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Naturally Harmonic Vector Fields

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Abstract. This paper is a survey on recent results obtained in collaboration with M.T.K. Abbassi and D. Perrone. Let (M,g) be a compact Riemannian manifold. If we equip the tangent bundle TM with the Sasaki metric g^s, the only vector fields defining harmonic maps from (M,g) to (TM,g^s) are the parallel ones, as Nouhaud [14] and Ishihara [10] proved independently. The Sasaki metric is just a particular example of Riemannian g-natural metric. Equipping TM with an arbitrary Riemannian g-natural metric G and investigating the harmonicity of a vector field V of M, thought as a map from (M,g) to (TM,G), several interesting behaviours are found.

If V is a unit vector field, then it also defines a smooth map from M to the unit tangent sphere bundle T_1M. Being T_1M an hypersurface of TM, any Riemannian metric on TM induces one on the unit tangent sphere bundle. Denoted by $\tilde{g}^s$ the Sasaki metric on T_1M (the one induced on it by g^s), Han and Yim [11] characterized unit vector fields which define harmonic maps from (M,g) to $(T_1M,\widetilde{g^s})$. The variational problem related to the energy restricted to unit vector fields, $E:\mathfrak{X}^1(M)\to\mathbb{R}, V\mapsto E(V)$, has been studied by Wood in [18]. We equipped T_1M with an arbitrary Riemannian metric $\tilde{G}$ induced by a Riemannian g-natural metric G on TM, and we studied harmonicity properties of the map $V:(M,g)\to(T_1M,\tilde{G})$ corresponding to a unit vector field.

Keywords: tangent bundle, harmonic vector fields, g-natural metrics, Reeb vector field

MSC 2000 classification: primary 53C43, secondary 53C07, 53C15

Introduction

Let (M,g) be an n-dimensional Riemannian manifold. The so called *Sasaki metric* g^s is by far the simplest and most investigated among all possible Riemannian metrics on the *tangent bundle* TM.

In particular, as concerns harmonicity problems, any vector field $V\in\mathfrak{X}(M)$ over a compact Riemannian manifold (M,g) defines a smooth map from M to TM and so, it is natural to investigate the harmonicity of such a map from (M,g) to (TM,g^s). Nouhaud [14] found the expression of the energy associated to V and proved that parallel vector fields are all and the ones defining harmonic maps. Ishihara [10] obtained independently the same result, giving also the explicit expression of the tension field associated to a vector field V. More recently, Gil-Medrano [9] proved that critical points of the energy functional

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restricted to vector fields are again parallel vector fields. These results clearly show a very rigid behaviour of the Sasaki metric under the point of view of harmonicity of vector fields.

On the other hand, the Sasaki metric g^s is only one possible choice inside a wide family of Riemannian metrics on TM, known as *Riemannian g-natural metrics*, which are described by means of six independent smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$ [5]. As their name suggests, these metrics arise from a very "natural" construction starting from a Riemannian metric g over M. The introduction of g-natural metrics moves from the classification of natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles [13], or equivalently, from the description of all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles [12].

The rigidity of the Sasaki metric and its being g-natural make interesting to investigate harmonicity properties of a vector field V, when g^s is replaced by an arbitrary Riemannian g-natural metric G. This study has been made in [1] and permitted to find new examples of harmonic maps from M to TM, defined by non-parallel vector fields. Some of the main results are presented here in Section 2. Whenever it is possible, these results are compared to corresponding theorems about the Sasaki metric.

Consider now the set $\mathfrak{X}^1(M)$ of all unit smooth vector fields on M, which we suppose to be non-empty (this implies the vanishing of the Euler-Poincaré characteristic of M). We denote by $\widetilde{g^s}$ the Sasaki metric induced by g^s on the unit tangent sphere bundle T_1M.

Any $V \in \mathfrak{X}^1(M)$ defines a smooth map from (M,g) to $(T_1M, \tilde{g^s})$. Han and Yim [11] characterized unit vector fields which define harmonic maps from (M,g) to $(T_1M, \widetilde{g^s})$, by determining the associated tension field. Wood [18] determined the Euler-Lagrange equation for the variational problem related to the energy $E : \mathfrak{X}^1(M) \to \mathbb{R}, V \mapsto E(V)$, where $E(V)$ is the energy of the corresponding map $V : (M,g) \to (T_1M, \tilde{g^s})$. Previously, Wiegmink [16] already considered the variational problem related to the *total bending* of V which, up to a constant, coincides with the energy of V.

By definition, a *g-natural metric $\widetilde{G}$ on T_1M* is nothing but the restriction of a g-natural metric G on TM to its hypersurface T_1M. Although g-natural metrics on T_1M possess a simpler form than the ones on TM, they form a quite big class of metrics, depending on four independent real parameters (satisfying some inequalities in order to be Riemannian). Moreover, classic examples of Riemannian metrics on T_1M, like the Sasaki metric $\tilde{g}^s$ itself and the *Cheeger-Gromoll metric g_{CG}*, are special examples of Riemannian g-natural metrics.

Equipping T_1M with an arbitrary Riemannian g-natural metric $\widetilde{G}$, in [2] we investigated when a unit vector field $V \in \mathfrak{X}^1(M)$ determines a harmonic map $V : (M, g) \to (T_1M, \widetilde{G})$, and other problems related to the harmonicity of this map. In Section 4 below, the main results of [2] are reported.

1 Riemannian g-natural metrics on TM

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At any point (x, u) of its *tangent bundle* TM, the tangent space of TM splits into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

For any vector $X \in M_x$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}$ (the *horizontal lift* of X to $(x, u) \in TM$), such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions f on M. The map $X \to X^h$ is an isomorphism between the vector spaces M_x and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between M_x and $\mathcal{V}_{(x,u)}$. Each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

Kolář, Michor and Slovák introduced a large class of metrics, known as *g-natural metrics*, on the tangent bundle TM of a Riemannian manifold (M, g). The introduction of these metrics moves from the description of all first order natural operators $D : S^2_+T^* \rightsquigarrow (S^2T^*)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+T^*$ and S^2T^* denote the bundle functors of all Riemannian metrics and all symmetric $(0, 2)$-tensors over n-manifolds respectively. Details about the concept of naturality and related notions can be found in [12].

Every section $G : TM \to (S^2T^*)TM$ is called a (possibly degenerate) *metric*. As Kowalski and Sekizawa proved in [13], there is a bijective correspondence between the triples of so called first order natural F-metrics and first order natural (possibly degenerate) metrics G on the tangent bundle. This fundamental result makes possible to classify explicitly g-natural metrics on TM. In fact, it turns out that all g-natural metrics on the tangent bundle of a Riemannian manifold (M, g) are completely determined as follows:

1 Proposition ([5]). *Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are six smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$,*

$i = 1, 2, 3$, *such that for every* u, X, $Y \in M_x$, *we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2) g_x(X,Y) + (\beta_1 + \beta_3)(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g_x(X,Y) + \beta_1(r^2) g_x(X,u) g_x(Y,u), \end{cases} \tag{1}$$

where $r^2 = g_x(u,u)$. *For* $n = 1$, *the same holds with* $\beta_i = 0$, $i = 1, 2, 3$.

In the sequel, we shall use the following notations:

- $\phi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(\mathrm{t})$,
- $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$, for all $t \in \mathbb{R}^+$.

