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Historical synopsis of the Taylor remainder

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Abstract. In this paper we give an historical synopsis of various Taylor remainders and their different proofs (without being exhaustive). We overview the formulas and the proofs given by such names as Bernoulli, Taylor, MacLaurin, Lagrange, Lacroix, Cauchy, Schlömilch, Roche, Cox, Turquan, Bourget, Koenig, Darboux, Amigues, Teixeira, Peano, Blumenthal, Wolfe and Gonçalves. We end the paper with a new Taylor remainder which generalizes the well-known Lagrange remainder.

Keywords: Taylor formula, Taylor remainder, history of Taylor remainder, Lagrange remainder

MSC 2000 classification: primary 26A06, secondary 26A99

1 Introduction

The Taylor formula is a quite old topic, more than two centuries old, but mathematicians continue even nowadays to publish results in this direction, for a handful examples of this contemporary research, see e.g. [1, 14, 20].

By a *Taylor series* of a function f centered in the point x_0 we mean the following series expansion

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{1}$$

By *Taylor formula* we designate the representation of a function f as the sum of a polynomial and an extra term, namely

$$f(x) = P_n(x) + r_n(x) \tag{2}$$

where P_n is the *Taylor polynomial* of degree n

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

and $r_n(x)$ is the *Taylor remainder* given simply by $f(x) - P_n(x) =: r_n(x)$.

Throughout the paper, we will not use an unified presentation when dealing with Taylor formula, following the historical presentation in each particular article. It should be pointed out that some of the proofs given may not pass the modern standards of rigor, but following the historical presentation will highlight the ideas instead of the formalism.

Besides consulting the original references which we cited, we also relied in some old overviews of Taylor formula, some appearing in book format [15] and others in article form [17].

2 Historical Overview

In the 17th century series expansion of particular functions appeared by the hands of European mathematicians, viz. in 1668 J. Gregory published the series expansion of $\arctan x$ in his *Exercitationes Geometricae*, see also [30] for some historical background regarding the arctangent series expansion. In the same year N. Mercator gave the series expansion of $\ln(1+x)$ in his *Logarithmotechnia* and I. Newton obtained the series expansion for $(1+x)^\alpha$, $\sin x$, $\cos x$ and $\exp x$, which appeared in the correspondence with Leibniz in 1676, see the book [32] for more details.

What is less well-known is that the Indian Kerala school of Mādhava already knew the series expansion of $\arctan x$, $\sin x$ and $\cos x$ before 1540, for more details see [8, 9, 32].

We now survey some techniques to obtain Taylor series expansion as well as Taylor reminder for the Taylor formula. Some of the proofs are highly artificial, and in this sense we would like to give the following quotation [18]:

> It is also the one result that I was dreading lecturing, at least with the Lagrange form of the remainder, because in the past I have always

> found that the proof is one that I have not been able to understand properly. I don't mean by that that I couldn't follow the arguments I read. What I mean is that I couldn't reproduce the proof without committing a couple of things to memory, which I would then forget again once I had presented them. Briefly, an argument that appears in a lot of textbooks uses a result called the Cauchy mean value theorem, and applies it to a cleverly chosen function.
>
> TIMOTHY GOWERS, recipient of the Fields medal

2.1 Johan Bernoulli

It seems that the first mathematician to give a *general* formula for series expansion of a function was J. Bernoulli (see Remark 1 for some dispute of this statement) which published the formula

$$\int_0^x \varphi(x)\,\mathrm{d}x = x\varphi(x) - \frac{1}{2!}x^2\varphi'(x) + \frac{1}{3!}x^3\varphi''(x) - \cdots \tag{3}$$

in *Acta eruditorum* in 1694 (see also [4]). The formula (3) follows from integrating both sides of the identity

$$\varphi(x)\,\mathrm{d}x = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}\left[kx^{k-1}\varphi^{(k-1)}(x) + x^k\varphi^{(k)}(x)\right]\mathrm{d}x + \frac{(-1)^n x^n}{n!}\varphi^{(n)}(x)\,\mathrm{d}x$$

and imposing that the term I_n vanishes when $n \to \infty$, where

$$I_n := \frac{(-1)^n}{n!}\int_0^x x^n\varphi^{(n)}(x)\,\mathrm{d}x.$$

This argument appeared in the textbook of Arbogast in 1800, see [3, p.334]. It should also be noted that iterated integration by parts applied to $\int_0^x \varphi(x)\,\mathrm{d}x$ also yields (3).

Remark 1. In [37] it is stated that J. Gregory obtained Taylor's theorem in 1671, where the finding was communicated to J. Collins, secretary of the Royal Society. The evidence of such statement is preserved in the library of the University of St. Andrews which we cannot verify and see the actual formulation and idea of proof.

