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The classification of simple Jacobi–Ricci commuting algebraic curvature tensors

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Abstract. We classify algebraic curvature tensors such that the Ricci operator ρ is simple (i.e. ρ is complex diagonalizable and $\text{Spec}\{\rho\} = \{a\}$ or $\text{Spec}\{\rho\} = \{a_1 \pm a_2\sqrt{-1}\}$) and which are Jacobi–Ricci commuting (i.e. $\rho\mathcal{J}(v) = \mathcal{J}(v)\rho$ for all v).

Keywords: algebraic curvature tensor, Jacobi operator, Ricci operator

MSC 2000 classification: primary 53C20

This is dedicated to Professor Oldrich Kowalski

The study of curvature is fundamental in differential geometry. It is often convenient to work first in an abstract algebraic context and then subsequently to pass to the geometrical setting. We say that $\mathfrak{M} := (V, \langle\cdot,\cdot\rangle, A)$ is a *model* if $\langle\cdot,\cdot\rangle$ is a non-degenerate inner-product of signature (p, q) on a real vector space V of dimension $m = p + q$ and if $A \in \otimes^4 V^*$ is an *algebraic curvature tensor*, i.e. a 4-tensor which has the symmetries of the Riemann curvature tensor:

$$\begin{aligned} &A(v_1, v_2, v_3, v_4) = -A(v_2, v_1, v_3, v_4) = A(v_3, v_4, v_1, v_2), \\ &A(v_1, v_2, v_3, v_4) + A(v_2, v_3, v_1, v_4) + A(v_3, v_1, v_2, v_4) = 0\,. \end{aligned} \tag{1}$$

If P is a point of a pseudo-Riemannian manifold $\mathcal{M} = (M, g)$, then the associated model is defined by setting $\mathfrak{M}(\mathcal{M}, P) := (T_PM, g_P, R_P)$ where R_P is the curvature tensor of the Levi–Civita connection; every model is geometrically realizable in this fashion. Consequently the study of algebraic curvature tensors plays a central role in many geometric investigations.

If $\mathfrak{M}$ is a model, then *Jacobi operator* $\mathcal{J}$, the skew-symmetric curvature operator $\mathcal{R}$, and the *Ricci operator* ρ are defined by the identities:

$$\begin{aligned}\langle \mathcal{J}(x)y, z\rangle &= A(y, x, x, z),\\ \langle \mathcal{R}(x, y)z, w\rangle &= A(x, y, z, w),\\ \langle \rho x, y\rangle &= \mathrm{Tr}\left\{z \to \tfrac{1}{2}\mathcal{R}(z, x)y + \tfrac{1}{2}\mathcal{R}(z, y)x\right\}\ .\end{aligned}$$

One says $\mathfrak{M}$ is *Einstein* if $\rho = a\,\mathrm{id}$; a is called the *Einstein constant*.

The study of commutativity properties of natural operators defined by the curvature tensor was initiated by Stanilov [4, 5] and has proved to be a very fruitful one; we refer to [1] for a survey of the field and for a more complete bibliography than is possible to present here.

We begin with the following fundamental result which is established in [2] and which examines when the Jacobi operator or the skew-symmetric curvature operator commutes with the Ricci operator:

1 Lemma. *The following conditions are equivalent for a model* $\mathfrak{M}$*:*

(1) $\mathcal{J}(v)\rho = \rho\mathcal{J}(v)$ *for all* $v \in V$*.*

(2) $\mathcal{R}(v_1, v_2)\rho = \rho\mathcal{R}(v_1, v_2)$ *for all* $v_1, v_2 \in V$*.*

(3) $A(\rho v_1, v_2, v_3, v_4) = A(v_1, \rho v_2, v_3, v_4) = A(v_1, v_2, \rho v_3, v_4)$
$= A(v_1, v_2, v_3, \rho v_4)$ *for all* $v_1, v_2, v_3, v_4 \in V$*.*

One says that a model $\mathfrak{M}$ is *decomposable* if there is an orthogonal direct sum decomposition $V = V_1 \oplus V_2$ inducing a splitting $A = A_1 \oplus A_2$; $\mathfrak{M}$ is *indecomposable* if it is not decomposable. Let $\mathrm{Spec}(\rho) \subset \mathbb{C}$ be the spectrum of the Ricci operator. One has [3]:

2 Lemma. *If* $\mathfrak{M}$ *is an indecomposable Jacobi–Ricci commuting model, then* $\mathrm{Spec}(\rho) = \{a_1\}$ *or* $\mathrm{Spec}(\rho) = \{a_1 \pm a_2\sqrt{-1}\}$ *where* $a_2 > 0$.