Taking into account notations above, the following result holds:

2 Proposition. [5] A g-natural metric G on the tangent bundle of a Riemannian manifold (M, g) is Riemannian if and only if the following inequalities hold for all $t \in \mathbb{R}^+$:

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0. \tag{2}$$

Note that the *Sasaki metric* g^s is the Riemannian g-natural metric uniquely determined by

$$\alpha_1(t) = 1, \quad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0. \tag{3}$$

2 Harmonicity properties of $V : (M, g) \to (TM, G)$

Let $f : (M, g) \to (M', g')$ be a smooth map between Riemannian manifolds, with M compact. The *energy* of f is defined as the integral

$$E(f) := \int_M e(f) dv_g$$

where $e(f) = \frac{1}{2}||f_*||^2 = \frac{1}{2}\mathrm{tr}_g f^* g'$ is the so-called *energy density* of f. With respect to a local orthonormal basis of vector fields $\{e_1, .., e_n\}$ on M, it is possible to express the energy density as $e(f) = \frac{1}{2}\sum_{i=1}^n g'(f_* e_i, f_* e_i)$. Critical points of the energy functional on $C^\infty(M, M')$ are known as *harmonic maps*. They have been characterized in [8] as maps whose *tension field* $\tau(f) = \mathrm{tr}\nabla df$ vanishes. When (M, g) is a general Riemannian manifold (including the non-compact case), a map $f : (M, g) \to (M', g')$ is said to be harmonic if $\tau(f) = 0$. For further details about the energy functional, we can refer to [7].

Let now (M,g) be a compact Riemannian manifold of dimension n and (TM,G) its tangent bundle, equipped with an arbitrary Riemannian g-natural metric G. Each vector field $V \in \mathfrak{X}(M)$ defines a smooth map $V : (M,g) \to (TM,G)$, $p \mapsto V_p$. By definition, the *energy* $E(V)$ of V is the energy associated to this map. Therefore, $E(V) = \int_M e(V)dv_g$, where the density function $e(V)$ is given by

$$e_p(V) = \frac{1}{2}||V_{*p}||^2 = \frac{1}{2}\mathrm{tr}_g(V^* G)_p = \frac{1}{2}\sum_{i=1}^{n}(V^*G)_p(e_i,e_i), \tag{4}$$

$\{e_1,..,e_n\}$ being any local orthonormal basis of vector fields defined in a neighborhood of p. Explicitly, we get

$$\begin{aligned} e(V) \;&=\; \frac{1}{2}\Big\{ n(\alpha_1+\alpha_3)(r^2) + (\beta_1+\beta_3)(r^2)r^2 + 2\alpha_2(r^2)\mathrm{div}(V) \qquad (5)\\ &\quad +2\beta_2(r^2)V(r^2) + \alpha_1(r^2)||\nabla V||^2 + \frac{1}{4}\beta_1(r^2)||\mathrm{grad}\, r^2||^2 \Big\}, \end{aligned}$$

where $r = ||V||$. In the special case when $G = g^s$, by (3) it easily follows that (5) reduces to the well known formula

$$e(V) \;=\; \frac{n}{2} + \frac{1}{2}||\nabla V||^2. \tag{6}$$

2.1 Absolute minima for the energy

Let M be compact. It is well known that when TM is equipped with the Sasaki metric g^s, parallel vector fields are all and the ones absolute minima for the energy functional. In fact, integrating (6), one easily finds that the energy associated to the map $V : (M,g) \to (TM,g^s)$ admits the following very simple expression [14]:

$$E(V) = \frac{n}{2}\mathrm{vol}(M) + \frac{1}{2}\int_M ||\nabla V||^2 dv_g.$$

Up to a constant, expression above also corresponds to the total bending of V [16]. For any constant $\rho > 0$, we now put $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : ||V||^2 = \rho\}$. We then equip TM with an arbitrary Riemannian g-natural metric G and write $E(V) = \int_M e(V)dv_g$ for $V \in \mathfrak{X}^\rho(M)$. Integrating (5) we have that the energy of V is given by

$$E(V) = \frac{1}{2}[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3](\rho)\cdot \mathrm{vol}(M,g)+\frac{1}{2}\alpha_1(\rho)\cdot\int_M ||\nabla V||^2 dv_g \tag{7}$$

and so, we have the following

3 Theorem. *Let (M,g) be a compact Riemannian manifold. Equipping TM with an arbitrary Riemannian g-natural metric G, a vector field $V \in \mathfrak{X}^\rho(M)$ is an absolute minimum for the energy $E : \mathfrak{X}^\rho(M) \to \mathbb{R}$ restricted to $\mathfrak{X}^\rho(M)$ if and only if V is parallel.*

Note that a parallel vector field V necessarily has constant length.

2.2 The tension field associated to $V : (M,g) \to (TM,G)$

Let (M,g) be a Riemannian manifold and $V \in \mathfrak{X}(M)$. By definition, the *tension field* associated to the map $V : (M,g) \to (TM,G)$, is given by

$$\begin{array}{rcl} \tau(V) : M & \to & V^{-1}(TTM), \\ p & \mapsto & \mathrm{tr}(\nabla dV)_p. \end{array}$$

The tension field $\tau(V)$ associated to $V : (M,g) \to (TM,G)$ is rather complicated and has been calculated in [1]. It depends on functions determining the Levi-Civita connection of an arbitrary Riemannian g-natural metric G on TM. The explicit expression of $\tau(V)$ is the following:

$$\begin{aligned} \tau_p(V) = \Bigg\{ & -2A_1QV + 2C_1\mathrm{tr}[R(\nabla.V,V)\cdot] + C_3\sum_{i=1}^n e_i(r^2)e_i \\ & + 2C_2\nabla_V V + E_1\sum_{i=1}^n e_i(r^2)\nabla_{e_i}V + \Big[2A_2 - A_3 g(QV,V) + nA_4 \\ & + A_5 r^2 + 2C_4 g(\mathrm{tr}[R(\nabla.V,V)\cdot],V) + 2C_5\mathrm{div}V + C_6V(r^2) \\ & + E_2||\nabla V||^2 + \frac{1}{4}E_3\sum_{i=1}^n [e_i(r^2)]^2\Big] V\Bigg\}^h_p \\ & + \Bigg\{ -\bar{\Delta}V - B_1QV + 2D_1\mathrm{tr}[R(\nabla.V,V)\cdot] + D_3\sum_{i=1}^n e_i(r^2)e_i \\ & + 2D_2\nabla_V V + F_1\sum_{i=1}^n e_i(r^2)\nabla_{e_i}V + \Big[2B_3 - B_4 g(QV,V) + nB_5 \\ & + B_6 r^2 + 2D_4 g(\mathrm{tr}[R(\nabla.V,V)\cdot],V) + 2D_5\mathrm{div}V + D_6V(r^2) \\ & + F_2||\nabla V||^2 + \frac{1}{4}F_3\sum_{i=1}^n [e_i(r^2)]^2\Big] V\Bigg\}^v_p, \end{aligned} \tag{8}$$

where $r = ||V||$, $A_i, \ldots F_i$ are evaluated at r^2 and

$$\bar{\Delta}V = -\mathrm{tr}\nabla^2 V = \sum_i \left(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V\right)$$

is the so called *rough Laplacian* of (M,g) calculated at V. Functions $A_i, \dots F_i$ appearing in (8) come from the Levi-Civita connection of the Riemannian g-natural metric G and depend on α_i, β_i how explained in the following