2.2 Brooke Taylor

In 1715 it appears the book *Methodus Incrementorum* in which we can find the formula

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots \tag{4}$$

without any conditions for the validity of such representation. To obtain such formula, Taylor used the theory of finite differences, which can be summarized in the following way: by Δy, $\Delta^2 y$, etc. we denote the finite difference of $y = f(x)$ given by $\Delta f(x;\Delta x) := f(x+\Delta x) - f(x)$. Using this notion we can write

$$f(x+\Delta x) = f(x) + \Delta f(x;\Delta x) = y + \Delta y.$$

Via induction we obtain

$$f(x+k\Delta x) = \sum_{j=0}^{k} \binom{k}{j} \Delta^j y, \tag{5}$$

where $\Delta^n y := \Delta\left(\Delta^{n-1} y\right)$. Taking $k\Delta x = h$, from (5) we get

$$f(x+h) = y + h\frac{\Delta y}{\Delta x} + \frac{h(h-\Delta x)}{1\cdot 2}\frac{\Delta^2 y}{\Delta x^2} + \cdots . \tag{6}$$

When $\Delta x \to 0$ (or equivalently, when $k \to \infty$), due to the relation

$$\lim_{\Delta x\to 0} \frac{\Delta^k y}{\Delta x^k} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k},$$

we formally obtain (4).

Remark 2. The formula (4) follows in fact from the formula (3) after some easy manipulations (take $\varphi(x) = f'(h-x)$ in (3) and change x by h and h by $x+h$ and we obtain (4)). In [5, p.584] Bernoulli claimed priority in the discovery of the Taylor formula (see also [15]). Hence, maybe it should be historically correct to call the Taylor's formula for *Bernoulli-Taylor's formula.*

2.3 Colin Maclaurin

In the *Treatise of Fluxions* of 1742 of Maclaurin it appeared a new proof of Taylor formula, see [24, Nr. 751]. Admitting that the function f is infinitely differentiable and can be expanded into a series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{7}$$

the coefficients a_n can be obtained differentiating the equality (7), thus obtaining $f^{(n)}(0) = n!a_n$. This proof fails to justify the fact that we can differentiate termwise and also assumes *a priori* that we can expand the function in a series.

Remark 3. It seems that similar reasoning was already used 12 years before by James Stirling in his *Methodus Differentialis*, see the original Latin reference in [33, p.102] or an annotated English translation in [35, p.117].

2.4 Joseph-Louis Lagrange

2.4.1 Taylor Series

Another method to obtain the Taylor series was devised by Lagrange and appeared in a 1772 article in *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, see [22]. Supposing that the function u can be expanded in a series $u(x) = \sum_{k=0}^{\infty} a_k x^k$, then, by replacing x with $x+h$ and *rearranging* the terms, we obtain

$$u(x+h) = u(x) + Ah + Bh^2 + Ch^3 + \cdots , \tag{8}$$

where $A, B, C, \cdots$ are functions of x. Changing first x by $x + \omega$ in (8) and changing h by $h+\omega$ again in (8) we obtain two different expansion of $u(x+h+\omega)$. Using Newton's binomial formula, with some tedious computations, we obtain exactly the coefficients of Taylor series (1). The proof also has some problems, namely it assumes the existence of the series expansion and the permissibility of rearranging the terms of a series.

Remark 4. The aforementioned technique is based, in a sense, on the proof of the fact that if

$$P(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n \tag{9}$$

is a polynomial of degree n, the following formula is valid

$$P(x+h) = P(x) + hP'(x) + \frac{h^2}{2!}P''(x) + \ldots \frac{h^n}{n!}P^{(n)}(x). \tag{10}$$

Changing x to $x+h$ in (9) we obtain, after applying Newton's binomial theorem and rearranging the terms,

$$\begin{aligned} P(x+h) =& a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n + \\ & + h\left[n a_0 x^{n-1} + (n-1)a_1 x^{n-2} + \cdots + a_{n-1}\right] \\ & + \frac{h^2}{2!}\left[n(n-1)a_0 x^{n-2} + (n-1)(n-2)a_1 x^{n-3} + \ldots a_{n-2}\right] \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + \frac{h^k}{k!}\left[[n]_k a_0 x^{n-k} + [n-1]_k a_1 x^{n-k+1} + \ldots a_k\right] \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + a_0 h^n, \end{aligned}$$

where $[n]_k := n(n-1)\dots(n-k+1)$. From this equality, (10) is immediate.

2.4.2 Taylor remainder

It seems that Lagrange was the first to study the conditions to expand a function in Taylor series.

In 1797 in his *Théorie des Fonctions Analytique* the following formula was given

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + R_n, \tag{11}$$

where

$$R_n = \frac{h^n}{n!} f^{(n)}(x+\theta h) \tag{12}$$

with θ between 0 and 1. If $R_n \to 0$ when $n \to \infty$, then the Taylor series (1) is valid, else it does not hold.

The proof given by Lagrange of relation (12) was by two distinct methods.