Although ρ is self-adjoint, ρ need not be diagonalizable in the higher signature setting and in fact the Jordan normal form of ρ can be quite complicated. To simplify the discussion, we shall suppose ρ complex diagonalizable henceforth. Motivated by Lemmas 1 and 2, we make the following:

3 Definition. If $\mathfrak{M}$ is a model which has any of the (3) equivalent properties listed in Lemma 1, then $\mathfrak{M}$ is said to be a *Jacobi–Ricci commuting model*. If in addition, the Ricci operator ρ is complex diagonalizable and $\mathrm{Spec}(\rho) = \{a_1\}$ or $\mathrm{Spec}(\rho) = \{a_1 \pm a_2\sqrt{-1}\}$ for $a_2 > 0$, then $\mathfrak{M}$ is said to be a *simple Jacobi–Ricci commuting model*. If $\mathcal{M}$ is a pseudo-Riemannian manifold, then $\mathcal{M}$ is said to be *simple Jacobi–Ricci commuting* if $\mathfrak{M}(\mathcal{M}, P)$ is a simple Jacob–Ricci commuting model for all points P of M.

The following is immediate from the definitions we have given.

4 Lemma. *Let $\mathfrak{M}$ be a model.*

(1) $\mathfrak{M}$ is Einstein if and only if $\mathfrak{M}$ is a simple Jacobi–Ricci commuting model with $\operatorname{Spec}(\rho) = \{a_1\}$.

(2) Let $\mathfrak{M}$ be a simple Jacobi–Ricci commuting model which is not Einstein. Set $J = J_{\mathfrak{M}} := a_2^{-1}\{\rho - a_1 \operatorname{id}\}$. Then J is a self-adjoint complex structure on V, $A(Jx, y, z, w) = A(x, Jy, z, w) = A(x, y, Jz, w) = A(x, y, z, Jw)$, and $\rho = a_1 + a_2 J$.

In view of Lemma 4 (1), we shall assume $\mathfrak{M}$ is not Einstein henceforth. The following ansatz for constructing simple Jacobi–Ricci commuting models which are not Einstein will be crucial:

5 Definition. Let $\mathfrak{N} := (V_0, g, A_1, A_2)$ where g is a positive definite inner product on a finite dimensional real vector space V_0 and where A_1 and A_2 are Einstein algebraic curvature tensors with Einstein constants, respectively, a_1 and $a_2 > 0$. Extend g, A_1, and A_2 to be complex linear on the complexification $V_{\mathbb{C}} := V_0 \otimes_{\mathbb{R}} \mathbb{C}$. Let $V := V_0 \oplus V_0\sqrt{-1}$ be the underlying real vector space of $V_{\mathbb{C}}$, let $\langle \cdot, \cdot \rangle := \operatorname{Re} g(\cdot, \cdot)$, and let $A := \operatorname{Re}\{A_1 + \sqrt{-1}A_2\}$ define $\mathfrak{M}(\mathfrak{N}) := (V, \langle \cdot, \cdot \rangle, A)$.

The following classification result is the fundamental result of this paper:

6 Theorem. *Adopt the notation established above:*

(1) $\mathfrak{M}(\mathfrak{N})$ is a simple Jacobi–Ricci commuting model which is not Einstein, which has $\operatorname{Spec}\{\rho\} = \{2a_1 \pm 2a_2\sqrt{-1}\}$, and which has that $J_{\mathfrak{M}(\mathfrak{N})}$ is multiplication by $\sqrt{-1}$.