4 Proposition. **[5]** Let (M,g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its curvature tensor. Let G be a Riemannian g-natural metric on TM. Then the Levi-Civita connection $\bar\nabla$ of (TM,G) is characterized by

$$\begin{array}{ll}
(i) & (\bar\nabla_{X^h}Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u;X_x,Y_x)\} + v\{B(u;X_x,Y_x)\},\\
(ii) & (\bar\nabla_{X^h}Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u;X_x,Y_x)\} + v\{D(u;X_x,Y_x)\},\\
(iii) & (\bar\nabla_{X^v}Y^h)_{(x,u)} = h\{C(u;Y_x,X_x)\} + v\{D(u;Y_x,X_x)\},\\
(iv) & (\bar\nabla_{X^v}Y^v)_{(x,u)} = h\{E(u;X_x,Y_x)\} + v\{F(u;X_x,Y_x)\},
\end{array}$$

for all vector fields X, Y on M and $(x,u) \in TM$. Here, $h\{\cdot\}$ and $v\{\cdot\}$ respectively denote the horizontal and vertical lifts of a vector tangent to M and, for all $x \in M$ and vectors u, X_x, Y_x tangent to M at x, A, B, C, D, E and F are defined as follows:

$$\begin{aligned}
A(u;X_x,Y_x) \quad = \quad & A_1[R_x(X_x,u)Y_x + R_x(Y_x,u)X_x] + A_2[g_x(Y_x,u)X_x\\
& +g_x(X_x,u)Y_x] + A_3 g_x(R_x(X_x,u)Y_x,u)u + A_4 g_x(X_x,Y_x)u\\
& +A_5 g_x(X_x,u)g_x(Y_x,u)u,
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\frac{\alpha_1\alpha_2}{2\alpha},\\
A_2 &= \frac{\alpha_2(\beta_1+\beta_3)}{2\alpha},\\
A_3 &= \frac{\alpha_2\{\alpha_1[\phi_1(\beta_1+\beta_3) - \phi_2\beta_2] + \alpha_2(\beta_1\alpha_2 - \beta_2\alpha_1)\}}{\alpha\phi},\\
A_4 &= \frac{\phi_2(\alpha_1+\alpha_3)'}{\phi},\\
A_5 &= \frac{\alpha\phi_2(\beta_1+\beta_3)' + (\beta_1+\beta_3)\{\alpha_2[\phi_2\beta_2 - \phi_1(\beta_1+\beta_3)]}{\alpha\phi}\\
&\quad + \frac{(\alpha_1+\alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1)\}}{\alpha\phi},
\end{aligned} \tag{9}$$

$$\begin{aligned}
B(u;X_x,Y_x) = \; & B_1 R_x(X_x,u)Y_x + B_2 R_x(X_x,Y_x)u + B_3[g_x(Y_x,u)X_x\\
& +g_x(X_x,u)Y_x] + B_4 g_x(R_x(X_x,u)Y_x,u)u + B_5 g_x(X_x,Y_x)u\\
& +B_6 g_x(X_x,u)g_x(Y_x,u)u,
\end{aligned}$$

where

$$
\begin{aligned}
B_1 &= \frac{\alpha_2^2}{\alpha}, \\
B_2 &= -\frac{\alpha_1(\alpha_1+\alpha_3)}{2\alpha}, \\
B_3 &= -\frac{(\alpha_1+\alpha_3)(\beta_1+\beta_3)}{2\alpha}, \\
B_4 &= \frac{\alpha_2\{\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)(\beta_2\alpha_1-\beta_1\alpha_2)\}}{\alpha\phi}, \\
B_5 &= -\frac{(\phi_1+\phi_3)(\alpha_1+\alpha_3)'}{\phi}, \\
B_6 &= \frac{-\alpha(\phi_1+\phi_3)(\beta_1+\beta_3)'+(\beta_1+\beta_3)\{(\alpha_1+\alpha_3)[(\phi_1+\phi_3)\beta_1-\phi_2\beta_2]}{\alpha\phi} \\
&\quad + \frac{\alpha_2[\alpha_2(\beta_1+\beta_3)-(\alpha_1+\alpha_3)\beta_2]\}}{\alpha\phi},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
C(u;X_x,Y_x) = {} & C_1R(Y_x,u)X_x + C_2g_x(X_x,u)Y_x + C_3g_x(Y_x,u)X_x \\
& +C_4g_x(R_x(X_x,u)Y_x,u)u + C_5g_x(X_x,Y_x)u \\
& +C_6g_x(X_x,u)g_x(Y_x,u)u,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= -\tfrac{\alpha_1^2}{2\alpha}, \\
C_2 &= -\tfrac{\alpha_1(\beta_1+\beta_3)}{2\alpha}, \\
C_3 &= \tfrac{\alpha_1(\alpha_1+\alpha_3)'-\alpha_2(\alpha_2'-\frac{\beta_2}{2})}{\alpha}, \\
C_4 &= \tfrac{\alpha_1\{\alpha_2(\alpha_2\beta_1-\alpha_1\beta_2)+\alpha_1[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]\}}{2\alpha\phi}, \\
C_5 &= \tfrac{\phi_1(\beta_1+\beta_3)+\phi_2(2\alpha_2'-\beta_2)}{2\phi}, \\
C_6 &= \tfrac{\alpha\phi_1(\beta_1+\beta_3)'+\{\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)+\alpha_1[\phi_2\beta_2-(\beta_1+\beta_3)\phi_1]\}[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}]}{\alpha\phi} \\
&\quad + \tfrac{\{\alpha_2[\beta_1(\phi_1+\phi_3)-\beta_2\phi_2]-\alpha_1[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]\}(\alpha_2'-\frac{\beta_2}{2})}{\alpha\phi}
\end{aligned}
$$

$$
\begin{aligned}
D(u;X_x,Y_x) = {} & D_1R_x(Y_x,u)X_x + D_2g_x(X_x,u)Y_x + D_3g_x(Y_x,u)X_x \\
& +D_4g_x(R_x(X_x,u)Y_x,u)u + D_5g_x(X_x,Y_x)u \\
& +D_6g_x(X_x,u)g_x(Y_x,u)u,
\end{aligned}
$$

where

$$D_1 = \frac{\alpha_1\alpha_2}{2\alpha},$$
$$D_2 = \frac{\alpha_2(\beta_1+\beta_3)}{2\alpha},$$
$$D_3 = \frac{-\alpha_2(\alpha_1+\alpha_3)'+(\alpha_1+\alpha_3)(\alpha_2'-\frac{\beta_2}{2})}{\alpha},$$
$$D_4 = \frac{\alpha_1\{(\alpha_1+\alpha_3)(\alpha_1\beta_2-\alpha_2\beta_1)+\alpha_2[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]\}}{2\alpha\phi}$$
$$D_5 = -\frac{\phi_2(\beta_1+\beta_3)+(\phi_1+\phi_3)(2\alpha_2'-\beta_2)}{2\alpha\phi},$$
$$D_6 = \frac{-\alpha\phi_2(\beta_1+\beta_3)'+\{(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)+\alpha_2[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]\}[(\alpha_1+\alpha_3)'+\frac{\beta_1+\beta_3}{2}]}{\alpha\phi}$$
$$+\frac{\{(\alpha_1+\alpha_3)[\beta_2\phi_2-\beta_1(\phi_1+\phi_3)]+\alpha_2[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]\}(\alpha_2'-\frac{\beta_2}{2})}{\alpha\phi}$$

$$E(u;X_x,Y_x) = E_1[g_x(Y_x,u)X_x + g_x(X_x,u)Y_x] + E_2 g_x(X_x,Y_x)u$$
$$+E_3 g_x(X_x,u)g_x(Y_x,u)u$$

where

$$E_1 = \frac{\alpha_1(\alpha_2'+\frac{\beta_2}{2})-\alpha_2\alpha_1'}{\alpha},$$
$$E_2 = \frac{\phi_1\beta_2-\phi_2(\beta_1-\alpha_1')}{\phi},$$
$$E_3 = \frac{\alpha(2\phi_1\beta_2'-\phi_2\beta_1')+2\alpha_1'\{\alpha_1[\alpha_2(\beta_1+\beta_3)-\beta_2(\alpha_1+\alpha_3)]+\alpha_2[\beta_1(\phi_1+\phi_3)-\beta_2\phi_2]\}}{\alpha\phi}$$
$$+\frac{(2\alpha_2'+\beta_2)\{\alpha_1[\phi_2\beta_2-\phi_1(\beta_1+\beta_3)]+\alpha_2(\alpha_1\beta_2-\alpha_2\beta_1)\}}{\alpha\phi}$$

$$F(u;X_x,Y_x) = F_1[g_x(Y_x,u)X_x + g_x(X_x,u)Y_x] + F_2 g_x(X_x,Y_x)u$$
$$+F_3 g_x(X_x,u)g_x(Y_x,u)u$$

where

$$F_1 = \frac{-\alpha_2(\alpha_2'+\frac{\beta_2}{2})+(\alpha_1+\alpha_3)\alpha_1'}{\alpha},$$
$$F_2 = \frac{(\phi_1+\phi_3)(\beta_1-\alpha_1')-\phi_2\beta_2}{\phi},$$
$$F_3 = \frac{\alpha[(\phi_1+\phi_3)\beta_1'-2\phi_2\beta_2']+2\alpha_1'\{\alpha_2[\beta_2(\alpha_1+\alpha_3)-\alpha_2(\beta_1+\beta_3)]+(\alpha_1+\alpha_3)[\beta_2\phi_2-\beta_1(\phi_1+\phi_3)]\}}{\alpha\phi}$$
$$+\frac{(2\alpha_2'+\beta_2)\{\alpha_2[\phi_1(\beta_1+\beta_3)-\phi_2\beta_2]+(\alpha_1+\alpha_3)(\alpha_2\beta_1-\alpha_1\beta_2)\}}{\alpha\phi}.$$

