{p_2(x)}dx \\
&- \int_{|sv_0|>C_A} F(x,sv_0)dx - \int_{|sv_0|\leq C_A} F(x,sv_0)dx, \\
&\leq \frac{s^{p_1^+}}{p_1^-}\int_\Omega |\nabla v_0|^{p_1(x)}dx + \frac{s^{p_2^+}}{p_2^-}\int_\Omega |\nabla v_0|^{p_2(x)}dx \\
&- As^{p_M^+}\int_\Omega |v_0|^{p_M^+}dx - \int_{|sv_0|\leq C_A} F(x,sv_0)dx + A\int_{|sv_0|\leq C_A} |sv_0|^{p_M^+}dx \\
&\leq \frac{s^{p_1^+}}{p_1^-}\int_\Omega |\nabla v_0|^{p_1(x)}dx + \frac{s^{p_2^+}}{p_2^-}\int_\Omega |\nabla v_0|^{p_2(x)}dx \\
&- As^{p_M^+}\int_\Omega |v_0|^{p_M^+}dx + c_6,
\end{aligned}$$

which implies that

$$J(sv_0) \to -\infty \quad \text{as} \quad s \to +\infty.$$

Thus, there exist $s_0 > 1$ and $e = s_0v_0 \in X\backslash\overline{B_r(0)}$ such that $J(e) < 0$.

Thereby, the Proposition 4 guarantees that problem (1) has a nontrivial weak solution. This completes the proof.

Proof of Theorem 2. The proof is based on the Fountain Theorem. According to Lemma 2 and ($\mathbf{f_3}$), J is an even functional and satisfies condition (C). We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that:

(A1) $b_k := \inf\{J(u) :\ u \in Z_k, \|u\| = r_k\} \to +\infty$ as $k \to +\infty$;

(A2) $a_k := \max\{J(u) :\ u \in Y_k, \|u\| = \rho_k\} \leq 0$ as $k \to +\infty$.

In what follows, we will use the mean value theorem in the following form: for every $\gamma \in C_+(\overline{\Omega})$ and $u \in L^{\gamma(x)}(\Omega)$, there is $\zeta \in \Omega$ such that

$$\int_\Omega |u|^{\gamma(x)} dx = |u|_{\gamma(x)}^{\gamma(\zeta)}. \tag{21}$$

Indeed, it is easy to see that

$$1 = \int_\Omega \left(\frac{|u|}{|u|_{\gamma(x)}} \right)^{\gamma(x)} dx.$$

On the other hand, by the mean value theorem for integrals, there exists a positive constant $\kappa \in [\gamma^-, \gamma^+]$ depending on γ such that

$$\int_\Omega \left(\frac{|u|}{|u|_{\gamma(x)}} \right)^{\gamma(x)} dx = \left(\frac{1}{|u|_{\gamma(x)}} \right)^{\kappa} \int_\Omega |u|^{\gamma(x)} dx.$$

The continuity of γ ensures that there exists $\zeta \in \Omega$ such that $\gamma(\zeta) = \kappa$. Combining all together, we get (21).

(A1): For any $u \in Z_k$ such that $\|u\| = r_k$ is big enough to ensure that $\|u\|_{p_1(x)} \geq 1$ and $\|u\|_{p_2(x)} \geq 1$ (r_k will be specified below), by condition $(\mathbf{f_1})$ and (21) we have

$$\begin{aligned}
J(u) &= \int_\Omega \frac{1}{p_1(x)} |\nabla u|^{p_1(x)} dx + \int_\Omega \frac{1}{p_2(x)} |\nabla u|^{p_2(x)} dx - \int_\Omega F(x,u) dx \\
&\geq \frac{1}{p_M^+} \left(\|u\|_{p_1(x)}^{p_1^-} + \|u\|_{p_2(x)}^{p_2^-} \right) - c_7 \int_\Omega |u|^{q(x)} dx - c_8 \\
&\geq \frac{c^*}{p_M^+} \|u\|^{p_m^-} - c_7 |u|_{q(x)}^{q(\zeta)} - c_9, \text{ where } \zeta \in \Omega \\
&\geq \begin{cases} \frac{c^*}{p_M^+} \|u\|^{p_m^-} - c_7 - c_9 & \text{if } |u|_{q(x)} \leq 1 \\ \frac{c^*}{p_M^+} \|u\|^{p_m^-} - c_7 (\beta_k \|u\|)^{q^+} - c_9 & \text{if } |u|_{q(x)} > 1 \end{cases} \\
&\geq \frac{c^*}{p_M^+} \|u\|^{p_m^-} - c_7 (\beta_k \|u\|)^{q^+} - c_{10} \\
&= c^* \left(\frac{1}{p_M^+} \|u\|^{p_m^-} - c_{11} \beta_k^{q^+} \|u\|^{q^+} \right) - c_{10}.
\end{aligned}$$