(2) Let $\mathfrak{M}$ be a simple Jacobi–Ricci commuting model which is not Einstein with $\operatorname{Spec}(\rho) = \{2a_1 \pm 2a_2\sqrt{-1}\}$ for $a_2 > 0$. Then there exists $\mathfrak{N}$ so $\mathfrak{M}$ is isomorphic to $\mathfrak{M}(\mathfrak{N})$.

(3) If $\mathfrak{N} = (V_0, g, A_1, A_2)$ and $\tilde{\mathfrak{N}} = (\tilde{V}_0, \tilde{g}, \tilde{A}_1, \tilde{A}_2)$, then $\mathfrak{M}(\mathfrak{N})$ is isomorphic to $\mathfrak{M}(\tilde{\mathfrak{N}})$ if and only if there is an isomorphism $\theta : V_0 \to \tilde{V}_0$ and a skew-

adjoint linear transformation T of (V_0, g) with $|T| < 1$ so that:

$$\begin{aligned}
&\tilde{g}(\theta v, \theta w) = g(v, w) - g(Tv, Tw),\\
&\tilde{A}_1(\theta v, \theta w, \theta x, \theta y) = A_1(v, w, x, y) - A_1(Tv, Tw, x, y)\\
&\quad -A_1(Tv, w, Tx, y) - A_1(Tv, w, x, Ty) - A_1(v, Tw, Tx, y)\\
&\quad -A_1(v, Tw, x, Ty) - A_1(v, w, Tx, Ty) + A_1(Tv, Tw, Tx, Ty)\\
&\quad -A_2(Tv, w, x, y) - A_2(v, Tw, x, y) - A_2(v, w, Tx, y)\\
&\quad -A_2(v, w, x, Ty) + A_2(Tv, Tw, Tx, y) + A_2(Tv, Tw, x, Ty)\\
&\quad +A_2(Tv, w, Tx, Ty) + A_2(v, Tw, Tx, Ty)\\
&\tilde{A}_2(\theta v, \theta w, \theta x, \theta y) = A_2(v, w, x, y) - A_2(Tv, Tw, x, y)\\
&\quad -A_2(Tv, w, Tx, y) - A_2(Tv, w, x, Ty) - A_2(v, Tw, Tx, y)\\
&\quad -A_2(v, Tw, x, Ty) - A_2(v, w, Tx, Ty) + A_2(Tv, Tw, Tx, Ty)\\
&\quad +A_1(Tv, w, x, y) + A_1(v, Tw, x, y) + A_1(v, w, Tx, y)\\
&\quad +A_1(v, w, x, Ty) - A_1(Tv, Tw, Tx, y) - A_1(Tv, Tw, x, Ty)\\
&\quad -A_1(Tv, w, Tx, Ty) - A_1(v, Tw, Tx, Ty)
\end{aligned} \tag{2}$$

Theorem 6 completes the analysis in the algebraic setting. In the geometric setting, by contrast, the situation is still far from clear. However there is a geometrical example known [2] in signature $(2, 2)$ which may be described as follows; we refer to [2] for further details. Let (x_1, x_2, x_3, x_4) be coordinates on $\mathbb{R}^4$. Define a metric whose non-zero components are, up to the usual $\mathbb{Z}_2$ symmetries, given by:

$$\begin{aligned}
&g(\partial_1, \partial_3) = g(\partial_2, \partial_4) = 1, && g(\partial_3, \partial_4) = s(x_2^2 - x_1^2),\\
&g(\partial_3, \partial_3) = 2sx_1x_2, && g(\partial_4, \partial_4) = -2sx_1x_2\,.
\end{aligned} \tag{3}$$

7 Lemma. *Let $\mathcal{M}$ be as in Equation (3). Then $\mathcal{M}$ is a locally symmetric simple Jacobi–Ricci commuting manifold with* $\operatorname{Spec}(\rho) = \{\pm 2s\sqrt{-1}\}$ *of signature* $(2, 2)$*. The Ricci operator and non-zero curvatures are described by:*

$$\begin{aligned}
&R_{1314} = s, && R_{1323} = -s, && R_{1424} = s, && R_{2324} = -s,\\
&\rho\partial_1 = -2s\partial_2, && \rho\partial_2 = 2s\partial_1, && \rho\partial_3 = 2s\partial_4, && \rho\partial_4 = -2s\partial_3\,.
\end{aligned}$$

The remainder of this note is devoted to the proof of Theorem 6.