Lagrange Method 1. This method uses an auxiliary parameter z and appeared in *Théorie des Fonctions Analytique*, see [23, Nr. 35]. Define $P(x,z)$ by the equation

$$f(x+h) = \sum_{k=0}^{n-1} \frac{h^k z^k}{k!} f^{(k)}(x+h-hz) + h^n P(x,z) \tag{13}$$

and note that $P(x,0) = 0$. Differentiating (13) with respect to the variable z we obtain

$$h^n \frac{\partial P}{\partial z} = \frac{h^n z^{n-1}}{(n-1)!} f^{(n)}(x+h-hz),$$

which entails

$$P(x,z) = \frac{1}{(n-1)!} \int_0^z z^{n-1} f^{(n)}(x+h-hz)\,\mathrm{d}z.$$

Taking $z=1$ we obtain Taylor formula (11) with remainder R_n in integral form given by

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 z^{n-1} f^{(n)}(x+h-hz)\,\mathrm{d}z. \tag{14}$$

By the first-mean value theorem for integrals and assuming the continuity of the function $f^{(n)}(x+h-hz)$ in the interval $0 \le z \le 1$ we obtain exactly (12) from (14). It should be noted that Lagrange proved a particular case of the first-mean value theorem for integrals to obtain the result, since it seems that the first-mean value theorem for integrals was proved in 1821 by Cauchy, as claimed in [17].

LAGRANGE METHOD 2. In his 1806 *Leçons sur le Calcul des Fonctions*, Lagrange obtained the same result using the criterion for monotonicity of a function via inspection of the sign of the derivative. Namely, let us define the following functions

$$
\begin{aligned}
\varphi_n(\xi) =& f^{(n)}(x+\xi) - L\\
\varphi_{n-1}(\xi) =& f^{(n-1)}(x+\xi) - f^{(n-1)}(x) - L\xi\\
\varphi_{n-2}(\xi) =& f^{(n-2)}(x+\xi) - f^{(n-2)}(x) - f^{(n-1)}(x)\xi - L\frac{\xi^2}{2!}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\varphi_0(\xi) =& f(x+\xi) - f(x) - f'(x)\xi - \cdots - f^{(n-1)}(x)\frac{\xi^{n-1}}{(n-1)!} - L\frac{\xi^n}{n!}.
\end{aligned}
$$

It is immediate that φ_n is the derivative of φ_{n-1} and so on. Moreover $\varphi_k(0) = 0$ for all $k = 0, \cdots, n-1$ and $\varphi_n(0) = f^{(n)}(x) - L$. For definiteness, let $h > 0$ and take $S = \max_{\xi\in[0,h]} f^{(n)}(x+\xi)$ and $I = \min_{\xi\in[0,h]} f^{(n)}(x+\xi)$. If $\varphi_n(\xi)$ is non-negative in certain interval $[0, h]$ then $\varphi_0(\xi)$ is also non-negative, and similar reasoning for non-positive. Using the derivative criterion for monotonicity of functions we obtain

$$\frac{Ih^n}{n!} < f(x+h) - f(x) - f'(x)h - \cdots - f^{(n-1)}(x)\frac{h^{n-1}}{(n-1)!} < \frac{Sh^n}{n!},$$

which implies, from the intermediate value property of continuous functions, that there exists a θ such that

$$f(x+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(x) + \frac{h^n f^{(n)}(x+\theta h)}{n!}.$$

Similar reasoning applies when $h < 0$. It should be noted that the intermediate value theorem was proved later by Bolzano and Cauchy, but it was considered by Lagrange as evident.

2.5 Sylvestre François Lacroix

In the 1819 book *Traité du Calcul Differéntiel et du Calcul Intégral* [21], Lacroix gave a proof of the Lagrange integral remainder (14) using an idea of

D'Alembert. Nowadays a similar idea, using n-times iteration of the fundamental theorem of calculus and integration by parts, is widely used and is one of the easiest ways to obtain a Taylor remainder, see e.g. [38]. We now give the idea of proof in [21], which is not so straightforward and uses antiderivatives.

Let $U(x) = u(x+h)$ and $u(x)$ be functions. We can start to write

$$U = u + P, \tag{15}$$

where P depends on x and h and u only depends on x. Since $\frac{\mathrm{d}U}{\mathrm{d}h} = \frac{\mathrm{d}P}{\mathrm{d}h}$ we can write (15) as $U = u + \int \frac{\mathrm{d}U}{\mathrm{d}h}\,\mathrm{d}h$. Now we define $\frac{\mathrm{d}U}{\mathrm{d}h} := \frac{\mathrm{d}u}{\mathrm{d}x} + Q$ from which $Q = \int \frac{\mathrm{d}^2U}{\mathrm{d}h^2}\,\mathrm{d}h^2$ which entails that $\int \frac{\mathrm{d}U}{\mathrm{d}h}\,\mathrm{d}h = \frac{\mathrm{d}u}{\mathrm{d}x}h + \iint \frac{\mathrm{d}^2U}{\mathrm{d}h^2}\,\mathrm{d}h^2$ and this gives the formula $U = u + \frac{\mathrm{d}u}{\mathrm{d}x}h + \iint \frac{\mathrm{d}^2U}{\mathrm{d}h^2}\,\mathrm{d}h^2$. Letting $\frac{\mathrm{d}^2U}{\mathrm{d}h^2} = \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + R$ we obtain that $\frac{\mathrm{d}^3U}{\mathrm{d}h^3} = \frac{\mathrm{d}R}{\mathrm{d}h}$ from which we get $R = \int \frac{\mathrm{d}^3U}{\mathrm{d}h^3}\,\mathrm{d}h$. We then obtain