Now, a vector field $V \in \mathfrak{X}(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if $\tau_h(V) = \tau_v(V) = 0$, that is, the horizontal and vertical components of the tension field associated to $V : (M, g) \to (TM, G)$ vanish.

In some special cases, it is easy to rewrite $\tau_h(V)$ and $\tau_v(V)$ more explicitly in terms of functions α_i, β_i which determine G. When $G = g^s$ is the Sasaki metric, we obtain as a very particular case the following well known result:

5 Theorem ([10])**.** *$V : (M, g) \to (TM, g^s)$ is a harmonic map if and only if*

(i) $\mathrm{tr}[R(\nabla . V, V)\cdot] = 0$ *and*

(ii) $\bar{\Delta} V = 0$.

In particular, when M is compact, integrating condition *(ii)* in Theorem 5 one gets at once that $V : (M, g) \to (TM, g^s)$ is a harmonic map if and only if $\nabla V = 0$, that is, V is a parallel vector field.

We applied the expression of $\tau_h(V)$ and $\tau_v(V)$ given in (8) to investigate relationships between harmonicity of maps defined by some special vector fields and properties of g-natural metrics. In particular, for parallel vector fields we obtained the following

6 Theorem. *A parallel vector field V defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if its constant length $\rho = ||V||^2$ is a critical point of the function*

$$(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3. \tag{11}$$

In particular:

(i) For any Riemannian g-natural metric G on TM such that $(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3$ is constant, all parallel vector fields define harmonic maps from (M, g) to (TM, G).

(ii) For any Riemannian g-natural metric G on TM such that $[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(t) \neq 0$ for all t, parallel vector fields do not define harmonic maps from (M, g) to (TM, G).

By (3) it easily follows that case *(i)* of Theorem 6 applies to g^s.

Except when V is parallel, equations $\tau_h(V) = 0$ and $\tau_v(V) = 0$ remain rather difficult to manage in full generality, even for vector fields of constant length. For this reason, we considered the special case of a Riemannian g-natural metric G for which $\alpha_2(\rho) = \beta_2(\rho) = 0$. Note that $\alpha_2 = \beta_2 = 0$ has a clear geometric meaning, since it characterizes g-natural metrics on TM with respect to which horizontal and vertical distributions are mutually orthogonal. Under this assumption, writing down the horizontal and vertical components of the tension field, we respectively get

$$\begin{aligned}&\frac{\alpha_1}{2(\alpha_1+\alpha_3)}(\rho)\mathrm{tr}[R(\nabla.V,V)\cdot]+\frac{\beta_1+\beta_3}{2(\alpha_1+\alpha_3)}(\rho)\nabla_V V\\&-\left[\frac{\alpha_1(\beta_1+\beta_3)}{2(\alpha_1+\alpha_3)(\phi_1+\phi_3)}(\rho)g(\mathrm{tr}[R(\nabla.V,V)\cdot],V)+\frac{\beta_1+\beta_3}{2(\phi_1+\phi_3)}(\rho)\mathrm{div}V\right]V=0\end{aligned}\tag{12}$$

and

$$\begin{aligned}\bar{\Delta}V+\Bigg(\frac{(\beta_1+\beta_3)}{\alpha_1}(\rho)+n\frac{(\alpha_1+\alpha_3)'}{\phi_1}(\rho)+\rho\frac{\alpha_1(\beta_1+\beta_3)'-\beta_1(\beta_1+\beta_3)}{\alpha_1\phi_1}(\rho)\\+\frac{\alpha_1'-\beta_1}{\phi_1}(\rho)||\nabla V||^2\Bigg)V=0.\end{aligned}\tag{13}$$

In particular, (13) implies at once that $\bar{\Delta}V$ is collinear with V. Therefore, V is an eigenvector for the rough Laplacian $\bar{\Delta}$ and, since $\sqrt{\rho}=||V||$ is a constant, we have $\bar{\Delta}V=\frac{1}{\rho}||\nabla V||^2V$ and (13) implies

$$\left(\frac{1}{\rho}\alpha_1+\alpha_1'\right)(\rho)||\nabla V||^2+[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho)=0.\tag{14}$$

Now, by (12) and (13) it follows that very different behaviours occur for different Riemannian g-natural metrics, concerning the harmonicity of vector fields of constant length. The results are resumed in the following

7 Theorem. *Let (M,g) be a Riemannian manifold and G a Riemannian g-natural metric on TM satisfying $\alpha_2(\rho)=\beta_2(\rho)=0$, where $\rho>0$. Then, a vector field $V\in\mathfrak{X}^\rho(M)$ defines a harmonic map $V:(M,g)\to(TM,G)$ if and only if it satisfies (12) and (13). In particular:*

(i) If $\left(\frac{1}{\rho}\alpha_1+\alpha_1'\right)(\rho)=[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho)=0$, then $V:(M,g)\to(TM,G)$ is a harmonic map if and only if V is an eigenvector of $\bar{\Delta}$ and (12) holds.

(ii) If $\left(\frac{1}{\rho}\alpha_1+\alpha_1'\right)(\rho)\neq 0=[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho)$, then $V:(M,g)\to(TM,G)$ is a harmonic map if and only if V is parallel.

(iii) If $\left(\frac{1}{\rho}\alpha_1+\alpha_1'\right)(\rho)=0\neq[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho)$, then no vector fields $V\in\mathfrak{X}^\rho(M)$ define harmonic maps from (M,g) to $(TM,\bar{G})$.

(iv) If $\left(\frac{1}{\rho}\alpha_1+\alpha_1'\right)(\rho)\neq 0\neq[(n-1)(\alpha_1+\alpha_3)+\phi_1+\phi_3]'(\rho)$, then $V:(M,g)\to(TM,G)$ is a harmonic map if and only if (12) holds, $\bar{\Delta}V$ is collinear to V and (14) holds.

The point we want to stress is that since a general Riemannian g-natural metric G depends on six different smooth functions α_i, β_i (satisfying inequalities (2)), *in each of cases (i)-(iv) listed in Theorem* 7*, there are plenty of Riemannian g-natural metrics which furnish examples.* Some explicit examples have been illustrated in [1]. Such a variety of different behaviours is rather surprising, when compared with the fact that the tension field of $V : (M, g) \to (TM, g^s)$ vanishes only when V is parallel [10].