Proof of Theorem 6 (1). We generalize the discussion of [2]. Let $\{e_i\}$ be an orthonormal basis for V_0. Let $e_i^+ := e_i$ and $e_i^- := \sqrt{-1}e_i$ be an orthonormal basis for V; the vectors e_i^+ are spacelike and the vectors e_i^- are timelike so $\mathfrak{M}(\mathfrak{N})$ has neutral signature. Clearly the symmetries of Equation (1) hold for the complexification of A_1 and A_2 and, consequently, for $A_1 + \sqrt{-1}A_2$ and $A = \operatorname{Re}(A_1 + \sqrt{-1}A_2)$. Thus $\mathfrak{M}(\mathfrak{N})$ is a model and the non-zero components of

A relative to this basis are given by:

$$\begin{aligned}
&A(e_i^-,e_j^+,e_k^+,e_l^+)=A(e_i^+,e_j^-,e_k^+,e_l^+)=A(e_i^+,e_j^+,e_k^-,e_l^+)\\
&\quad=A(e_i^+,e_j^+,e_k^+,e_l^-)=-A_2(e_i,e_j,e_k,e_l),\\
&A(e_i^+,e_j^-,e_k^-,e_l^-)=A(e_i^-,e_j^+,e_k^-,e_l^-)=A(e_i^-,e_j^-,e_k^+,e_l^-)\\
&\quad=A(e_i^-,e_j^-,e_k^-,e_l^+)=A_2(e_i,e_j,e_k,e_l),\\
&A(e_i^+,e_j^+,e_k^+,e_l^+)=A(e_i^-,e_j^-,e_k^-,e_l^-)=A_1(e_i,e_j,e_k,e_l),\\
&A(e_i^+,e_j^+,e_k^-,e_l^-)=A(e_i^+,e_j^-,e_k^+,e_l^-)=A(e_i^-,e_j^+,e_k^+,e_l^-)\\
&\quad=A(e_i^+,e_j^-,e_k^-,e_l^+)=A(e_i^-,e_j^+,e_k^-,e_l^+)=A(e_i^-,e_j^-,e_k^+,e_l^+)\\
&\quad=-A_1(e_i,e_j,e_k,e_l)\,.
\end{aligned}\tag{4}$$

Let $\rho:=\rho_A$ and $\rho_i:=\rho_{A_i}$. We sum over k in the following expansions to see:

$$\begin{aligned}
\langle\rho e_i^+,e_j^+\rangle&=A(e_i^+,e_k^+,e_k^+,e_j^+)-A(e_i^+,e_k^-,e_k^-,e_j^+)=2\rho_1(e_i,e_j)=2a_1\delta_{ij},\\
\langle\rho e_i^-,e_j^-\rangle&=A(e_i^-,e_k^+,e_k^+,e_j^-)-A(e_i^-,e_k^-,e_k^-,e_j^-)=-2\rho_1(e_i,e_j)=-2a_1\delta_{ij},\\
\langle\rho e_i^+,e_j^-\rangle&=A(e_i^+,e_k^+,e_k^+,e_j^-)-A(e_i^+,e_k^-,e_k^-,e_j^-)=-2\rho_2(e_i,e_j)=-2a_2\delta_{ij}.
\end{aligned}$$

This shows that $\rho e_i^\pm=2a_1e_i^\pm\pm 2a_2e_i^\mp$. Thus we may view ρ as acting by complex scalar multiplication by $\lambda:=2a_1+2a_2\sqrt{-1}$ on $V_{\mathbb{C}}$. This implies that the underlying real operator is complex diagonalizable and $\operatorname{Spec}(\rho)=\{\lambda,\bar\lambda\}$. Since the tensors A_i were extended to be complex multi-linear, we have

$$\begin{aligned}
&A(\rho v_1,v_2,v_3,v_4)=\operatorname{Re}\{A_1(\lambda v_1,v_2,v_3,v_4)+\sqrt{-1}A_2(\lambda v_1,v_2,v_3,v_4)\}\\
=\;&\operatorname{Re}\{\lambda A_1(v_1,v_2,v_3,v_4)+\sqrt{-1}\lambda A_2(v_1,v_2,v_3,v_4)\}\\
=\;&\operatorname{Re}\{A_1(v_1,\lambda v_2,v_3,v_4)+\sqrt{-1}A_2(v_1,\lambda v_2,v_3,v_4)\}=A(v_1,\rho v_2,v_3,v_4)\,.
\end{aligned}$$