$$U = u + \frac{\mathrm{d}u}{\mathrm{d}x}h + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\frac{h^2}{2!} + \iiint \frac{\mathrm{d}^3U}{\mathrm{d}h^3}\,\mathrm{d}h^3.$$

The process can be now iterated and finally we obtain

$$U = u + \frac{\mathrm{d}u}{\mathrm{d}x}h + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\frac{h^2}{2!} + \cdots + \frac{\mathrm{d}^{n-1}u}{\mathrm{d}x^{n-1}}\frac{h^{n-1}}{(n-1)!} + \underbrace{\int\cdots\int}_{n\text{-times}} \frac{\mathrm{d}^nU}{\mathrm{d}h^n}\,\mathrm{d}h^n.$$

Defining $H = \frac{\mathrm{d}^nU}{\mathrm{d}h^n}$ and after some calculations based on the Cauchy formula for repeated integration we obtain the desired result, see details in [21, p.396].

2.6 Augustin Louis Cauchy

In 1826 the so-called *Taylor remainder in Cauchy form*

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(x+\theta h) \tag{16}$$

for the Taylor formula

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + R_n$$

appeared in *Exercices de Mathématiques*, see [10, p.41]. Cauchy applied Lagrange's finite-increment theorem $\varphi(x+h) = \varphi(x) + h\varphi'(x+\theta h)$ to the function

$$\varphi(z) = f(x+h) - \sum_{k=0}^{n-1} \frac{(x+h-z)^k}{k!}f^{(k)}(z)$$

entailing

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + R_n$$

with R_n given by (16), since

$$\varphi'(z) = -\frac{(x+h-z)^{n-1}}{(n-1)!} f^{(n)}(z).$$

due to "telescopic" cancellations.

2.7 Oscar Schlömilch

The so-called *Schlömilch remainder of Taylor formula* appeared in the textbook of Schlömilch *Handbuch der Differentialrechnung* published in 1847-1848. This remainder in its general form is given by

$$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} \frac{\varphi(x+h) - \varphi(x)}{\varphi'(x+\theta h)} f^{(n)}(x+\theta h), \tag{17}$$

for the Taylor formula

$$f(x+h) = f(x) + hf'(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + R_n.$$

The remainder (17) is obtained applying Cauchy's finite increment theorem

$$\frac{\psi(x+h) - \psi(x)}{\varphi(x+h) - \varphi(x)} = \frac{\psi'(x+\theta h)}{\varphi'(x+\theta h)}, \quad 0 < \theta < 1, \tag{18}$$

to the function

$$\psi(z) = f(x+h) - f(z) - (x+h-z)f'(z) - \cdots - \frac{(x+h-z)^{n-1}}{(n-1)!} f^{(n-1)}(z).$$

Applying the particular function $\varphi(z) = (x+h-z)^p$ to (17) we get the remainder

$$R_n = \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(x+\theta h), \tag{19}$$

which is denoted by *Schlömilch-Roche remainder* (cf. Remark 5). The proof relies on the Cauchy finite increment theorem, which in its original formulation of Cauchy (and used by Schlömilch) required that φ' and ψ' were continuous functions. Later on, Ossian Bonnet improved the result and the assumption on the continuity of the derivatives were dropped, this improvement already appeared in 1868 in Joseph Alfred Serret's *Cours de Calcul Différentiel et Intégral.*

Remark 5. In a private letter to the editor Joseph Liouville of the *J. Math. Pures Appl.* Schlömilch called the attention to the fact that the formula (19), which was published in that journal by E. Roche, was a particular case of the general formula obtained some years before. The proof of Roche was different in spirit since it was based on integral calculus, as can be seen in the next section.

2.8 Édouard Albert Roche

In 1858 (see [26]) the *Schlömilch-Roche remainder* (19) was obtained using the integral remainder (14) and the first-mean value theorem for integrals. Namely

$$\begin{aligned} R_n &= \frac{h^n}{(n-1)!}\int_0^1 z^{p-1}z^{n-p}f^{(n)}(x+h-hz)\,\mathrm{d}z \\ &= \frac{\Theta^{n-p}h^n}{(n-1)!p}f^{(n)}(x+h-\Theta h), \end{aligned}$$

which is nothing else than (19) taking $\theta = 1-\Theta$.