2.3 Critical points for the energy restricted to vector fields

Given a compact Riemannian manifold (M, g), we want to investigate conditions under which the map $V : (M, g) \to (TM, G)$ associated to a vector field $V \in \mathfrak{X}(M)$, is a critical point for the energy functional $E : \mathfrak{X}(M) \to \mathbb{R}$, that is, only considering variations among maps defined by vector fields. Gil-Medrano solved the corresponding problem when TM is equipped with the Sasaki metric, proving the following

8 Theorem ([9])**.** *$V : (M, g) \to (TM, g^s)$ is a critical point for the energy functional $E : \mathfrak{X}(M) \to \mathbb{R}$ if and only if V is parallel.*

We equipped TM with an arbitrary Riemannian g-natural metric G and considered a vector field $V \in \mathfrak{X}(M)$ and an arbitrary smooth variation $\{V_t\} \subset \mathfrak{X}(M)$ of V, with $|t| < \varepsilon$ and $V_0 = V$. We proved that V is a critical point for $E : \mathfrak{X}(M) \to \mathbb{R}$ if and only if

$$T(V) := \alpha_2 \tau_h(V) + \beta_2 g(\tau_h(V), V)V + \alpha_1 \tau_v(V) + \beta_1 g(\tau_v(V), V)V = 0, \quad (15)$$

where, following the notations introduced in the previous subsection, $\tau_h(V)$ and $\tau_v(V)$ respectively denote the horizontal and vertical components of the tension field associated to $V : (M, g) \to (TM, G)$.

It can be noted that equation (15) has a clear geometric meaning. In fact, it expresses the *vanishing of the projection of the tension field $\tau(V)$, made with respect to the Riemannian g-natural metric G, into the vertical distribution.*

Clearly, if $V : (M, g) \to (TM, G)$ is a harmonic map, then $\tau_h(V) = \tau_v(V) = 0$ and so, (15) holds. It is also worthwhile to emphasize that in the special situation when $\alpha_2 = \beta_2 = 0$, $T(V) = 0$ is equivalent to requiring that $\tau_v(V) = 0$.

Since the critical point condition $T(V) = 0$ has a tensorial character, it also makes sense when (M, g) is not compact. Hence, we can give the following

9 Definition. Let (M, g) be a Riemannian manifold. A vector field V on M is said to be $\mathfrak{X}$*-harmonic* (with respect to a fixed Riemannian g-natural metric G on TM) if and only if it satisfies $T(V) = 0$.

If $G = g^s$, then $T(V) = 0$ reduces to the well known formula $\bar{\Delta}V = 0$ and this easily implies Theorem 8.

We now determine $\mathfrak{X}$-harmonic vector fields, under some special assumptions either on the vector fields or on the Riemannian g-natural metric G.

Suppose first $V \in \mathfrak{X}(M)$ is a parallel vector field. Then,

$$T(V) = -[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho)\, V.$$

Hence, $T(V) = 0$ coincides with the necessary and sufficient condition we found in Theorem 6 for the harmonicity of $V : (M, g) \to (TM, G)$, and in Subsection 3 for critical points of the energy E restricted to parallel vector fields. Therefore, we obtain the following

10 Theorem. *Let (M, g) be a Riemannian manifold and G any Riemannian g-natural metric on TM. For a parallel vector field V on M, the following statements are equivalent:*

(a) $V : (M, g) \to (TM, G)$ is a harmonic map;

(b) V is $\mathfrak{X}$-harmonic;

(c) V is a critical point for E in the set of all parallel vector fields on M;

(d) $\rho = ||V||^2$ is a critical point for the function $[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]$.

Theorem 10 includes as special case the Sasaki metric g^s, for which (d) is trivially satisfied and so, all parallel vector fields define harmonic maps.

Consider now a vector field $V \in \mathfrak{X}^\rho(M)$. Then, $T(V) = 0$ if and only if

$$\begin{aligned} &\alpha_1(\rho)\bar{\Delta}V + \left\{\beta_1(\rho)g(\bar{\Delta}V, V) + [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho)\right. \\ &\left. +(2\alpha_2' - \beta_2)(\rho)\mathrm{div}V + (\alpha_1' - \beta_1)(\rho)||\nabla V||^2\right\} V = 0. \end{aligned} \tag{16}$$

By (16) it follows at once that $\bar{\Delta}V$ is collinear to V. Therefore, since V has constant length $||V|| = \sqrt{\rho}$, we have $\bar{\Delta}V = \frac{1}{\rho}||\nabla V||^2 V$ and from (16) we get

$$\begin{aligned} &\left(\frac{1}{\rho}\alpha_1 + \alpha_1'\right)(\rho)||\nabla V||^2 + (2\alpha_2' - \beta_2)(\rho)\mathrm{div}V \\ &\qquad + [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0. \end{aligned} \tag{17}$$

Thus, we have the following

11 Theorem. *Let (M, g) be a Riemannian manifold and G any Riemannian g-natural metric on TM. A vector field $V \in \mathfrak{X}^\rho(M)$ is $\mathfrak{X}$-harmonic if and only if $\bar{\Delta}V$ is collinear to V and (17) holds.*

Equation (17) is difficult to solve in full generality. In a special case, requiring $\alpha_2(\rho) = \beta_2(\rho) = 0$, we proved the following

12 Proposition. Let (M, g) be a Riemannian manifold and $V \in \mathfrak{X}^\rho(M)$. For any Riemannian g-natural metric G on TM, satisfying $\alpha_2(\rho) = \alpha_2'(\rho) = \beta_2(\rho) = 0$,

(i) V is $\mathfrak{X}$-harmonic if and only if (13) holds.

(ii) V defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if it is $\mathfrak{X}$-harmonic and satisfies (12).

Proposition 12 shows explicitly that for a well determined and wide class of Riemannian g-natural metrics, $\mathfrak{X}$-harmonic vector fields do not necessarily define harmonic maps (as it occurs for the Sasaki metric).

2.4 Applications to Reeb and Hopf vector fields

In some cases, a distinguished vector field plays a special role in describing the geometry of a Riemannian manifold. This occurs for the Reeb vector field of a contact metric manifold and, as a very special case, for Hopf vector fields on the unit sphere. We now investigate the harmonicity of these vector fields with respect to Riemannian g-natural metrics on the tangent bundle.

We briefly recall that given a smooth manifold M of odd dimension $n = 2m + 1$, a *contact structure* (η, φ, ξ) over M is composed by a global 1-form η (the *contact form*) such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on M, a global vector field ξ (the *Reeb* or *characteristic vector field*) and a global tensor φ, of type (1,1), such that

$$\eta(\xi) = 1\,, \quad \varphi\xi = 0\,, \quad \eta\varphi = 0\,, \quad \varphi^2 = -I + \eta \otimes \xi\,. \tag{18}$$

A Riemannian metric g is said to be *associated* to the contact structure (η, φ, ξ), if it satisfies

$$\eta = g(\xi, \cdot)\,, \quad d\eta = g(\cdot, \varphi\cdot)\,, \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot)\,. \tag{19}$$

We denote a contact metric manifold by (M, η, g) or $(M, \eta, g, \xi, \varphi)$. By (18) and (19) it follows at once that $\xi \in \mathfrak{X}^1(M)$. For further details, references and information about contact metric manifolds, we can refer to [6].

In [15], D. Perrone introduced and studied *H-contact spaces*. They are contact metric manifolds $(M, \eta, g, \xi, \varphi)$, whose Reeb vector field ξ is a critical point for the energy functional E restricted to the space $\mathfrak{X}^1(M)$ of all unit vector fields on (M, g), considered as smooth maps from (M, g) into the *unit tangent sphere bundle* T^1M, equipped with the Riemannian metric induced on T^1M by the Sasaki metric g^s of TM. In particular, the following characterization was proved:

13 Theorem ([15]). *$(M, \eta, g, \xi, \varphi)$ is H-contact if and only if ξ is an eigenvector of the Ricci operator.*

By Theorem 13 it follows that the class of H-contact manifolds is very large. In fact, several well studied classes of contact metric manifolds, like η-Einstein spaces, K-contact spaces, (k, μ)-spaces and strongly locally ϕ-symmetric spaces are all examples of H-contact spaces. Using the special properties of the Reeb vector field ξ and the expression (8) for the tension field, we can derive the following necessary condition for the harmonicity of ξ:

14 Theorem. *Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold and G an arbitrary Riemannian g-natural metric on TM. If $\xi : (M, g) \to (TM, G)$ is a harmonic map, then (M, η, g) is H-contact.*

Under some assumptions on the Riemannian g-natural metric G, we are able to completely characterize harmonicity of $\xi : (M, g) \to (TM, G)$. In particular, the following result holds:

15 Theorem. *Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold and G any Riemannian g-natural metric on TM, satisfying $\alpha_2(1) = \beta_2(1) = 0$. Then ξ defines a harmonic map $\xi : (M, g) \to (TM, G)$ if and only if M is H-contact,* $\mathrm{tr}[R(\nabla.\xi, \xi)\cdot] = 0$ *and*

$$(\mathrm{tr}h^2 + 2m)\,(\alpha_1 + \alpha_1')(1) + [2m(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(1) = 0, \qquad (20)$$

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ is the Lie derivative of φ.