This establishes one equality of Lemma 1 (3); the other equalities follow similarly and hence $\mathfrak{M}(\mathfrak{N})$ is Jacobi–Ricci commuting as well. $\boxed{QED}$

We shall need the following technical result before establishing the second assertion of Theorem 6. Although well known, we include the proof for the sake of completeness and to establish notation:

8 Lemma. *Let J be a self-adjoint map of $(V,\langle\cdot,\cdot\rangle)$ so that $J^2=-\operatorname{id}$. Then there exists an orthonormal basis $\{e_1^\pm,\dots,e_p^\pm\}$ for V so that $Je_i^\pm=\pm e_i^\mp$.*

Proof. We assume that $p=1$ as the general result then follows by induction. Let $\{f^\pm\}$ be an orthonormal basis for V where f^+ is spacelike and f^- is timelike. As J is trace-free and self-adjoint,

$$J=\begin{pmatrix} a & b\\ -b & -a\end{pmatrix}.$$

Since $J^2 = -\operatorname{id}$, $b^2 - a^2 = 1$. Let $e(\theta) := \cosh\theta f^+ + \sinh\theta f^-$. Then

$$\begin{aligned}
&\langle Je(\theta), e(\theta)\rangle \\
= \;& \langle (a\cosh\theta + b\sinh\theta)f^+ + (-b\cosh\theta - a\sinh\theta)f^-, \cosh\theta f^+ + \sinh\theta f^-\rangle \\
= \;& a\cosh^2\theta + b\cosh\theta\sinh\theta + b\cosh\theta\sinh\theta + a\sinh^2\theta \\
= \;& \tfrac{1}{2}\{(a+b)e^{2\theta} + (a-b)e^{-2\theta}\}\,.
\end{aligned}$$

Since $a^2 - b^2 = -1$, $a + b$ and $a - b$ have opposite signs. Thus for some value of θ, we have $\langle Je(\theta), e(\theta)\rangle = 0$. Set $e_i^+ = e(\theta)$ and $e_i^- = Je(\theta)$. $\boxed{QED}$

Proof of Theorem 6 (2). Let $\mathfrak{M} = (V, \langle\cdot,\cdot\rangle, A)$ be a simple Jacobi–Ricci commuting model. Assume that the Ricci operator ρ is complex diagonalizable and that $\operatorname{Spec}(\rho) = \{2a_1 \pm 2a_2\sqrt{-1}\}$ for $a_2 > 0$. By Lemma 8, there is an orthonormal basis $\{e_i^\pm\}$ for V so $Je_i^\pm = \pm e_i^\mp$. Set

$$\begin{aligned}
A_1(e_i, e_j, e_k, e_l) &:= A(e_i^+, e_j^+, e_k^+, e_l^+), \\
A_2(e_i, e_j, e_k, e_l) &:= -A(e_i^-, e_j^+, e_k^+, e_l^+)\,.
\end{aligned}$$

We may then derive the relations of Equations (4) from Lemma 4 (2). We check that A_1 and A_2 are algebraic curvature tensors by verifying that:

$$\begin{aligned}
&A_1(e_i, e_j, e_k, e_l) = A(e_i^+, e_j^+, e_k^+, e_l^+) = -A(e_j^+, e_i^+, e_k^+, e_l^+) \\
&\quad = -A_1(e_j, e_i, e_k, e_l), \\
&A_1(e_i, e_j, e_k, e_l) = A(e_i^+, e_j^+, e_k^+, e_l^+) = A(e_k^+, e_l^+, e_i^+, e_j^+) \\
&\quad = A_1(e_k, e_l, e_i, e_j), \\
&A_1(e_i, e_j, e_k, e_l) + A_1(e_j, e_k, e_i, e_l) + A_1(e_k, e_i, e_j, e_l) \\
&\quad = A(e_i^+, e_j^+, e_k^+, e_l^+) + A(e_j^+, e_k^+, e_i^+, e_l^+) + A(e_k^+, e_i^+, e_j^+, e_l^+) = 0, \\
&A_2(e_i, e_j, e_k, e_l) = -A(e_i^-, e_j^+, e_k^+, e_l^+) = A(e_j^+, e_i^-, e_k^+, e_l^+) \\
&\qquad = -A(e_j^-, e_i^+, e_k^+, e_l^+) = -A_2(e_j, e_i, e_k, e_l), \\
&A_2(e_k, e_l, e_i, e_j) = -A(e_k^-, e_l^+, e_i^+, e_j^+) = -A(e_i^+, e_j^+, e_k^-, e_l^+) \\
&\qquad = -A(e_i^-, e_j^+, e_k^+, e_l^+) = A_2(e_i, e_j, e_k, e_l), \\
&A_2(e_i, e_j, e_k, e_l) + A_2(e_j, e_k, e_i, e_l) + A_2(e_k, e_i, e_j, e_l) \\
&\qquad = -A(e_i^-, e_j^+, e_k^+, e_l^+) - A(e_j^-, e_k^+, e_i^+, e_l^+) - A(e_k^-, e_j^+, e_i^+, e_l^+) \\
&\qquad = -A(e_i^-, e_j^+, e_k^+, e_l^+) - A(e_j^+, e_k^+, e_i^-, e_l^+) - A(e_k^+, e_j^+, e_i^-, e_l^+) = 0.
\end{aligned}$$

We verify A_1 and A_2 are Einstein by summing over k to compute

$$\begin{aligned}
&(\rho_1 e_i, e_l) = A(e_i^+, e_j^+, e_j^+, e_l^+) \\
&\quad = \tfrac{1}{2}A(e_i^+, e_j^+, e_j^+, e_l^+) - \tfrac{1}{2}A(e_i^+, e_j^-, e_j^- e_l^+) = \tfrac{1}{2}\langle \rho e_i^+, e_l^+\rangle = a_1\delta_{il}. \\
&(\rho_2 e_i, e_l) = -A(e_i^-, e_j^+, e_j^+, e_l^+) \\
&\quad = -\tfrac{1}{2}A(e_i^-, e_j^+, e_j^+, e_l^+) + \tfrac{1}{2}A(e_i^-, e_j^-, e_j^- e_l^+) = -\tfrac{1}{2}\langle \rho e_i^-, e_l^+\rangle = a_2\delta_{il}.
\end{aligned}$$

The desired result now follows. $\boxed{QED}$

Replacing e_i^- by $-e_i^-$ in Equation (4) yields an isomorphism between the models $\mathfrak{N}(V,(\cdot,\cdot),A_1,A_2)$ and $\mathfrak{N}(V,(\cdot,\cdot),A_1,-A_2)$; it is for this reason that we may always assume the Einstein constant of A_2 is positive. This reflects that complex conjugation defines a field isomorphism of $\mathbb{C}$ taking $\lambda \to \bar{\lambda}$ or, equivalently, by replacing a_2 by $-a_2$ in the construction. More important, however, is the fact that the splitting $V = V_+ \oplus V_-$ where $V_\pm := \text{Span}\{e_i^\pm\}$ which is crucial to our discussion is highly non-unique. Let $\mathfrak{N} = (V_0, g, A_1, A_2)$ and let $\tilde{\mathfrak{N}} = (\tilde{V}_0, \tilde{g}, \tilde{A}_1, \tilde{A}_2)$. Let $\mathfrak{M} = \mathfrak{M}(\mathfrak{N})$ and $\tilde{\mathfrak{M}} = \mathfrak{M}(\tilde{\mathfrak{N}})$. Let J and $\tilde{J}$ be the associated complex structures on V and on $\tilde{V}$, respectively. We then have maximal spacelike subspaces $V_+ := V_0$ and $\tilde{V}_+ := \tilde{V}_0$ of V and $\tilde{V}$, respectively, so that for all x, y, z, w in V_0 and for all $\tilde{x}$, $\tilde{y}$, $\tilde{z}$, $\tilde{w}$ in $\tilde{V}_0$,