In 1860 Roche [27] gave a new proof of (19) based on differential calculus and ideas already present in the *Cours d'Analyse* of Sturm. Taking

$$R_n = f(z) - f(x) - (z-x)f'(x) - \cdots - \frac{(z-x)^n}{n!}f^{(n)}(x)$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(R_n - \frac{(z-x)^{p+1}C}{n!(p+1)}\right) = \underbrace{\frac{(z-x)^p}{n!}\big(C-\varphi(x)\big)}_{\psi(x)},$$

where C is a constant and $\varphi(x) = (z-x)^{n-p}f^{(n+1)}(x)$. Taking, for definiteness, $x < z$, $M = \sup_{\xi\in[x,z]}\varphi(\xi)$ and $m = \inf_{\xi\in[x,z]}\varphi(\xi)$, we obtain

$$\frac{(z-x)^{p+1}m}{n!(p+1)} < R_n < \frac{(z-x)^{p+1}M}{n!(p+1)}$$

since the derivative will be positive (or negative) and $\left(R_n - \frac{(z-x)^{p+1}C}{n!(p+1)}\right)\Big|_{x=z} = 0$. When $\varphi(x)$ is a continuous function, then there exists N such that $R_n = \frac{(z-x)^{p+1}N}{n!(p+1)}$. Similar considerations can be made for $z < x$.

In 1864 Roche [28] also obtained a generalization of Taylor formula, which is a particular case of the Gomes Teixeira formula (30).

2.9 Homersham Cox

In 1851 appeared the proof of Cox [11] which, as stated in [37], is the base of the majority of modern textbook proofs. The advantage of this proof is that it relies solely on Rolle's theorem which has a nice geometric and cinematic interpretation. Define

$$F(z) := -f(x+h) + \sum_{k=0}^{n-1} \frac{(x+h-z)^k}{k!} f^{(k)}(z) + \frac{(x+h-z)^p}{h^p} \left[f(x+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(x) \right]. \quad (20)$$

Noting that $F(x) = F(x+h)$, by Rolle's theorem we have $F'(x+\theta h) = 0$ with $0 < \theta < 1$, where $F'(z)$ is given by

$$F'(z) = \frac{(x+h-z)^{n-1}}{(n-1)!} f^{(n)}(z) - \frac{p(x+h-z)^{p-1}}{h^p} \left[f(x+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(x) \right]. \quad (21)$$

Choosing $z = x + \theta h$ in (21), when $p = n$ we recover the original proof of Cox, obtaining the Lagrange remainder (11) and when p is arbitrary we recover the Schlömilch-Roche remainder (19).

2.10 Turquan

In 1863 the Lagrange remainder (12) was obtained in [34] using iteration and Lagrange's finite-increment theorem

$$g(x+h) = g(x) + hg'(x+\theta h). \quad (22)$$

Let us take

$$f(x+h) = f(x) + \frac{h}{1} f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + h^n R$$

from which we get that there exists $0 < \theta_1 < 1$ such that

$$f'(x+\theta_1 h) = f'(x) + \frac{h^1}{2!} f''(x) + \cdots + \frac{h^{n-2}}{(n-1)!} f^{(n-1)}(x) + h^{n-1} R. \quad (23)$$

Applying the same reasoning we obtain

$$f''(x+\theta_1\theta_2 h) = \frac{f''(x)}{2\theta_1} + \frac{h}{3!\theta_1} f'''(x) + \ldots + \frac{h^{n-3}}{(n-1)!\theta_1} f^{(n-1)}(x) + \frac{h^{n-2}}{\theta_1} R, \quad (24)$$

and taking $h = 0$ in (24) it follows that $\theta_1 2! = 1$, from which

$$f''(x+\theta h) = f''(x) + \frac{h}{3} f'''(x) + \ldots + \frac{h^{n-3}}{1 \cdot 3 \cdots (n-1)} f^{(n-1)}(x) + 1 \cdot 2 \cdot h^{n-2} R.$$

Iterating the previous process, redefining in each line the value of θ, we obtain

$$\begin{aligned} f'''(x+\theta h) &= f'''(x) + \ldots + 3! R \cdot h^{n-3} \\ f^{\mathrm{IV}}(x+\theta h) &= f^{\mathrm{IV}}(x) + \ldots + 4! R \cdot h^{n-4} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f^{(n)}(x+\theta h) &= n! R, \end{aligned}$$

from which we get (12).