It is interesting to compare this result with the well known characterization of unit vector fields U defining a harmonic map $U : (M, g) \to (T^1M, g^s)$, obtained by Han and Yim:

16 Theorem ([11]). *A unit vector field U defines a harmonic map U : $(M, g) \to (T^1M, g^s)$ if and only if*

(i) $\bar{\Delta}U$ is collinear to U, and

(ii) $\mathrm{tr}[R(\nabla.U, U)\cdot] = 0$.

Therefore, by Theorems 15 and 16, for the wide class of Riemannian g-natural metrics satisfying $\alpha_2(1) = \beta_2(1) = 0$ and (20), we obtain at once the following

17 Corollary. *Let G be any Riemannian g-natural metric on TM, satisfying $\alpha_2(1) = \beta_2(1) = 0$ and (20). Then, $\xi : (M, g) \to (TM, G)$ is a harmonic map if and only if $\xi : (M, g) \to (T^1M, g^s)$ is a harmonic map.*

Sasakian manifolds are probably the most known and investigated class of contact metric manifolds. It is then natural to investigated harmonicity of the map defined by the Reeb vector field ξ of a Sasakian manifold $(M, \eta, g, \xi, \varphi)$. We proved the following result:

18 Theorem. *Let $(M, \eta, g, \xi, \varphi)$ be a Sasakian manifold, $dimM = 2m+1$ and G any Riemannian g-natural metric on TM, satisfying $\alpha_2(1) = \beta_2(1) = 0$. Then, ξ defines a harmonic map $\xi : (M, g) \to (TM, G)$ if and only if*

$$2m(\alpha_1 + \alpha_1')(1) + [2m(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(1) = 0. \tag{21}$$

As concerns $\mathfrak{X}$-harmonicity of the Reeb vector field, calculating $T(\xi)$ we obtain:

19 Theorem. *Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold and G an arbitrary Riemannian g-natural metric on TM. If ξ is $\mathfrak{X}$-harmonic, then M is H-contact. Conversely, if M is H-contact, then ξ is $\mathfrak{X}$-harmonic if and only if* (20) *holds.*

Note that (20) is not fulfilled by the Sasaki metric on TM, as it follows from (3). Therefore, we have the following

20 Corollary. *When $(M, \eta, g, \xi, \varphi)$ is an arbitrary contact metric manifold and TM is equipped with g^s, then the Reeb vector field ξ is never $\mathfrak{X}$-harmonic. In particular, $\xi : (M, g) \to (TM, g^s)$ is never a harmonic map.*

On the other hand, it is easy to exhibit examples of Riemannian g-natural metrics which satisfy (20). For example, (20) holds for all Riemannian g-natural metrics belonging to the two-parameters family satisfying

$$\begin{cases} \alpha_1(t) = k_1 e^{-t}, \\ \alpha_3(t) = k_2 - \alpha_1(t), \\ \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0, \end{cases}$$

where k_1, k_2 are positive constants.

Next, we recall that *Hopf vector fields* on the unit sphere S^{2m+1}, equipped with its canonical metric g_0, are all and the ones Killing unit vector fields on S^{2m+1} [17]. Moreover, a Hopf vector field $\bar{\xi}$ can always be considered as the Reeb vector field of a suitable Sasakian structure $(S^{2m+1}, \bar{\eta}, g_o, \bar{\xi}, \bar{\varphi})$, where $\bar{\eta} = g_0(\cdot, \bar{\xi})$ and $\bar{\varphi} = -\nabla\bar{\xi}$. Rewriting results above for these special contact metric structures, we then have

21 Corollary. *For all Riemannian g-natural metrics on TS^{2m+1}, satisfying $\alpha_2(1) = \beta_2(1) = 0$, a Hopf vector field $\bar{\xi}$ defines a harmonic map $\bar{\xi} : (S^{2m+1}, g_0) \to (TS^{2m+1}, G)$ if and only if* (21) *holds.*

22 Corollary. *For all Riemannian g-natural metrics on TS^{2m+1}, a Hopf vector field $\bar{\xi}$ is $\mathfrak{X}$-harmonic if and only if* (21) *holds.*

Comparing Corollaries 21 and 22 we can conclude that even for Hopf vector fields, in general $\mathfrak{X}$-harmonicity is not sufficient for the harmonicity of the corresponding map.

3 Riemannian g-natural metrics on T_1M

We briefly recall that the *tangent sphere bundle of radius* $r > 0$ over a Riemannian manifold (M, g), is the hypersurface

$$T_rM = \{(x, u) \in TM | g_x(u, u) = r^2\}.$$

The tangent space of T_rM, at a point $(x, u) \in T_rM$, is given by

$$(T_rM)_{(x,u)} = \{X^h + Y^v : X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \quad (22)$$

When $r = 1$, T_1M is called *the unit tangent (sphere) bundle.*

We call *g-natural metrics on T_1M* the restrictions of g-natural metrics of TM to its hypersurface T_1M. These metrics possess a simpler form. Precisely, at it was shown in [4], every Riemannian g-natural metric $\widetilde{G}$ on T_1M is necessarily induced by a Riemannian g-natural G on TM of the special form

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (a + c)\, g_x(X, Y) + \beta\, g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = b\, g_x(X, Y), \\ G_{(x,u)}(X^v, Y^v) = a\, g_x(X, Y), \end{cases} \quad (23)$$

for three real constants a, b, c and a smooth function $\beta : [0, \infty) \to \mathbb{R}$. It is easily seen that G is obtained by the general expression (1), when

$$\alpha_1 = a, \quad \alpha_2 = b, \quad \alpha_3 = c, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 = \beta, \quad (24)$$

Such a metric $\widetilde{G}$ on T_1M only depends on the value $d := \beta(1)$ of β at 1. ¿From (2) and (24) it follows that $\widetilde{G}$ is Riemannian if and only if

$$a > 0, \quad \alpha := a(a + c) - b^2 > 0 \quad \text{and} \quad \phi := a(a + c + d) - b^2 > 0. \quad (25)$$

Notice that the Sasaki metric $\tilde{g}^s$ belongs to the class of Riemannian g-natural metrics on T_1M and satisfies $b = 0$.

By a simple calculation, using the Schmidt's orthonormalization process, it is easy to check that the vector field on TM defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a + c + d)\phi}} [-b.u^h + (a + c + d).u^v], \quad (26)$$

for all $(x, u) \in TM$, is unit normal at any point of T_1M.

We now define the "tangential lift" X^{t_G}, with respect to G, of a vector $X \in M_x$ to $(x, u) \in T_1M$, as the tangential projection of the vertical lift of X to (x, u) with respect to N^G, that is,

$$X^{t_G} = X^v - G_{(x,u)}(X^v, N^G_{(x,u)})\; N^G_{(x,u)} = X^v - \sqrt{\frac{\phi}{a + c + d}} g_x(X, u)\; N^G_{(x,u)}. \quad (27)$$

If $X \in M_x$ is orthogonal to u, then $X^{t_G} = X^v$.

The tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^{t_G}, where X, $Y \in M_x$. Using this fact, the Riemannian metric $\widetilde{G}$ on T_1M, induced from G, is completely determined by the identities

$$\begin{cases} \widetilde{G}_{(x,u)}(X^h, Y^h) &= (a+c)\, g_x(X,Y) + d\, g_x(X,u) g_x(Y,u), \\ \widetilde{G}_{(x,u)}(X^h, Y^{t_G}) &= b\, g_x(X,Y), \\ \widetilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) &= a\, g_x(X,Y) - \frac{\phi}{a+c+d}\, g_x(X,u) g_x(Y,u), \end{cases} \tag{28}$$

for all $(x,u) \in T_1M$ and $X, Y \in M_x$. It should be noted that, by (28), horizontal and vertical lifts are orthogonal with respect to $\widetilde{G}$ if and only if $b = 0$.

4 Harmonicity properties of $V : (M,g) \to (T_1M, \widetilde{G})$

4.1 The energy of a unit vector field

A unit vector field V also defines a map $V : (M,g) \to (T_1M, \widetilde{G})$, where $\widetilde{G}$ is the metric on T_1M induced by G. However, since $(T_1M, \widetilde{G})$ is isometrically immersed into (TM, G), the energy density of $V : (M,g) \to (T_1M, \widetilde{G})$ coincides with the one of $V : (M,g) \to (TM,G)$. The general expression for the energy density of $V : (M,g) \to (TM,G)$ is given by (5). Let now $\widetilde{G}$ be an arbitrary Riemannian g-natural metric on T_1M. Hence, $\widetilde{G}$ is described by (28), where a, b, c and d are four real numbers, satisfying (25). $\widetilde{G}$ is induced by a Riemannian g-natural metric G on TM which can be chosen of the special form (23). With respect to this particular G, (5) becomes

$$2e(V) = n(a+c) + d + a\,||\nabla V||^2 + 2b\,\mathrm{div}V \tag{29}$$

and integrating over M, we get

$$E(V) = \frac{1}{2}[n(a+c)+d] \cdot \mathrm{vol}(M,g) + \frac{a}{2} \cdot \int_M ||\nabla V||^2 dv_g. \tag{30}$$

Since $a > 0$, (30) implies that

$$E(V) \geq \frac{1}{2}[n(a+c)+d] \cdot \mathrm{vol}(M,g) = \frac{1}{2}[(n-1)(a+c)+a+c+d] \cdot \mathrm{vol}(M,g) > 0, \tag{31}$$

for all $V \in \mathfrak{X}^1(M)$. The equality holds in (31) if and only if V is parallel. Therefore, we have the following

23 Theorem. *Let (M,g) be a compact Riemannian manifold. Equipping T_1M with an arbitrary Riemannian g-natural metric $\widetilde{G}$, a unit vector field V is an absolute minimum for the energy $E : \mathfrak{X}^1(M) \to \mathbb{R}$ restricted to $\mathfrak{X}^1(M)$ if and only if V is parallel.*

4.2 Critical points for the energy restricted to unit vector fields

Given a compact Riemannian manifold (M, g), a vector field $V \in \mathfrak{X}^1(M)$ is said to be *harmonic* if and only if the corresponding map $V : (M, g) \to (T_1M, \widetilde{G})$ is a critical point for the energy functional $E : \mathfrak{X}^1(M) \to \mathbb{R}$, that is, only considering variations among maps defined by unit vector fields.

The critical point condition for $E : \mathfrak{X}^1(M) \to \mathbb{R}$ can be deduced directly from (7) (we may also refer to [2] for a direct proof). The conclusion is resumed in the following

24 Theorem. *Let (M, g) be a compact Riemannian manifold. When T_1M is equipped with an arbitrary Riemannian g-natural metric $\widetilde{G}$, a unit vector field V is harmonic if and only if $\bar{\Delta}V$ is collinear to V.*

In literature, critical points of $E : \mathfrak{X}^1(M) \to \mathbb{R}$ have been already investigated, considering T_1M equipped with the Sasaki metric $\widetilde{g}^s$ [16],[18]. Theorem 24 extends the same characterization to an arbitrary Riemannian g-natural metric $\widetilde{G}$. It is important to stress that the collinearity of $\bar{\Delta}V$ and V *only depends on g and not on the particular Riemannian g-natural metric $\widetilde{G}$ on T_1M*. In particular, V is harmonic when T_1M is equipped with any $\widetilde{G}$ if and only if so is when T_1M is equipped with $\widetilde{g}^s$.

4.3 The tension field associated to $V : (M, g) \to (T_1M, \widetilde{G})$

Taking into account the form of the Levi-Civita connection of (TM, G) when G is a special Riemannian g-natural metric satisfying (23), via standard calculations one obtains the following

25 Theorem. *Let (M, g) be a compact Riemannian manifold and $V \in \mathfrak{X}^1(M)$ a unit vector field. When TM is equipped with a special Riemannian g-natural metric G satisfying* (23)*, the tension field $\tau(V)$ of $V : (M, g) \to (TM, G)$ is given by*

$$\tau(V)(x) = (\tau_h(V)(x))^h + (\tau_v(V)(x))^v, \tag{32}$$

with

$$\tau_h(V) = \frac{ab}{\alpha}QV - \frac{a^2}{\alpha}S(V) - \frac{ad}{\alpha}\nabla_V V + \left[\frac{bd}{\alpha} - \frac{a^2bd}{\alpha\phi}\varrho(V,V) \right.$$
$$\left. + \frac{\alpha b\beta' - abd^2}{\alpha\phi} + \frac{a^3d}{\alpha\phi}g(S(V),V) + \frac{ad}{\phi}\mathrm{div}V\right] V, \tag{33}$$

$$\tau_v(V) = -\bar{\Delta}V - \frac{b^2}{\alpha}QV + \frac{ab}{\alpha}S(V) + \frac{bd}{\alpha}\nabla_V V + \Big[-\frac{(a+c)d}{\alpha} + \frac{ab^2 d}{\alpha\phi}\varrho(V,V) + \frac{b^2d^2 - \alpha\varphi\beta'}{\alpha\phi} - \frac{a^2bd}{\alpha\phi}g(S(V),V) - \frac{bd}{\phi}\mathrm{div}V \Big] V, \tag{34}$$

where we put $\varphi = a+c+d$ and β' is evaluated at 1. Then, $V : (M,g) \to (TM,G)$ is a harmonic map if and only if $\tau(V) = 0$, that is, $\tau_h(V) = \tau_v(V) = 0$.

26 Remark. Since the condition $\tau(V) = 0$ has a tensorial character, as usual wit can be assumed as a *definition* of harmonic maps even when M is not compact, and Theorem 25 extends at once to the non-compact case.

We now denote by $\tau_1(V)$ the tension field associated to $V : (M,g) \to (T_1M, \widetilde{G})$. Since $(T_1M, \widetilde{G})$ is isometrically immersed into (TM,G) via the inclusion, $\tau_1(V)$ is nothing but the tangential projection of $\tau(V)$ on T_1M. Explicitly, we obtained in [2] the following

27 Theorem. *Let (M,g) be a compact Riemannian manifold and $V \in \mathfrak{X}^1(M)$ a unit vector. When T_1M is equipped with an arbitrary Riemannian g-natural metric $\widetilde{G}$, the tension field $\tau_1(V)$ of $V : (M,g) \to (T_1M, \widetilde{G})$ is given by*

$$\tau_1(V)(x) = (\tau_{1h}(V)(x))^h + (\tau_{1v}(V)(x))^v \tag{35}$$

where

$$\tau_{1h}(V) = \frac{ab}{\alpha}QV - \frac{a^2}{\alpha}S(V) - \frac{ad}{\alpha}\nabla_V V + \Big[-\frac{b(ad+b^2)}{\alpha\varphi}\varrho(V,V) - \frac{b}{\varphi}g(\bar{\Delta}V,V) + \frac{d}{\varphi}\mathrm{div}V + \frac{a(ad+b^2)}{\alpha\varphi}g(S(V),V) \Big] V, \tag{36}$$

$$\tau_{1v}(V) = -\bar{\Delta}V - \frac{b^2}{\alpha}QV + \frac{ab}{\alpha}S(V) + \frac{bd}{\alpha}\nabla_V V + \Big[\frac{b^2}{\alpha}\varrho(V,V) + g(\bar{\Delta}V,V) - \frac{ab}{\alpha}g(S(V),V) \Big] V. \tag{37}$$

Thus, $V : (M,g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if $\tau_{1h}(V) = \tau_{1v}(V) = 0$.