$$\begin{array}{ll}
V_+ \perp JV_+, & \tilde{V}_+ \perp \tilde{J}\tilde{V}_+, \\
A_1(v,w,x,y) = A(v,w,x,y), & \tilde{A}_1(\tilde{v},\tilde{w},\tilde{x},\tilde{y}) = \tilde{A}(\tilde{v},\tilde{w},\tilde{x},\tilde{y}), \\
A_2(v,w,x,y) = A(Jv,w,x,y), & \tilde{A}_2(\tilde{v},\tilde{w},\tilde{x},\tilde{y}) = -\tilde{A}(\tilde{J}\tilde{v},\tilde{w},\tilde{x},\tilde{y}), \\
g(x,y) = \langle x,y\rangle, & \tilde{g}(\tilde{x},\tilde{y}) = \langle \tilde{x},\tilde{y}\rangle\,.
\end{array}$$

Suppose that Θ is an isomorphism from $\mathfrak{M}$ to $\tilde{\mathfrak{M}}$. We may then identify $V = \tilde{V}$ and $J = \tilde{J}$. The decomposition $V = V_+ \oplus JV_+$ defines orthogonal projections $\pi_\pm$. Since $\tilde{V}_+$ is spacelike, π_+ defines an isomorphism θ from $\tilde{V}_+$ to V_+. Let

$$T = -J \circ \pi_- \circ \theta^{-1} : V_+ \to \tilde{V}_+ \to V_- \to V_+\,.$$

We may then represent any element of $\tilde{V}_+$ in the form $v + JTv$ for $v \in V_+$.

9 Lemma. *Adopt the notation established above:*

(1) $\tilde{V}_+ \perp J\tilde{V}_+$ if and only if T is skew-adjoint.

(2) The induced metric on $\tilde{V}_+$ is positive definite if and only if $|T| < 1$.

Proof. We have that J is self-adjoint and that $J^2 = -\,\text{id}$. Consequently, we have the following implications which establish Assertion (1):

$$\begin{aligned}
&\tilde{V}_+ \perp J\tilde{V}_+. \\
\Leftrightarrow\ &\langle v + JTv, Jw + JJTw\rangle = 0 \text{ for all } v,w \in V_+. \\
\Leftrightarrow\ &-\langle v, Tw\rangle - \langle Tv, w\rangle = 0 \text{ for all } v,w \in V_+. \\
\Leftrightarrow\ &T \text{ is skew-adjoint.}
\end{aligned}$$

We argue similarly to prove Assertion (2):

$$
\begin{aligned}
&\langle v+JTv, v+JTv\rangle > 0 \text{ for all } 0 \neq v \in V_+.\\
\Leftrightarrow\ & g(v,v) - g(Tv,Tv) > 0 \text{ for all } 0 \neq v \in V_+.\\
\Leftrightarrow\ & |Tv|^2 < |v|^2 \text{ for all } 0 \neq v \in V_+.\\
\Leftrightarrow\ & |T| < 1.
\end{aligned}
$$

QED

Proof of Theorem 6 (3). Suppose $\Theta : \mathfrak{M}(V_0, A_1, A_2) \to \mathfrak{M}(\tilde{V}_0, \tilde{A}_1, \tilde{A}_2)$ is an isomorphism. We use Θ to identify V with $\tilde{V}$ and to parametrize $\tilde{V}_+$ in the form $\{v + JTV\}$ where T is a skew-adjoint linear map of V_0 with $|T| < 1$. We then the following identities for all v, w, x, and y:

$$
\begin{aligned}
&\tilde{g}(v,w) = \langle v+JTv, w+JTw\rangle,\\
&\tilde{A}_1(v,w,x,y) = A(v+JTv, w+JTw, x+JTx, y+JTy),\\
&\tilde{A}_2(v,w,x,y) = A(J(v+JTv), w+JTw, x+JTx, y+JTy)\,.
\end{aligned}
\tag{5}
$$

Lemma 1 (2) and Equation (5) imply that Equation (2) holds. This establishes one implication of Theorem 6 (3). As the arguments are reversible, the converse implication holds as well. QED

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