2.11 Justin Bourget

In 1870 another formula for the Taylor remainder was obtained in [7]. Using Cauchy's finite increment theorem (18) we get

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi'(x+\theta(x_0-x))}{\psi'(x+\theta(x_0-x))}, \quad 0 < \theta < 1, \quad (25)$$

whenever $\varphi(x_0) = \psi(x_0) = 0$. Let us take

$$\varphi(x) = f(x_0) - \sum_{k=0}^{n-1} \frac{(x_0-x)^k}{k!} f^{(k)}(x)$$

and $\psi(x) = \omega(x_0 - x)$ where ω is a differentiable function satisfying the condition $\omega(0) = 0$. Direct computations show that

$$\varphi'(x) = -\frac{(x_0-x)^{n-1}}{(n-1)!} f^{(n)}(x)$$

and $\psi'(x) = -\omega'(x_0 - x)$. Taking all previous remarks into account we arrive at

$$\begin{aligned} f(x_0) &= \sum_{k=0}^{n-1} \frac{(x_0-x)^k}{k!} f^{(k)}(x) \\ &\quad + \frac{\omega(x_0-x)}{\omega'\big((x_0-x)(1-\theta)\big)} \frac{(1-\theta)^{n-1}(x_0-x)^{n-1}}{(n-1)!} f^{(n)}((x_0-x)(1-\theta)). \quad (26) \end{aligned}$$

Taking $x_0 = x + h$ in (26) we obtain

$$f(x+h) = \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(x) + R_n,$$

where

$$R_n = \frac{\omega(h)}{\omega'(h(1-\theta))} \frac{(1-\theta)^{n-1} h^{n-1}}{(n-1)!} f^{(n)}(h(1-\theta)). \tag{27}$$

Specifying $\omega(x) = x^p$ we obtain the Schlömilch-Roche remainder (19), from which follows Lagrange and Cauchy remainders.

2.12 Jules Koenig

In 1874 Koenig [19] developed a new way to obtain the Taylor series. He starts with the fact that the series

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$$

is convergent. Multiplying term by term the previous series for some bounded numbers will not change the nature of the series, taking the quantities $\varphi_0(h)$, $\varphi_1(h)$, $\varphi_2(h), \dots$, we obtain

$$\varphi_0(h) + \varphi_1(h)\frac{z}{1} + \varphi_2(h)\frac{z^2}{2!} + \dots,$$

which can be considered a function of two variables $\Phi(x, z)$. Studying under what conditions this two variable function depends in fact of only one variable $z + h$ leads to the Taylor function, for details see [19]. This is a nice idea but it gives no expression for the remainder.

2.13 Jean Gaston Darboux

In 1876 (see [12]), Darboux obtained the following formula

$$\begin{aligned} \varphi^{(n)}(0) \left[f(x+h) - f(x)\right] &= \sum_{k=1}^{n} (-1)^{k+1} h^k \left[\varphi^{(n-k)}(1) f^{(k)}(x+h) - \varphi^{(n-k)}(0) f^{(k)}(x)\right] \\ &\quad + (-1)^n h^{n+1} \int_0^1 \varphi(t) f^{(n+1)}(x+ht)\,\mathrm{d}t, \end{aligned} \tag{28}$$

where $\varphi(t)$ is a polynomial of degree n. The aforementioned formula was obtained applying the Newton-Leibniz formula

$$\int_0^1 \Phi'(t)\,\mathrm{d}t = \Phi(1) - \Phi(0)$$

to the unwieldy function

$$\Phi(t) = \sum_{k=0}^{n} (-1)^k \varphi^{(n-k)}(t) h^k f^{(k)}(x+ht).$$

From formula (28) we can obtain, choosing $\varphi(t) = t^n(t-1)^n$ and changing n to $2n$, the following expression

$$\begin{aligned} f(x+h) - f(x) & \\ &= \sum_{k=1}^{2n} \frac{(-1)^{k+1}\binom{2n}{k} h^k}{(2n-k+1)\cdots 2n} \left[f^{(k)}(x+h) - (-1)^k f^{(k)}(x) \right] \\ &\qquad + \frac{(-1)^n h^{2n+1}}{(2n)!} \int_0^1 t^n (1-t)^n f^{(2n+1)}(x+ht)\,\mathrm{d}t. \end{aligned}$$

2.14 E. Amigues

In 1880, using a different technique, the remainder (27) was obtained in [2], where some strong assumptions are imposed *a priori*, namely it is supposed that the function f can be expanded as a Taylor series as $f(x+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x)$. To obtain a remainder for the n-term of the Taylor polynomial, taking A as a constant which will be found latter and some prescribed function φ satisfying the condition $\varphi(0) = 0$, we write $f(x+h)$ as

$$f(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} f^{(k)}(x) + A\varphi(h) + \underbrace{\left[-A\varphi(h) + \sum_{k=n+1}^{\infty} \frac{h^k}{k!} f^{(k)}(x) \right]}_{:=F(x)}$$

due to the assumption of series expansion of f. We now choose A in such a way that $F(x) = 0$, which gives the remainder $R_n = A\varphi(h)$. To obtain A we proceed in the following way: by replacing x and h by z and $x+h-z$, respectively, we get

$$F(z) = -A\varphi(x+h-z) + \sum_{k=n+1}^{\infty} \frac{(x+h-z)^k}{k!} f^{(k)}(z).$$

Since $F(x) = 0$, due to the definition of A, and the fact that $F(x+h) = 0$ (since we supposed that $\varphi(0) = 0$), by Rolle's theorem we obtain that there exists $0 < \theta < 1$ for which $F'(x + \theta h) = 0$. The derivative of F is simply

$$F'(z) = A\varphi'(x + h - z) - \frac{(x + h - z)^n}{n!} f^{(n+1)}(z)$$

due to "telescopic" cancellations. From the above considerations we obtain the Taylor remainder (27)

$$R_n = A\varphi(h) = \frac{\varphi(h)}{\varphi'(h(1-\theta))} \frac{h^n(1-\theta)^n}{n!} f^{(n+1)}(x + \theta h), \tag{29}$$

where for the function φ it is required only that $\varphi(0) = 0$ and $\varphi'(x)$ is continuous between 0 and h.