Because of the tensorial character of the condition $\tau_1(V) = 0$, we can use it as a *definition* of harmonic maps $(M,g) \to (T_1M, \widetilde{G})$ even when M is not compact.

We can derive by $\tau_{1h}(V) = \tau_{1v}(V) = 0$ a set of equivalent conditions, which permit a better understanding of the geometrical meaning of these equations, proving the following

28 Theorem. *Let (M,g) be a Riemannian manifold, V a unit vector field and $\widetilde{G}$ an arbitrary Riemannian g-natural metric on T_1M. Then, $V : (M,g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if V is a harmonic vector field and*

$$bQV - aS(V) = \left\{ b\,||\nabla V||^2 - d\,\mathrm{div}V \right\} V + d\,\nabla_V V. \tag{38}$$

Under some special assumptions, condition (38) simplifies. For example, the following result holds:

29 Corollary. *Let (M,g) be a Riemannian manifold and $V \in \mathfrak{X}^1(M)$ a unit vector. Suppose T_1M is equipped with a Riemannian g-natural metric $\widetilde{G}$ such that horizontal and tangential distributions are orthogonal. Then, $V : (M,g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if*

(i) $S(V) = \frac{d}{a}\{(\mathrm{div}V)V - \nabla_V V\}$, *and*

(ii) $\bar{\Delta}V$ *is collinear to* V.

In the special case of the Sasaki metric $\widetilde{g^s}$, by Theorem 28 we obtain at once the already cited Theorem 16 by Han and Yim.

Moreover, Theorem 28 permits to extend the characterization given in the case of $\widetilde{g^s}$ to a two-parameter family of Riemannian g-natural metrics on T_1M (including $\widetilde{g^s}$). In fact, we have at once the following

30 Corollary. *Let (M,g) be a Riemannian manifold and $V \in \mathfrak{X}^1(M)$. If T_1M is equipped with a Riemannian g-natural metric $\widetilde{G}$ with $b = d = 0$, then $V : (M,g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if*

(i) $S(V) = 0$, *and*

(ii) $\bar{\Delta}V$ *is collinear to* V.

4.4 Unit Killing vector fields characterized in terms of harmonicity

As we shall see, natural harmonicity permits to characterize unit Killing vector fields over a Riemannian manifold (M,g). It is well known that $V \in \mathfrak{X}(M)$ is a *Killing vector field* if and only if the local one-parameter group of V consists of local isometries of g. Moreover, a vector field V is Killing if and only if $\mathfrak{L}_V g = 0$, where $\mathfrak{L}$ denotes the Lie derivative. As the definition already shows, Killing vector fields are intimately related to the Riemannian metric g. They have been extensively studied by many authors, and have shown several interesting applications. In [2], we provided a characterization of a unit Killing vector field V, in terms of harmonicity of the map $V : (M, g) \to (T_1M, \tilde{G})$, by proving the following

31 Theorem. *A unit vector field V is Killing if and only if the harmonicity of the map $V : (M, g) \to (T_1M, \widetilde{G})$ depends only on g and not on the particular Riemannian g-natural metric $\widetilde{G}$ on T_1M.*

If $V \in \mathfrak{X}^1(M)$ is not a Killing vector field, then the harmonicity of the map $V : (M, g) \to (T_1M, \widetilde{G})$ explicitly depends on the choice of the Riemannian g-natural metric. Examples will be given in the following Subsection, in the framework of contact metric geometry.

4.5 Harmonicity of the Reeb vector field as unit vector field

We now consider the special case when $V = \xi$ is the Reeb vector field of a contact metric manifold (M, η, g).

We first recall that a *K-contact* manifold is a contact metric manifold such that ξ is a Killing vector field with respect to g. Equivalently, M is K-contact if and only if $h = 0$. Any Sasakian manifold is K-contact and the converse also holds for three-dimensional spaces. For further details, references and information about contact metric manifolds, we refer to [6].

Now, for the Reeb vector field ξ, by Theorem 28 we obtain at once the following

32 Theorem. *Let (M, η, g) be a contact metric manifold and $\widetilde{G}$ an arbitrary Riemannian g-natural metric on T_1M. Then, $\xi : (M, g) \to (T_1M, \widetilde{G})$ is a harmonic map if and only if*

(i) $a\,\mathrm{tr}[R(\nabla_{\cdot}\xi, \xi)\cdot] = -2b\,(\mathrm{tr}h^2)\xi$, and

(ii) $Q\xi$ is collinear to ξ.

In particular, Theorem 32 yields

33 Corollary. *Consider a contact metric manifold (M, η, g), an arbitrary Riemannian g-natural metric $\widetilde{G}$ on T_1M and the map $\xi : (M, g) \to (T_1M, \widetilde{G})$ defined by the Reeb vector field. Then, ξ is a harmonic map if and only if M is H-contact and $a\mathrm{tr}[R(\nabla_{\cdot}\xi, \xi)\cdot] = -2b\,\mathrm{tr}h^2$.*

In particular:

(a) when M is K-contact, ξ is a harmonic map if and only if $\mathrm{tr}[R(\nabla_{\cdot}\xi, \xi)\cdot] = 0$;

(b) when M is Sasakian, ξ is a harmonic map.

By statement *(a)* of Corollary 33, if M is K-contact, then the harmonicity of the map $\xi : (M, g) \to (T_1M, \widetilde{G})$ does not depend on $\widetilde{G}$. Indeed, this is a special case of Theorem 31, because by definition the Reeb vector field of a K-contact space is Killing.

If we restrict ourselves to Riemannian g-natural metrics with $b = 0$ (that is, for which horizontal and tangential distributions are orthogonal), then harmonicity of $\xi : (M, g) \to (T_1M, \widetilde{G})$ does not depend on the particular $\widetilde{G}$. More precisely, we have the following

34 Corollary. *Consider a contact metric manifold* (M, η, g) *and a Riemannian g-natural metric* $\widetilde{G}$ *on* T_1M *with* $b = 0$*. Then, the following properties are equivalent:*

(i) $\xi : (M, g) \to (T_1M, \widetilde{G})$ *is a harmonic map;*

(ii) $\mathrm{tr}[R(\nabla_{\cdot}\xi, \xi)\cdot] = 0$ *and* M *is* H*-contact;*

(iii) $\xi : (M, g) \to (T_1M, \widetilde{g}^s)$ *is a harmonic map.*

35 Remark. All the results obtained for a unit vector field V, thought as a map $V : (M, g) \to (T_1M, \widetilde{G})$, can be easily adapted and rewritten for vector fields of *any* constant length $\rho > 0$, interpreted as maps $(M, g) \to (T_\rho M, \widetilde{G}')$. Notice that the underlying geometry of the target space is the same, since $(T_\rho M, \widetilde{G}')$ is isometric to $(T_1M, \widetilde{G})$ for a suitable Riemannian g-natural metric $\widetilde{G}$. Such an easy generalization is however useful and natural in some contexts. For example, given any Riemannian g-natural metric $\widetilde{G}$ on T_1M, the *geodesic flow vector field* defines a map $(T_1M, \widetilde{G}) \to (T_\rho T_1M, \tilde{\widetilde{G}})$, where $\tilde{\widetilde{G}}$ is any Riemannian g-natural metric on $T_\rho T_1M$ constructed from $\widetilde{G}$. The study of the harmonicity of the geodesic flow vector field has been made in [3].

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