We fix r_k as follows

$$r_k = \left(c_{11} q^+ \beta_k^{q^+} \right)^{\frac{1}{p_m^- - q^+}},$$

then

$$J(u) \geq c^* \left(\frac{1}{p_M^+} (c_{10} q^+ \beta_k^{q^+})^{\frac{p_m^-}{p_m^- - q^+}} - c_{11} \beta_k^{q^+} (c_{11} q^+ \beta_k^{q^+})^{\frac{q^+}{p_m^- - q^+}} \right) - c_{10}$$
$$\geq c^* r_k^{p_m^-} \left(\frac{1}{p_M^+} - \frac{1}{q^+} \right) - c_{10}.$$

From Lemma 1, we know that $\beta_k \to 0$. Then since $1 < p_m^- \leq p_M^+ < q^+$, it follows $r_k \to +\infty$, as $k \to +\infty$. Consequently, $J(u) \to +\infty$ as $\|u\| \to +\infty$ with $u \in Z_k$. The assertion (A1) is valid.

(A2): Since $dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, there exists $d_k > 0$, for all $u \in Y_k$ with $\|u\|$ big enough to ensure that $\|u\|_{p_1(x)} \geq 1$ and $\|u\|_{p_2(x)} \geq 1$, we have

$$\begin{aligned} \Phi(u) &\leq \frac{1}{p_1^-} \int_\Omega |\nabla u|^{p_1(x)} dx + \frac{1}{p_2^-} \int_\Omega |\nabla u|^{p_2(x)} dx \\ &\leq \frac{1}{p_m^-} \|u\|_{p_1(x)}^{p_1^+} + \frac{1}{p_m^-} \|u\|_{p_2(x)}^{p_2^+} \\ &\leq \frac{c_1}{p_m^-} \|u\|^{p_1^+} + \frac{c_2}{p_m^-} \|u\|^{p_2^+} \\ &\leq \frac{\max(c_1, c_2)}{p_m^-} \|u\|^{p_M^+} \leq d_k |u|_{p_M^+}^{p_M^+}. \end{aligned} \tag{22}$$

Now, from $(\mathbf{f_4})$, there exists $R_k > 0$ such that for all $|s| \geq R_k$, we have $F(x,s) \geq 2d_k |s|^{p_M^+}$. From $(\mathbf{f_1})$, there exists a positive constant M_k such that

$$|F(x,s)| \leq M_k \quad \text{for all } (x,s) \in \Omega \times [-R_k, R_k].$$

Then for all $(x,s) \in \Omega \times \mathbb{R}$ we have

$$F(x,s) \geq 2d_k |s|^{p_M^+} - M_k. \tag{23}$$

Combining (22) and (23), for $u \in Y_k$ such that $\|u\| = \rho_k > r_k$, we infer that

$$\begin{aligned} J(u) &= \Phi(u) - \int_\Omega F(x,u) dx \\ &\leq -d_k |u|_{p_M^+}^{p_M^+} + M_k |\Omega| \\ &\leq -\frac{\max(c_1, c_2)}{p_m^-} \|u\|^{p_M^+} + M_k |\Omega|. \end{aligned}$$

Therefore, for ρ_k large enough ($\rho_k > r_k$), we get from the above that

$$a_k := \max\{J(u) : \ u \in Y_k, \|u\| = \rho_k\} \leq 0.$$

The assertion (A2) holds. Finally we apply the Fountain Theorem to achieve the proof of Theorem 2.

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