2.15 Francisco Gomes Teixeira

In 1886 a very general formula was given by Gomes Teixeira in [16]. The formula reads as follows:

$$\begin{aligned}&\frac{f(x+h) - f(x) - hf'(x) - \cdots - \frac{h^\ell}{\ell!} f^{(\ell)}(x)}{F(x+h) - F(x) - hF'(x) - \cdots - \frac{h^k}{k!} F^{(k)}(x)} \\ &\qquad = \frac{\frac{h^{\ell+1}}{(\ell+1)!} f^{(\ell+1)}(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + R_n(f,x)}{\frac{h^{k+1}}{(k+1)!} F^{(k+1)}(x) + \cdots + \frac{h^{m-1}}{(m-1)!} F^{(m-1)}(x) + R_m(F,x)},\end{aligned} \tag{30}$$

where

$$\begin{aligned}R_n(f,x) &= \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(x+\theta h),\\ R_m(F,x) &= \frac{h^m(1-\theta)^{m-1}}{(m-1)!} F^{(m)}(x+\theta h),\end{aligned}$$

with θ between 0 and 1. The proof is a matter of tedious computations and applying the Lagrange finite increment formula (22) to the cumbersome function

$$\begin{aligned}g(z) = \sum_{s=0}^{\ell} \frac{h^k}{k!} f^{(k)}(x) - \sum_{s=0}^{n-1} \frac{(x+h-z)^k}{k!} f^{(k)}(z)&\\ - \left[\sum_{s=0}^{k} \frac{h^s}{s!} F^{(k)}(x) - \sum_{s=0}^{m-1} \frac{(x+h-z)^k}{k!} F^{(k)}(z)\right] \times A&\end{aligned}$$

where A is given by

$$A = \frac{f(x+h) - f(x) - hf'(x) - \cdots - \frac{h^\ell}{\ell!} f^{(\ell)}(x)}{F(x+h) - F(x) - hF'(x) - \cdots - \frac{h^k}{k!} F^{(k)}(x)}.$$

From the Gomes Teixeira formula (30) follows the Schlömilch-Roche remainder (19), the Cauchy finite-increment formula (18) and even the Peano remainder (31). For example, the formula (19) follows from specializing (30) as

$$\frac{f(h) - f(0) - hf'(0) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)}{F(h) - F(0) - hF'(0) - \cdots - \frac{h^{p-1}}{(p-1)!} F^{(n-1)}(0)} = \frac{\frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h)}{\frac{h^p(1-\theta)^{p-1}}{(p-1)!} F^{(p)}(\theta h)}$$

and taking $F(x) = x^p$.

2.16 Giuseppe Peano

In 1889 the *Peano form of Taylor remainder* was given in [25]. It gives qualitative information regarding $r_n(x,h)$, namely

$$r_n(x,h) = o\left(h^n\right) \quad \text{when} \quad h \to 0. \tag{31}$$

The proof of (31) given by Peano is based on the fact that, writing

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \cdots + \frac{h^n}{n!} f^{(n)}(x_0) + \frac{h^n}{n!} R_n$$

we obtain

$$R_n = \frac{f(x_0+h) - f(x_0) - hf'(x_0) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x_0) - \frac{h^n}{n!} f^{(n)}(x_0)}{\frac{h^n}{n!}}. \tag{32}$$

From (32) we see that R_n is the quotient between two functions that are null, as well as their derivatives up to order $(n-2)$, when $h = 0$. Iterating the Cauchy finite increment formula (18), we obtain

$$R_n = \frac{f^{(n-1)}(x_0 + h_1) - f^{(n-1)}(x_0)}{h_1} - f^{(n)}(x_0) \tag{33}$$

where $h_1 = \theta h$ and $0 < \theta < 1$. From the continuity of $f^{(n-1)}(x)$ we get (31) from (33).

Remark 6. Note that a similar formula as (33) was obtained in [14] using the L'Hospital rule repeatedly.

2.17 Leonard Mascot Blumenthal

In 1926 (see [6]) it appeared the following Taylor remainder:

$$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} \frac{\begin{vmatrix} \psi(a) & \varphi(a) \\ \psi(b) & \varphi(b) \end{vmatrix}}{\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix}} F^{(n)}(a+\theta h) \tag{34}$$

for the Taylor formula

$$F(b) = F(a) + (b-a)F'(a) + \frac{(b-a)^2}{2!}F''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}F^{(n-1)}(a) + R_n,$$

where it was assumed that

$$\begin{vmatrix} \varphi'(\xi) & \psi'(\xi) \\ \varphi(b) & \psi(b) \end{vmatrix} \neq 0.$$

The proof of *Blumenthal remainder* (34) follows from taking into account the following ingredients:

(1) defining $f(x) := F(b) - F(x) - (b-x)F'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}F^{(n-1)}(x)$;

(2) noticing that $f'(\xi) = \frac{-(b-\xi)^{n-1}}{(n-1)!}F^{(n)}(\xi)$ and that $f(a) = R_n$ and $f(b) = 0$;

(3) applying the following existence result:

Let f, φ and ψ be continuous and differentiable functions in (a, b). Then there exists ξ in the interval such that

$$\begin{vmatrix} f'(\xi) & \varphi'(\xi) & \psi'(\xi) \\ f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \end{vmatrix} = 0.$$

Remark 7. Formula (34) permits to obtain Schlömilch remainder taking $\psi = a \neq 0$.

2.18 José Vicente Gonçalves

In 1953 a new Taylor remainder appeared in the textbook [36]. For the Taylor formula

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + r_n(x,a)$$

then the remainder is given by

$$r_n(x,a) = \frac{(x-a)^{n+1}}{(n+1)!}\frac{f^{(n)}(\xi) - f^{(n)}(a)}{\xi - a}, \quad x \neq a, \quad \xi \in (a,x), \tag{35}$$

where $f : I \longrightarrow \mathbb{R}$ is a continuous function in the open interval I and n-times differentiable in the point $a \in I$. The strength of the Gonçalves remainder (35) lies in the fact that the remainder is given with respect to the n-derivative of the function whereas the other remainders are given using information regarding the $(n+1)$-derivative of the function. We know about this formula by 2nd hand source ([13, 31]). See §3 for a proof of this result, since we give a generalization of the Gonçalves remainder.

2.19 James Wolfe

In 1954 a new proof of the Lagrange Taylor remainder (12) appeared in [39] and is used in the recent textbook [29]. The proof relies on studying the function

$$g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(t-c)^k + \frac{M(t-c)^n}{n!} - f(t),$$

where M is the unique solution to

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{M(x-c)^n}{n!}.$$

We have $g(c) = g'(c) = \cdots g^{(n-1)}(c) = 0$ and by the definition of M we also have that $g(x) = 0$. Since $g(x) = g(c) = 0$, by Rolle's theorem, there exists θ between c and x such that $g'(\theta) = 0$. Applying repeatedly Rolle's theorem we get that there exists ξ such that $g^{(n)}(\xi) = M - f^{(n)}(\xi)$.

3 New Taylor Remainder

In the spirit of Schlömilch remainder (17) and basing ourselves on some idea of J. Santos Guerreiro, we can obtain a new generalization of *Gonçalves remainder* (35), which depends on an auxiliary function, namely:

Theorem 1. *Let $f : I \longrightarrow \mathbb{R}$ be a continuous function in the open interval I and n-times differentiable function in I. Moreover, let $\varphi : I \longrightarrow \mathbb{R}$ be a n-times differentiable function such that $\varphi^{(k)}(a) = 0$ for $j = 1, \dots, n-1$ and $\varphi^{(k)}(y) \neq 0$ for all y different from a and x and $j = 1, \dots, n-1$. Then, for all $x \in I$ we have*

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(x), \tag{36}$$

with

$$r_n(x) = \frac{\varphi(x) - \varphi(a)}{\varphi^{(n)}(\xi)} \left(f^{(n)}(\xi) - f^{(n)}(a) \right), \tag{37}$$

where $x \neq a$ and ξ is between a and x.

Proof. We first note that

$$r_n(a) = r'_n(a) = \cdots = r_n^{(n-1)}(a) = 0 = \varphi'(a) = \cdots = \varphi^{(n-1)}(a).$$

On the one hand, by the Cauchy finite increment formula (18) we have

$$\begin{aligned} \frac{r_n(x) - r_n(a)}{\varphi(x) - \varphi(a)} &= \frac{r'_n(\theta_1) - r'_n(a)}{\varphi'(\theta_1) - \varphi'(a)} \\ &= \cdots \\ &= \frac{r_n^{(n-1)}(\theta_{n-1}) - r_n^{(n-1)}(a)}{\varphi^{(n-1)}(\theta_{n-1}) - \varphi^{(n-1)}(a)} \\ &= \frac{r_n^{(n)}(\xi)}{\varphi^{(n)}(\xi)} \end{aligned} \tag{38}$$

where $\xi := \theta_n$. On the other hand, differentiating the equality (36) n-times we obtain $f^{(n)}(x) - f^{(n)}(a) = r_n^{(n)}(x)$ which, together with (38), entails (37). QED

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